Square Lattice Self-Avoiding Walks and Polygons

A.J. Guttmann¹ and A.R. Conway²

¹Department of Mathematics and Statistics, The University of Melbourne, Parkville, Victoria 3010, Australia

tonyg@ms.unimelb.edu.au

²Silicon Genetics, 2601 Spring Street, Redwood City, CA 94063, USA

Received January 3, 2001

AMS Subject Classification: 05A15

Dedicated to the memory of Gian-Carlo Rota, whose works have inspired and informed a generation of mathematicians

Abstract. We give an algorithm for the enumeration of self-avoiding walks on the (anisotropic) square lattice. Application of the algorithm on a 1024 processor Intel Paragon supercomputer resulted in a 51 term series. For (isotropic) square lattice self-avoiding polygons, a related algorithm has produced a 90 term series. Careful analysis provides compelling evidence for simple rational values of the exponents in both the dominant and subdominant terms in the asymptotic form of the coefficients. We also advance compelling arguments — but not a proof — that the generating function for SAW is *not* differentiable finite. The corresponding result for SAP has recently been proved.

Keywords: enumeration, self-avoiding walks, polygons

References

1. H.W.J. Blöte, T.W. Burkhardt, and I. Guim, Universality class of trails in two dimensions, *J. Phys. A: Math. Gen.* **30** (1997) 413–421.
2. R. Brak, A.J. Guttmann, and S.G. Whittington, On the behaviour of collapsing linear and branched polymers, *J. Math. Chem.* **8** (1991) 255–268.
3. J.L. Cardy and A.J. Guttmann, Universal amplitude combinations for self-avoiding walks, polygons and trails, *J. Phys. A* **26** (1993) 2485–2494.
4. T.C. Choy, R.J. Fleming, and S.R. Shannon, Corrections to scaling in two-dimensional polymer statistics, *Phys. Rev. B* **53** (1996) 2175–2178.
5. A.R. Conway and A.J. Guttmann, Algebraic techniques for enumerating self-avoiding walks on the square lattice, *J. Phys. A: Math. Gen.* **26** (1993) 1519–1534.
6. A.R. Conway and A.J. Guttmann, Square lattice self-avoiding walks and corrections to scaling, *Phys. Rev. Lett.* **77** (1996) 5284–5287.

7. I.G. Enting, Series expansions from the finite lattice method, *J. Phys. A*. **10** (1977) 801–805.
8. I.G. Enting and A.J. Guttmann, Polygons on the square, L and Manhattan lattices, *J. Phys. A: Math. Gen.* **18** (1985) 1007–1017.
9. I.G. Enting and A.J. Guttmann, The size and number of rings on the square lattice, *J. Phys. A: Math. Gen.* **21** (1988) L165–172.
10. I.G. Enting and A.J. Guttmann, The number of convex polygons on the square and honeycomb lattices, *J. Phys. A: Math. Gen.* **21** (1988) L467–474.
11. I.G. Enting and A.J. Guttmann, Polygons on the honeycomb lattice, *J. Phys. A: Math. Gen.* **22** (1989) 1371–1384.
12. I.G. Enting and A.J. Guttmann, On the solvability of some statistical mechanical systems, *Phys. Rev. Lett.* **76** (1996) 344–347.
13. I.G. Enting and A.J. Guttmann, unpublished.
14. M.E. Fisher and M.F. Sykes, Excluded-volume problem and the Ising model of ferromagnetism, *Phys. Rev.* **114** (1959) 45–58.
15. D.S. Gaunt, S.G. Whittington, and T.C. Yu, Free-energy of self-interacting uniform stars, *J. Phys. A: Math. Gen.* **30** (1997) 4607–4614.
16. A.J. Guttmann, On two dimensional self-avoiding random walks, *J. Phys. A: Math. Gen.* **17** (1984) 455–468.
17. A.J. Guttmann, On the critical behaviour of self-avoiding walks, *J. Phys. A: Math. Gen.* **20** (1987) 1839–1854.
18. A.J. Guttmann, Asymptotic analysis of coefficients, In: *Phase Transitions and Critical Phenomena*, Vol. 13, C. Domb and J. Lebowitz, Eds., Academic Press, London, 1989, pp. 1–234.
19. A.J. Guttmann, Indicators of solvability for lattice models, *Discrete Math.* **217** (2000) 167–189.
20. A.J. Guttmann, D.L. Hunter, N. Jan, S. Joseph, D. MacDonald, and L.L. Mosely, Self-avoiding walks on the simple-cubic lattice, *J. Phys. A*, **33** (2000) 5973–5983.
21. A.J. Guttmann and I. Jensen, Self-avoiding polygons on the square lattice, *J. Phys. A: Math. Gen.* **32** (1999) 4867–4876.
22. A.J. Guttmann, P.D. Roberts, M.F. Sykes, and M.G. Watts, The asymptotic behaviour of self-avoiding walks and returns on a lattice, *J. Phys. A: Math. Gen.* **5** (1972) 653–660.
23. A.J. Guttmann and J.S. Wang, The extension of self-avoiding random walk series in 2 dimensions, *J. Phys. A: Math. Gen.* **24** (1991) 3107–3109.
24. A.J. Guttmann and S.G. Whittington, Two-dimensional lattice embeddings of connected graphs of cyclomatic index two, *J. Phys. A* **11** (1978) 721–729.
25. J.M. Hammersley, The number of polygons on a lattice, *Proc. Camb. Phil. Soc.* **57** (1961) 516–523.
26. J.M. Hammersley and J.W. Morton, Poor Man’s Monte Carlo, *J. Royal. Stat. Soc. Ser. B* **16** (1954) 23–38.
27. J.M. Hammersley and D.J.A. Welsh, Further results on the rate of convergence to the connective constant of the hypercubical lattice, *Q. J. Math. (Oxford, 2nd series)* **13** (1962) 108–110.
28. T. Hara and G. Slade, Self-avoiding walk in five or more dimensions. I. The critical behaviour, *Commun. Math. Phys.* **147** (1992) 101–136.
29. T. Hara and G. Slade, The lace expansion for self-avoiding walks in five or more dimensions, *Rev. Math. Phys.* **4** (1992) 235–327.
30. T. Hara, G. Slade, and A.D. Sokal, New lower bounds on the self-avoiding walk connective constant, *J. Stat. Phys.* **72** (1993) 479–517.

31. P.C. Hemmer and S. Hemmer, An average self-avoiding random walk on the square lattice lasts 71 steps, *J. Chem. Phys.* **81** (1984) 584–585.
32. B.J. Hiley and M.F. Sykes, Probability of initial ring-closure in the restricted random walk of a macromolecule, *J. Chem. Phys.* **34** (1961) 1531–1537.
33. E. Hille and R.S. Phillips, *Functional Analysis and Semigroups*, A.M.S. Colloq. Publ. 31, Providence, R.I.
34. B.D. Hughes, *Random walks and random environments*, Vol. 1, Clarendon Press, Oxford, 1995.
35. D.L. Hunter, N. Jan, K. Kelly, and D. MacDonald, Self-avoiding walks in two to five dimensions. Exact enumerations and series study, *J. Phys A: Math. Gen.* **25** (1992) 1429–1440.
36. H. Kesten, On the number of self-avoiding walks, *J. Math. Phys.* **4** (1963) 960–969.
37. H. Kesten, On the number of self-avoiding walks II, *J. Math. Phys.* **5** (1964) 1128–1137.
38. R.S. Lehman and G.H. Weiss, A study of the restricted random walk, *J. SIAM* **6** (1958) 257–278.
39. N. Madras, A rigorous bound on the critical exponent for the number of lattice trees, animals and polygons, *J. Stat. Phys.* **78** (1995) 681–699.
40. N. Madras and G. Slade, *The Self-Avoiding Walk*, Birkhäuser, Boston–Basel–Berlin, 1993.
41. J.P. Maffar, B. Masand, S. Redner, and U. Wilensky, An extension of the two-dimensional self-avoiding walk series on the square lattice, *J. Phys. A: Math. Gen.* **25** (1992) L365–369.
42. B. Nienhuis, Exact critical points and critical exponents of $O(n)$ models in two dimensions, *Phys. Rev. Lett.* **49** (1982) 1062–1065.
43. G.L. O’Brien, Monotonicity of the number of self-avoiding walks, *J. Stat. Phys.* **59** (1990) 969–979.
44. E. Orlandini, E.J.J. van Rensburg, M.C. Tesi, and S.G. Whittington, Interacting self-avoiding walks and polygons in three dimensions, *J. Phys. A: Math. Gen.* **29** (1996) 2451–2464.
45. W.J.C. Orr, Statistical treatment of polymer solutions at infinite dilution, *Trans. Faraday Soc.* **43** (1947) 12–27.
46. A. Rechnitzer, Some problems in the counting of lattice animals, polyominoes, polygons and walks, Ph.D. Thesis, The University of Melbourne, 2001.
47. H. Saleur, Conformal invariance for polymers and percolation, *J. Phys A: Math. Gen.* **20** (1987) 455–470.
48. C.E. Soteros and S.G. Whittington, Contacts in self-avoiding walks and polygons, *J. Phys. A: Math. Gen.* **34** (2001) 4009–4040.
49. M.F. Sykes, Some counting theorems in the theory of the Ising model and the excluded volume problem, *J. Math. Phys.* **2** (1961) 52–62.