Recent progress on $q$-identities

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joint work with

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In this talk, I wish to present some recent work with my students and colleagues on $q$-identities, including:

1. **Some partition identities related to Euler’s partition theorem**
   We obtained a unification of two refinements of Euler’s partition theorem respectively due to Bessenrodt and Glaisher; With the aid of “rooted partition”, we reformulated two Ramanujan’s identities by two weighted forms of Euler’s partition theorem.

2. **A Franklin type involution for squares**
   We constructed a Franklin type involution for squares which leads to Bessenrodt and Pak’s partition theorem. Furthermore, our involution implies many other partition theorems.
(continued)

3 Zeilberger’s algorithm on partitions
We used the Algorithm Z on partitions due to Zeilberger, in a variant form, to give a combinatorial proof of Ramanujan’s \( \psi_1 \) summation formula.

4 Rogers-Ramanujan type identities
We obtained two Rogers-Ramanujan type identities for overpartitions, which imply a refined version of the anti-lecture hall theorem of Corteel and Savage. We also found involutions for three Rogers-Ramanujan-Gordon type identities obtained by Andrews on the generating functions for partitions with part difference and parity restrictions.
Overview

(continued)

(5) **Ramanujan’s third order mock theta functions**
We gave two partition identities for Ramanujan’s third order mock theta functions $\phi(-q)$ and $\psi(-q)$ which lead to two classical identities of Ramanujan on third order mock theta functions.

(6) **Congruences for bipartitions with odd parts distinct**
We considered congruences for the number of bipartitions with odd parts distinct. We also found combinatorial interpretations for two congruences modulo 2 and 3.

(7) **The method of combinatorial telescoping**
We presented a method for proving $q$-identities by combinatorial telescoping in the sense that one can transform a bijection or classification of combinatorial objects into a telescoping relation.
Outline

1. Some partition identities related to Euler’s partition theorem
2. A Franklin type involution for squares
3. Zeilberger’s algorithm on partitions
4. Rogers-Ramanujan type identities
5. Ramanujan’s third order mock theta functions
6. Congruences for bipartitions with odd parts distinct
7. The method of combinatorial telescoping
Euler’s partition theorem

**Theorem (Euler)**

The number of partitions of $n$ with **distinct parts** is equal to the number of partitions of $n$ with **odd parts**.

Let $D$ ($O$) denote the set of partitions with distinct parts (odd parts). Euler’s partition theorem could follow from the following identity:

$$ \sum_{\lambda \in D} q^{\lvert \lambda \rvert} = (-q; q)_\infty = \frac{1}{(q; q^2)_\infty} = \sum_{\mu \in O} q^{\lvert \mu \rvert}. $$

**Example**

There are three partitions of 5 with **distinct parts**: $(5), (4, 1), (3, 2)$, and there are also three partitions of 5 with **odd parts**: $(5), (3, 1, 1), (1, 1, 1, 1, 1)$. 
Sylvester’s refinement

There are many refinements of Euler’s partition theorem. The first refinement is due to Sylvester. Sylvester’s refinement involves the following two statistics:

- The number of chains of the partition $\lambda$ with distinct parts. 
  \[ n_c(\lambda) \]
  A chain in a partition with distinct parts is a maximal sequence of parts consisting of consecutive integers.
- The number of different parts of the partition $\mu$ with odd parts. 
  \[ n_d(\mu) \]

**Example**

*The partition $(8, 7, 5, 3, 2, 1)$ has three chains, that is, 8-7, 5, 3-2-1. The partition $(8, 6, 6, 5, 2, 2)$ has four different parts.*
Sylvester established the following refinement by constructing a bijection which we call Sylvester’s bijection.

**Theorem (Sylvester, Amer. J. Math., 1882)**

The number of partitions of $n$ into distinct parts with *exactly* $k$ chains is equal to the number of partitions of $n$ into odd parts (repetitions allowed) with *exactly* $k$ different parts. In the notation of generating functions, we have

$$
\sum_{\lambda \in D} z^{n_c(\lambda)} q^{\lambda} = \sum_{\mu \in O} z^{n_d(\mu)} q^{\mu}.
$$
Fine's first refinement

Fine gave the following celebrated refinement of Euler’s partition theorem.

**Theorem (Fine, Math. Surveys, 1988)**

The number of partitions of $n$ into distinct parts with largest part $k$ is equal to the number of partitions of $n$ into odd parts such that the largest part plus twice the number of parts equals $2k + 1$. In the notation of generating functions, we have

$$\sum_{\lambda \in D} x^{\lambda_1} q^{\lambda} = \sum_{\mu \in O} x^{(\mu_1 - 1)/2 + \ell(\mu)} q^{\mu}.$$  

**Remark:** Fine’s refinement can follow from Sylvester’s bijection just as observed by Bessenrodt (Discrete Math., 1994) and Kim-Yee (Ramanujan J., 1999).
Fine’s second refinement

Let \( r(\lambda) \) denote the rank of a partition \( \lambda \), which is defined as the largest part minus the number of parts, that is, \( r(\lambda) = \lambda_1 - \ell(\lambda) \). Fine presented another beautiful refinement of Euler’s partition theorem.


The number of partitions of \( n \) into distinct parts with the rank \( 2k \) or \( 2k + 1 \) is equal to the number of partitions of \( n \) into odd parts with the largest part \( 2k + 1 \). In the notation of generating functions, we have

\[
\sum_{\lambda \in D} x \left( r(\lambda) + \frac{1 + (-1)^{r(\lambda)}}{2} \right) q^{\mid \lambda \mid} = \sum_{\mu \in O} x^{\mu_1} q^{\mid \mu \mid}.
\]

**Remark:** Andrews (J. Natl. Acad. Math. India, 1983) gave an inductive proof and Pak (Math. Intelligencer, 2003) gave a combinatorial proof by constructing a new bijection which we call Pak’s iterated Dyson’s map.
Bessenrodt’s refinement

Bessenrodt has shown that Sylvester’s bijection implies another refinement, which is a limiting case of the lecture hall theorem due to Bousquet-Mélou and Erikssonin. Let $\ell_a(\lambda)$ denote the alternating sum of $\lambda = (\lambda_1, \lambda_2, \ldots)$, namely, $\ell_a(\lambda) = \lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 + \cdots$.

**Theorem (Bessenrodt, Discrete Math., 1994)**

The number of partitions of $n$ into distinct parts with alternating sum $\ell$ is equal to the number of partitions of $n$ with $\ell$ odd parts. In terms of generating functions, we have

$$\sum_{\lambda \in D} y^{\ell_a(\lambda)} q^{|\lambda|} = \sum_{\mu \in O} y^{\ell(\mu)} q^{|\mu|}.$$
Glaisher’s refinement

From a different angle, Glaisher has also given a refinement. Let $\ell_o(\lambda)$ denote the number of odd parts in $\lambda$, and let $n_o(\mu)$ denote the number of different parts in $\mu$ with odd multiplicities.

**Theorem (Glaisher, Messenger Math., 1883)**

The number of partitions of $n$ into distinct parts with $k$ odd parts is equal to the number of partitions of $n$ with odd parts such that there are exactly $k$ different parts repeated odd times. In terms of generating functions, we have

$$\sum_{\lambda \in D} x^{\ell_o(\lambda)} q^{\ell_1(\lambda)} = \sum_{\mu \in O} x^{n_o(\mu)} q^{\ell_1(\mu)}.$$

**Remark:** Glaisher’s refinement could not follow from Sylvester’s bijection and it can follow from a new bijection due to Glaisher, which we call Glaisher’s bijection.
Two-variable refinement

We gave the following unification of two refinements of Euler’s partition theorem respectively due to Bessenrodt and Glaisher.

**Theorem (Chen-Gao-Ji-Li, Ramanujan J., to appear)**

The number of partitions of $n$ into distinct parts with $\ell$ odd parts and alternating sum $m$ is equal to the number of partitions of $n$ into exactly $m$ odd parts and $\ell$ parts repeated odd times. In terms of generating functions, we have

$$
\sum_{\lambda \in D} x^{\ell_o(\lambda)} y^{\ell_a(\lambda)} q^{\mid \lambda \mid} = \sum_{\mu \in O} x^{n_o(\mu)} y^{\ell(\mu)} q^{\mid \mu \mid}.
$$
Two-variable refinement

Remarks:

- Setting $x = 1$, this refinement reduces to Bessenrodt's refinement which follows from Sylvester’s bijection.
- Setting $y = 1$, this refinement reduces to Glaisher’s refinement which follows from Glaisher’s bijection.
- Neither Sylvester’s bijection nor Glaisher’s bijection implies this result.

To prove this two-variable refinement, we construct another bijection for Euler’s partition theorem in which a specialization of Bessenrodt’s insertion algorithm (Discrete. Math., 1995) is the main ingredient.
Ramanujan’s identities

The following Ramanujan’s identities are related to Euler’s partition theorem just as anticipated by Andrews.

\[
\sum_{n=0}^{\infty} [(-q; q)_{\infty} - (-q; q)_n] = (-q; q)_{\infty} \left[ -\frac{1}{2} + \sum_{d=1}^{\infty} \frac{q^d}{1 - q^d} \right]
+ \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^{(n+1)/2}}{(-q; q)_n},
\]

\[
\sum_{n=0}^{\infty} \left[ \frac{1}{(q; q^2)_{\infty}} - \frac{1}{(q; q^2)_n} \right] = (-q; q)_{\infty} \left[ -\frac{1}{2} + \sum_{d=1}^{\infty} \frac{q^{2d}}{1 - q^{2d}} \right]
+ \frac{1}{2} \sum_{n=0}^{\infty} \frac{q^{(n+1)/2}}{(-q; q)_n}.
\]
**Background**

- **Andrews (Adv. Math., 1986)** gave the first algebraic proofs of these two identities and asked for their combinatorial proofs.

- Recently, **Andrews, Jiménez-Urroz and Ono (Duke Math. J., 2001)** reproved these two identities.

- **Chapman (J. Combin. Theory Ser. A, 2002)** also proposed that it would be interesting to find combinatorial proofs of these two identities.
Rooted partitions

To give combinatorial interpretations of the first sums on the right-hand sides of Ramanujan’s identities, we introduce the notion of rooted partitions.

**Definition**

A rooted partition of \( n \) can be formally defined as a pair of partitions \((\lambda, \mu)\), where \( |\lambda| + |\mu| = n \) and \( \mu \) is a nonempty partition with equal parts.

**Example**

There are twelve rooted partitions of 4:

\[
(\emptyset, (4)) \quad ((1), (3)) \quad ((3), (1)) \quad ((2), (2)) \\
(\emptyset, (2, 2)) \quad ((1, 1), (2)) \quad ((2, 1), (1)) \quad ((2), (1, 1)) \\
((1, 1, 1), (1)) \quad ((1, 1), (1, 1)) \quad ((1), (1, 1, 1)) \quad (\emptyset, (1, 1, 1, 1)).
\]
Combinatorial interpretations

We gave combinatorial interpretations of the following two terms with the aid of rooted partitions.


The following relation holds

\[
(−q; q)\sum_{d=1}^{∞} \frac{q^d}{1−q^d} = \sum_{\lambda \in O} 2\ell(\lambda)q^{|\lambda|} - \sum_{\mu \in D} \ell(\mu)q^{|\mu|}.
\]


The following relation holds

\[
(−q; q)\sum_{d=1}^{∞} \frac{q^{2d}}{1−q^{2d}} = \sum_{\lambda \in O} \ell(\lambda)q^{|\lambda|} - \sum_{\mu \in D} \ell(\mu)q^{|\mu|}.
\]
Connections to Euler’s partition theorem

Based on the above two lemmas, we may reformulate Ramanujan’s identities by the following two weighted forms of Euler’s partition theorem.


\[
\sum_{\mu \in D} \left( \ell(\mu) + \mu_1 + \frac{1 - (-1)^{r(\mu)}}{2} \right) q^{\mu} = \sum_{\lambda \in \mathcal{O}} 2\ell(\lambda) q^{\lambda}.
\]
Connections to Euler’s partition theorem


\[
\sum_{\mu \in D} \left( \ell(\mu) + \frac{1 - (-1)^{r(\mu)}}{2} \right) q^{\mid\mu\mid} = \sum_{\lambda \in O} \left( \ell(\lambda) - \frac{\lambda_1 - 1}{2} \right) q^{\mid\lambda\mid}.
\]

These two weighted forms can be deduced from weighted forms of Euler’s partition theorem coming from Sylvester’s bijection and Pak’s iterated Dyson’s map respectively.
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Bessenrodt and Pak’s partition theorem

Recently, Bessenrodt and Pak established a nice partition theorem analogous to Euler’s pentagonal number theorem by constructing an involution.

**Theorem (Bessenrodt-Pak, European J. Combin., 2004)**

Let $P_{do}(n)$ denote the set of partitions of $n$ into distinct parts with the smallest part being odd,

$$\sum_{\lambda \in P_{do}(n)} (-1)^{\ell(\lambda)} = \begin{cases} (-1)^k, & \text{if } n = k^2, \\ 0, & \text{otherwise}. \end{cases}$$

**Remark:** This partition theorem implies a theorem of Fine concerning the parity of the number of partitions in $P_{do}(n)$. 

Franklin type involution for squares

Chen and Liu (Adv. Appl. Math., to appear) constructed a Franklin type involution for squares which implies Bessenrodt and Pak’s partition theorem. Furthermore, it has many other applications.

**Main idea of our involution:** Let $D_k$ be the set of partitions $\pi$ into $k$ distinct parts such that $\pi_i - \pi_{i+1} \leq 2$ and the smallest part is 1. Let $E_k$ denote the set of partitions $\sigma$ into even parts less than or equal to $2k$, namely $\sigma_1 \leq 2k$. For a partition $\lambda$ in $P_{do}(n)$, we will split it into a pair of partitions in $D_k \times E_k$ and then build an involution on the set $D_k \times E_k$ such that the fixed points are squares.
Alladi’s first partition theorem

The first application of our involution gives a combinatorial proof of a partition theorem of Alladi.

**Theorem (Alladi, Ramanujan J., 2009)**

For \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell) \in P_{do} \), define \( \delta_i \) to be the least integer \( \geq (\lambda_i - \lambda_{i+1})/2 \), where \( \lambda_{\ell+1} \) is defined to be 0. Define the weight of \( \lambda \) by

\[
\omega_g(\lambda) = (-1)^\ell \prod_{i=1}^{\ell} a^{\delta_i}.
\]

Then we have

\[
\sum_{\lambda \in P_{do}(n)} \omega_g(\lambda) = \left\{ \begin{array}{ll}
(-a)^k, & \text{if } n = k^2, \\
0, & \text{otherwise}.
\end{array} \right.
\]
Alladi’s first partition theorem

Alladi’s weighted partition theorem is a combinatorial explanation of the following Ramanujan’s partial theta identity

$$1 + \sum_{k=1}^{\infty} \frac{(-q; q)_{k-1}(-a)^k q^{k(k+1)/2}}{(aq^2; q^2)_k} = \sum_{k=0}^{\infty} (-a)^k q^{k^2}.$$ 

Remarks:

- Berndt, Kim and Yee (J. Combin. Theory Ser. A., 2009) first gave an involution to prove this Ramanujan’s partial theta identity in terms of parity sequence.
- Later, Yee (Ramanujan J., 2009) gave another combinatorial proof of this identity.
- However, none of them seems to imply Alladi’s partition theorem.
Alladi’s second partition theorem

The second application of our involution serves as a combinatorial interpretation of another partition theorem of Alladi.

**Theorem (Alladi, Ramanujan J., 2010)**

For \( \lambda \in P_{do}(n) \), let \( \ell_o(\lambda) \) denote the number of odd parts of \( \lambda \). Define the weight of \( \lambda \) by

\[
\omega_o(\lambda) = (-1)^{\ell_o(\lambda)} a^{\ell_o(\lambda)}.
\]

Then we have

\[
\sum_{\lambda \in P_{do}(n)} \omega_o(\lambda) = \begin{cases} 
(-a)^k, & \text{if } n = k^2, \\
0, & \text{otherwise}.
\end{cases}
\]

**Remark:** We love weights as much as Krishna Alladi.
Andrews’ partial theta identity

Alladi’s partition theorem is the combinatorial interpretation of the following Andrews’ partial theta identity,

$$\sum_{n=0}^{\infty} q^{2n} (q^{2n+2}; q^2)_\infty (aq^{2n+1}; q^2)_\infty = \sum_{k=0}^{\infty} (-a)^k q^{k^2}.$$ 

For the special case of $a = -1$, Andrews’ identity connects to a theorem of Andrews. Our involution can also prove Andrews' theorem.


Let $P_{de}(n)$ denote the set of partitions of $n$ into distinct parts with the smallest part being even. We have

$$\sum_{\lambda \in P_{de}(n)} (-1)^{\ell(\lambda)} = \begin{cases} 1, & \text{if } n = k^2, \\ 0, & \text{otherwise.} \end{cases}$$
Generalizations of Andrews’ partial theta identity

Furthermore, we extended Andrews’ partial theta identity to a more general case by using our involution.


\[
\sum_{n=0}^{\infty} q^{2mn}(q^{2mn+2m}; q^2)_{\infty} (aq^{2mn+1}; q^2)_{\infty}
\]

\[
= 1 + \sum_{k=1}^{\infty} (-a)^k q^{k^2} \prod_{j=1}^{k} (1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}).
\]

**Remark:** This identity reduces to Andrews’ partial theta identity when \(m = 1\).
Generalizations of Andrews’ partial theta identity

Recently, Ismail and Stanton gave a further generalization of Andrews’ identity, and provided a combinatorial interpretation.

**Theorem (Ismail-Stanton, preprint)**

For any positive integer \( m \),

\[
\sum_{n=0}^{\infty} q^{2mn} (q^{2mn+2m}; q^{2m})_\infty \frac{(aq^{2mn+1}; q^2)_\infty}{(abq^{2mn+1}; q^2)_\infty} = 1 + \sum_{k=1}^{\infty} (abq)^k (1/b; q^2)_k \prod_{j=1}^{k} (1 + q^{2j} + q^{4j} + \cdots + q^{2(m-1)j}).
\]

**Remark:** Their proof is based on \( q \)-binomial theorem. Our identity is the special case \( b \to 0 \) of Ismail and Stanton’s identity.
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The Algorithm Z

Zeilberger found the following algorithm which is called the Algorithm Z by Andrews and Bressoud.


There is a bijection between the set of pairs of partitions \((\alpha, \beta)\) and the set of pairs of partitions \((\mu, \nu)\), where

- \(\alpha\) has \(i\) parts, \(\beta\) has \(j\) parts;
- \(\mu\) has \(i + j\) parts, \(\nu\) has \(j\) parts with each part \(\leq i\);
- \(|\alpha| + |\beta| = |\mu| + |\nu|\).
**Applications of the Algorithm Z**

- The Algorithm Z gives a combinatorial interpretation of the Gauss coefficient $\left[ \begin{array}{c} i+j \\ i \end{array} \right]$ satisfying that
  \[
  \frac{1}{(q; q)_{i+j}} \left[ \begin{array}{c} i+j \\ i \end{array} \right] = \frac{1}{(q; q)_i(q; q)_j}.
  \]

- Andrews and Bressoud (Discrete Math., 1984) have found combinatorial proofs of some classical $q$-identities by this algorithm.


- Fu (Discrete Math., 2007) applied this algorithm to give a combinatorial interpretation of the Lebesgue identity.
Applications of the Algorithm Z

We gave a combinatorial interpretation of $q$-binomial theorem using the Algorithm Z:

$$\sum_{n \geq 0} \frac{(-a/b; q)_n}{(q; q)_n} (bz)^n = \frac{(-az; q)_{\infty}}{(bz; q)_{\infty}}.$$ 

**Theorem (Chen-Chen-Fu-Zang, Ramanujan J., to appear)**

*There is a bijection between the set of pairs of partitions $(\alpha, \beta)$ and the set of pairs of partitions $(\mu, \nu)$, where*

- $\alpha$ has $i$ distinct parts, $\beta$ has $n - i$ parts;
- $\mu$ has $n$ parts, $\nu$ has $i$ distinct parts with each part $\leq n - 1$;
- $|\alpha| + |\beta| = |\mu| + |\nu|.$
Applications of the Algorithm $Z$
A variation of the Algorithm Z

We gave a variation of the Algorithm Z which plays a key role in our combinatorial proof of Ramanujan’s summation formula.

Theorem (Chen-Chen-Fu-Zang, Ramanujan J., to appear)

There is a bijection between the set of pairs of partitions \((\alpha, \beta)\) and the set of pairs of partitions \((\mu, \nu)\), where

- \(\alpha\) has \(s\) distinct parts with each part \(\geq m\) and \(\beta\) has \(t\) parts with each part \(\geq k + s + t - 1\);
- \(\mu\) has \(s + t\) distinct nonnegative parts with \(\mu_s - \mu_{s+1} \geq m + 1\), \(\nu\) has \(t\) distinct parts with \(k \leq \nu_i \leq k + s + t - 1\) for each \(1 \leq i \leq t\);
- \(|\alpha| + |\beta| = |\mu| + |\nu|\).
**Ramanujan’s \(1\psi_1\) summation**

Ramanujan’s summation for \(1\psi_1\) is usually stated in the following form:

**Theorem (Ramanujan)**

\[
1\psi_1(a; b; q, z) = \sum_{n=-\infty}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty},
\]

where \(|b/a| < |z| < 1, |q| < 1\).

Combinatorial proofs of Ramanujan’s $\psi_1$ summation

The combinatorial proofs have appeared only recently.

- Corteel and Lovejoy (J. Combin. Theory Ser. A, 2002) have found a bijective proof of the constant term identity for this summation.
We provided a new combinatorial proof of Ramanujan’s \( 1\psi_1 \) sum based on a variation of the Algorithm Z and \( q \)-binomial theorem. Our bijection is devised for the following restatement of Ramanujan’s formula:

\[
\frac{(-q/a; q)_{\infty} (-b/az; q)_{\infty}}{(q; q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-a; q)_n}{(b; q)_n} z^n = \frac{(-b/a; q)_{\infty} (-az; q)_{\infty} (-q/az; q)_{\infty}}{(b; q)_{\infty} (z; q)_{\infty}}.
\]
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Rogers-Ramanujan’s identities

The analytic statement of the Rogers-Ramanujan identities was independently discovered by Rogers, Ramanujan and Schur.

**Theorem (Rogers-Ramanujan’s identities)**

\[
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_n} = \frac{1}{(q, q^4; q^5)_\infty},
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q; q)_n} = \frac{1}{(q^2, q^3; q^5)_\infty}.
\]
Combinatorial interpretations

MacMahon and Schur gave the following combinatorial statement of the Rogers-Ramanujan identities independently.

**Theorem (The first Rogers-Ramanujan identity)**

The number of partitions of $n$ in which the difference between any two parts is at least 2 is equal to the number of partitions of $n$ into parts not congruent to $0, \pm 2$ modulo 5.

**Theorem (The second Rogers-Ramanujan identity)**

The number of partitions of $n$ in which each part exceeds 1 and the difference between any two parts is at least 2 is equal to the number of partitions of $n$ into parts not congruent to $0, \pm 1$ modulo 5.
Rogers-Ramanujan-Gordon’s identity

In 1961, Gordon obtained the following celebrated combinatorial generalization of the Rogers-Ramanujan identities by building an involution.

Theorem (Gordon, Amer. J. Math., 1961)

Supposed 1 ≤ a ≤ k are positive integers, and let \( B_{k,a}(n) \) denote the number of partitions of \( n \) of the form \((b_1, b_2, \ldots, b_\ell)\) where \( b_j - b_{j+k-1} \geq 2 \) and at most \( a - 1 \) of the \( b_i \) are equal to one, and let \( A_{k,a}(n) \) denote the number of partitions of \( n \) into parts not congruent to 0, \( \pm a \) modulo \( 2k + 1 \). Then

\[
B_{k,a}(n) = A_{k,a}(n).
\]

Remark: Setting \( k = a = 2 \) in this theorem gives the first Rogers-Ramanujan identity and \( k = a + 1 = 2 \) gives the second Rogers-Ramanujan identity.
Rogers-Ramanujan-Andrews’s identity

After Gordon’s involution for this theorem in 1961, Andrews provided a generating function proof of the Rogers-Ramanujan-Gordon identity in the following equivalent form:


$$
\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1^2 + N_2^2 + \cdots + N_{k-1}^2 + N_a + \cdots + N_k}}{(q; q)_{n_1} (q; q)_{n_2} \cdots (q; q)_{n_{k-1}}}
= \frac{(q^a; q^{2k+1})_\infty (q^{2k+1-a}; q^{2k+1})_\infty (q^{2k+1}; q^{2k+1})_\infty}{(q; q)_\infty},
$$

where $n_i = N_i - N_{i+1}$ and $1 \leq a \leq k$. 

Restrictions on Rogers-Ramanujan-Gordon’s identity

Recently, Andrews considered the parity restrictions on partitions in connection with the Rogers-Ramanujan-Gordon identity. He derived the following three identities.

**Theorem (Andrews, Ramanujan J., to appear)**

Suppose $k \geq a \geq 1$ are integers such that $k$ and $a$ are both even. Let $W_{k,a}(n)$ denote the number of partitions enumerated by $B_{k,a}(n)$ with further restriction that even parts appear an even number of times. Then we have

$$
\sum_{n \geq 0} W_{k,a}(n) q^n = \frac{(-q; q^2)_\infty (q^a; q^{2k+2})_\infty (q^{2k+2-a}; q^{2k+2})_\infty (q^{2k+2}; q^{2k+2})_\infty}{(q^2; q^2)_\infty}.
$$
Restrictions on Rogers-Ramanujan-Gordon’s identity

**Theorem (Andrews, Ramanujan J., to appear)**

Suppose $k \geq a \geq 1$ are integers such that $k$ and $a$ are both odd. Let $W_{k,a}(n)$ denote the number of those partitions enumerated by $B_{k,a}(n)$ with further restriction that even parts appear an even number of times. Then for all $n \geq 0$, we have

$$
\sum_{n \geq 0} W_{k,a}(n)q^n = \frac{(q^2; q^4)_{\infty}(q^a; q^{2k+2})_{\infty}(q^{2k+2-a}; q^{2k+2})_{\infty}(q^{2k+2}; q^{2k+2})_{\infty}}{(q; q)_{\infty}}.
$$
Restrictions on Rogers-Ramanujan-Gordon’s identity

Theorem (Andrews, Ramanujan J., to appear)

Suppose $k \geq a \geq 1$, $k$ is odd and $a$ is even. Let $\overline{W}_{k,a}(n)$ denote the number of those partitions enumerated by $B_{k,a}(n)$ with further restriction that odd parts appear an even number of times. Then for all $n \geq 0$, we have

$$
\sum_{n \geq 0} \overline{W}_{k,a}(n) q^n = \frac{(q^a; q^{2k+2})_\infty (q^{2k+2-a}; q^{2k+2})_\infty (q^{2k+2}; q^{2k+2})_\infty}{(-q; q^2)_\infty (q; q)_\infty}.
$$

Remark: Chen, Sang and Shi (2010, arXiv:1006.4081) built three involutions for Andrews' identities in which Gordon’s involution is the main ingredient.
**Overpartition analogue of Rogers-Ramanujan-Andrews’ identity**

On the other hand, we employed Andrews’ multiple series transformation to derive two Rogers-Ramanujan type identities for overpartitions. They can be considered as analogous to Andrews’ Rogers-Ramanujan identities.


\[
\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} \frac{q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2}N_1+N_2+\cdots+N_{k-1}}{(q; q)_{N_1-N_2-N_3-\cdots-N_{k-1}-N_{k-1}} (q; q)_{N_{k-1}}} (-q; q)_{N_1} \\
= (q; q)_\infty (q; q^{2k})_\infty (q^{2k-1}; q^{2k})_\infty (q^{2k}; q^{2k})_\infty (q; q)_\infty.
\]
Overpartition analogue of Rogers-Ramanujan-Andrews’ identity


\[
\sum_{N_1 \geq N_2 \geq \cdots \geq N_{k-1} \geq 0} q^{N_1(N_1+1)/2+N_2^2+\cdots+N_{k-1}^2+N_2+\cdots+N_{k-1}} (-q; q)_{N_1} \\
(q; q)_{N_1-N_2} \cdots (q; q)_{N_{k-2}-N_{k-1}} (q; q)_{N_{k-1}} (-q; q)_{N_{k-1}} \\
= \frac{(-q; q)_{\infty} (q; q^{2k-1})_{\infty} (q^{2k-2}; q^{2k-1})_{\infty} (q^{2k-1}; q^{2k-1})_{\infty}}{(q; q)_{\infty}}.
\]

**Remark:** As an application of these two identities, we found a finite version of the anti-lecture hall theorem of Corteel and Savage.
Overpartitions

Let us review some notation on overpartitions and anti-lecture hall compositions.

**Definition**

An overpartition of \( n \) is a non-increasing sequence of positive integers whose sum is \( n \) in which the first occurrence of a number may be overlined.

**Example**

There are 8 overpartitions of 3. They are

\[(3), (2, 1), (1, 1, 1), (\bar{3}), (\bar{2}, 1), (2, \bar{1}), (\bar{2}, \bar{1}), (\bar{1}, 1, 1).\]
Anti-lecture hall compositions

**Definition**

An *anti-lecture hall composition* of length $k$ is defined to be an integer sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that

\[
\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \cdots \geq \frac{\lambda_{k-1}}{k-1} \geq \frac{\lambda_k}{k} \geq 0.
\]

**Example**

For $k = 5$, $(5, 4, 5, 6, 4)$ is an anti-lecture hall composition.
Anti-lecture hall theorem

**Theorem (Corteel and Savage, Discrete Math., 2003)**

Let $A_k$ denote the set of anti-lecture hall compositions of length $k$ and $A$ denote the set of anti-lecture hall compositions. Then

$$\sum_{\lambda \in A_k} q^{|\lambda|} = \prod_{i=1}^{k} \frac{1 + q^i}{1 - q^{i+1}}.$$  

It follows that

$$\sum_{\lambda \in A} q^{|\lambda|} = \prod_{i=1}^{\infty} \frac{1 + q^i}{1 - q^{i+1}}.$$
Finite version of anti-lecture hall theorem

In the language of overpartitions, the above identity can be stated as the following partition theorem:

**Theorem (Anti-lecture hall theorem)**

The number of anti-lecture hall compositions of $n$ equals the number of overpartitions of $n$ with the non-overlined parts larger than 1.

By using our Rogers-Ramanujan type identities for overpartitions, we derive the following finite version.


The number of anti-lecture hall compositions of $n$ of the form $\lambda = (\lambda_1, \lambda_2, \ldots)$ such that $\lambda_1 \leq k - 2$ equals the number of overpartitions of $n$ in which the non-overlined parts are not congruent to 0, $\pm 1$ modulo $k$. 
Outline

1. Some partition identities related to Euler’s partition theorem
2. A Franklin type involution for squares
3. Zeilberger’s algorithm on partitions
4. Rogers-Ramanujan type identities
5. Ramanujan’s third order mock theta functions
6. Congruences for bipartitions with odd parts distinct
7. The method of combinatorial telescoping
### Definitions

Ramanujan defined the following three mock theta functions of order 3 which have many remarkable analytic properties.

**Definition**

\[
\begin{align*}
    f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q; q)_n^2}, \\
    \phi(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q^2; q^2)_n}, \\
    \psi(q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q; q^2)_n}.
\end{align*}
\]
Ramanujan’s identities

Ramanujan also found the following two relations.

**Theorem (Ramanujan’s identities)**

\[
\phi(-q) - 2\psi(-q) = f(q), \\
\phi(-q) + 2\psi(-q) = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}^2}.
\]

The first proofs were given by Watson (J. London Math. Soc., 1936). Fine (Basic Hypergeometric Series and Applications, 1988, p. 60) found another proof by using transformation formulas.
Generalizations of Ramanujan’s mock theta functions

Andrews (Quart. J. Math., 1966) defined the following functions as generalizations of Ramanujan’s mock theta functions.

**Definition**

\[
\begin{align*}
f(\alpha; q) &= \sum_{n=0}^{\infty} \frac{q^{n^2-n} \alpha^n}{(-q; q)_n(-\alpha; q)_n}, \\
\phi(\alpha; q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-\alpha q; q^2)_n}, \\
\psi(\alpha; q) &= \sum_{n=1}^{\infty} \frac{q^{n^2}}{\alpha(q^2)_n}.
\end{align*}
\]

When \(\alpha = q\), the above functions reduce to Ramanujan’s mock theta functions.
Generalizations of Ramanujan’s identities

Andrews also extended Ramanujan’s identities to the functions $f(\alpha; q)$, $\phi(\alpha; q)$ and $\psi(\alpha; q)$.

**Theorem (Andrews, Quart. J. Math., 1966)**

\[
\begin{align*}
\phi(-\alpha; -q) - (1 + \alpha q^{-1})\psi(-\alpha; -q) & = f(\alpha; q), \\
\phi(-\alpha; -q) + (1 + \alpha q^{-1})\psi(-\alpha; -q) & = \frac{(q; q)_\infty}{(-q; q)_\infty (-\alpha; q)_\infty}.
\end{align*}
\]

Clearly, the above identities specialize to Ramanujan’s identities by setting $\alpha = q$. 
**Fine’s partition identity for \( f(q) \)**

The connection between Ramanujan’s mock theta functions and the theory of partitions was first explored by Fine. Fine derived the following partition identity for \( f(q) \) from his transformation formula.

**Theorem (Fine, Basic Hypergeometric Series and Applications, 1988, p. 55)**

Let \( p_{do}(n) \) denote the number of partitions of \( n \) into distinct parts with the smallest part being odd. Then

\[
(-q; q)_\infty f(q) = 1 + 2 \sum_{n \geq 1} p_{do}(n)q^n.
\]

**Remark:** Recently, Chen, Ji and Liu (2010, arXiv:1006.3194) gave a simple involution for this identity.
Andrews’ partition identity for $\mathcal{F}_1(q)$

Andrews has also found a similar partition identity for Ramanujan’s seventh order mock theta function $\mathcal{F}_1(q)$.


Let $F_o(n)$ denote the number of partitions of $n$ in which no even parts are repeated and no even part is smaller than a repeated odd part, and if an odd part $2j - 1$ is repeated then each odd positive integer smaller than $2j - 1$ appears in the partition as a repeated part. Then

$$(-q; q)_{\infty} \mathcal{F}_1(q^2) = \sum_{n \geq 1} F_o(n)q^n,$$

where

$$\mathcal{F}_1(q) = \sum_{n=1}^{\infty} \frac{q^{n^2}}{(q^n; q)_n}.$$
Partition identities for $\phi(-q)$ and $\psi(-q)$

We also obtained the following two partition identities for $\phi(-q)$ and $\psi(-q)$ by building two involutions.


\[
(-q; q)\phi(-q) = 1 + \sum_{n=1}^{\infty} p_{do}(n)q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2},
\]

\[
2(-q; q)\psi(-q) = -\sum_{n=1}^{\infty} p_{do}(n)q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2}.
\]
Partition identities for $\phi(-q)$ and $\psi(-q)$

The above two partition identities can be restated as the forms analogous to Fine’s identity for $f(q)$ by employing Bessenrodt and Pak’s partition theorem.


Let $p_{do}^{e}(n)$ ($p_{do}^{o}(n)$) denote the number of partitions of $n$ into even (odd) distinct parts with the smallest part being odd. Then

$$(-q; q)_{\infty} \phi(-q) = 1 + 2 \sum_{n \geq 1} p_{do}^{e}(n)q^{n},$$

$$(-q; q)_{\infty} \psi(-q) = - \sum_{n \geq 1} p_{do}^{o}(n)q^{n}.$$
Applications

As an application, the above partition identities can lead to Ramanujan’s identities. It follows that

\[(−q; q)∞ \phi(−q) − 2(−q; q)∞ \psi(−q)\]

\[= 1 + \sum_{n=1}^{∞} p_{do}(n)q^n + \sum_{k=1}^{∞} (-1)^k q^{k^2}\]

\[+ \sum_{n=1}^{∞} p_{do}(n)q^n − \sum_{k=1}^{∞} (-1)^k q^{k^2}\]

\[= 1 + 2 \sum_{n \geq 1} p_{do}(n)q^n\]

\[= (−q; q)∞ f(q),\]

which implies the first Ramanujan’s identity by dividing both sides by \((−q; q)∞\).
Applications

The second Ramanujan’s identity could be derived as follows. In terms of two partition identities for $\phi(-q)$ and $\psi(-q)$, we have

\[
(-q; q)_{\infty} \phi(-q) + 2(-q; q)_{\infty} \psi(-q)
\]

\[
= 1 + \sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2}
\]

\[
- \sum_{n=1}^{\infty} p_{do}(n) q^n + \sum_{k=1}^{\infty} (-1)^k q^{k^2}
\]

\[
= 1 + 2 \sum_{k \geq 1} (-1)^k q^{k^2} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}},
\]

where the last equality follows from Gauss’ identity:

\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2} = \frac{(q; q)_{\infty}}{(-q; q)_{\infty}}.
\]

This yields the second Ramanujan’s identity after dividing both sides by $(-q; q)_{\infty}$. 
Partition identities for $f(\alpha q; q)$, $\phi(-\alpha q; -q)$ and $\psi(-\alpha q; -q)$

Similarly, we derived the following partition identities for $f(\alpha q; q)$, $\phi(-\alpha q; -q)$ and $\psi(-\alpha q; -q)$ based on our involutions which imply Andrews’ identities.


Let $P_{do}$ denote the set of partitions into distinct parts with the smallest part being odd. Then

\[
(-\alpha q; q)_{\infty} f(\alpha q; q) = 1 + 2 \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|},
\]

\[
(-\alpha q; q)_{\infty} \phi(-\alpha q; -q) = 1 + \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^k,
\]

\[
(-\alpha; q)_{\infty} \psi(-\alpha q; -q) = - \sum_{\nu \in P_{do}} \alpha^{l(\nu)} q^{|\nu|} + \sum_{k=1}^{\infty} (-1)^k q^k.
\]
Outline

1 Some partition identities related to Euler’s partition theorem

2 A Franklin type involution for squares

3 Zeilberger’s algorithm on partitions

4 Rogers-Ramanujan type identities

5 Ramanujan’s third order mock theta functions

6 Congruences for bipartitions with odd parts distinct

7 The method of combinatorial telescoping
Definitions

We considered the arithmetic properties of the number of bipartitions with odd parts distinct.

**Definition**

A bipartition $\pi$ of $n$ means a pair of partitions $(\alpha, \beta)$ with $|\alpha| + |\beta| = n$.

**Remark:** A bipartition $\pi = (\alpha, \beta)$ with odd parts distinct means that both odd parts of $\alpha$ and odd parts of $\beta$ are distinct.

**Example**

There are 11 bipartitions of 4 with odd parts distinct:

$((4), \emptyset)$ $((3, 1), \emptyset)$ $((2, 2), \emptyset)$ $((3), (1))$ $((2, 1), (1))$ $((2), (2))$

$((1), (2, 1))$ $((1), (3))$ $(\emptyset, (2, 2))$ $(\emptyset, (3, 1))$ $(\emptyset, (4))$. 
Background

Our work is mainly inspired by the following three papers.

- Andrews et al. (Ramanujan J., to appear) have investigated arithmetic properties of the number of partitions with even parts distinct.
- Hirschhorn and Sellers (Ramanujan J., 2010) considered arithmetic properties of the number of partitions with odd parts distinct.
Ramanujan type congruences

Let \( pod_2(n) \) denote the number of bipartitions of \( n \) with odd parts distinct. By convention, we set \( pod_2(0) = 1 \).

\[
\sum_{n=0}^{\infty} pod_2(3n + 2)q^n = 3 \frac{(q^2; q^2)_\infty^4 (q^6; q^6)_\infty}{(q; q)_\infty^6 (q^4; q^4)_\infty}. 
\]


Remark: The key idea of the proof is using the 3-dissection of \( 1/\psi(-q) \):

\[
\frac{1}{\psi(-q)} = \frac{\psi(-q^9)}{\psi(-q^3)^4} \left( A(-q^3)^2 + qA(-q^3)\psi(-q^9) + q^2\psi(-q^9)^2 \right),
\]

where

\[
A(q) = \frac{(q^2; q^2)_\infty (q^3; q^3)_\infty^2}{(q; q)_\infty (q^6; q^6)_\infty}. 
\]
From the above theorem, we obtained the following congruence relation.


For all $n \geq 0$,

$$\text{pod}_{-2}(3n + 2) \equiv 0 \pmod{3}.$$ 

**Remark:** We introduced two ranks for bipartitions $\pi = (\alpha, \beta)$ with odd parts distinct which can be used to explain the above congruence.

The first rank $\widehat{r(\pi)}$ is defined to be the difference between the number of parts of $\alpha$ and the number of parts of $\beta$, i.e., $\widehat{r(\pi)} = \ell(\alpha) - \ell(\beta)$.

The second rank $\overline{r(\pi)}$ is defined to be the difference between the largest part of $\alpha$ and the largest part of $\beta$, i.e., $\overline{r(\pi)} = \alpha_1 - \beta_1$. 

**Combinatorial interpretations for congruence modulo 3**
Infinite families of congruences modulo 3


For all \( \alpha \geq 1 \) and \( n \geq 0 \),

\[
pod_{-2} \left( 3^{2\alpha+1} n + \frac{23 \times 3^{2\alpha} - 7}{8} \right) \equiv 0 \pmod{3}.
\]

Proof. Using 3-dissection of \( \psi(q) \) to obtain that

\[
\sum_{n=0}^{\infty} (-1)^{n+1} pod_{-2}(3n + 1)q^n \equiv \psi(q)^2 \pmod{3}.
\]

Then our result follows from the following well known fact:
The positive integer \( n \) can not be the sum of two triangular numbers if and only if there exists a prime \( p \) congruent to 3 modulo 4 that has an odd exponent in the canonical factorization of \( 4n + 3 \).
Infinite families of congruence modulo 3


For all $\alpha \geq 1$ and $n \geq 0$,

$$\text{pod}_{-2}(3^{2\alpha+1}n + \frac{7 \times 3^{2\alpha} + 1}{4}) \equiv 0 \pmod{3},$$

$$\text{pod}_{-2}(3^{2\alpha+1}n + \frac{11 \times 3^{2\alpha} + 1}{4}) \equiv 0 \pmod{3}.$$

**Proof.** Using the 3-dissection of $1/\psi(-q)$ to prove the following relations,

$$\text{pod}_{-2}(27n + 16) \equiv \text{pod}_{-2}(27n + 25) \equiv 0 \pmod{3}$$

and

$$\text{pod}_{-2}(3n + 1) \equiv \text{pod}_{-2}(27n + 7) \pmod{3}.$$
Infinite families of congruence modulo 5


For all \( \alpha \geq 1 \) and \( n \geq 0 \),

\[
\text{pod}_2 \left( 5^{\alpha + 1} n + \frac{11 \times 5^{\alpha} + 1}{4} \right) \equiv 0 \pmod{5},
\]

\[
\text{pod}_2 \left( 5^{\alpha + 1} n + \frac{19 \times 5^{\alpha} + 1}{4} \right) \equiv 0 \pmod{5}.
\]

Proof. Using the Ramanujan’s \( q \)-expansion of \( q \psi(q)^3 \psi(q^5) - 5q^2 \psi(q) \psi(q^5)^3 \) to derive the following relations:

\[
\text{pod}_2(25n + 14) \equiv \text{pod}_2(25n + 24) \equiv 0 \pmod{5}
\]

and

\[
\text{pod}_2(25n + 19) \equiv -\text{pod}_2(5n + 4) \pmod{5}.
\]
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The combinatorial telescoping

Let us consider a summation of the form

$$\sum_{k=0}^{\infty} (-1)^k f(k).$$

Suppose that for each $k$,

$$f(k) = \sum_{\alpha \in A_k} w(\alpha)$$

is the weighted count of a set $A_k$. 
The combinatorial telescoping

We have the following theorem:


If we can find two sets $B_k$ and $H_k$ with a weight assignment $w$ such that there is a weight preserving bijection

$$\phi_k : A_k \rightarrow B_k \cup H_k \cup H_{k+1},$$

where $\cup$ stands for disjoint union. Then we have

$$\sum_{k=0}^{\infty} (-1)^k f(k) = \sum_{k=0}^{\infty} (-1)^k g(k),$$

where

$$g(k) = \sum_{\alpha \in B_k} w(\alpha).$$
The combinatorial telescoping

Proof. Since $\phi_k$ is weight preserving and set

$$ h(k) = \sum_{\alpha \in H_k} w(\alpha), $$

then the bijection in the above theorem implies that

$$ f(k) = g(k) + h(k) + h(k + 1). $$

If we set

$$ f'(k) = (-1)^k f(k), \quad g'(k) = (-1)^k g(k), \quad h'(k) = (-1)^k h(k). $$

Then we have

$$ f'(k) = g'(k) + h'(k) - h'(k + 1). $$

Summing the above identity over $k$, we deduce the following relation

$$ \sum_{k=0}^{\infty} (-1)^k f(k) = \sum_{k=0}^{\infty} (-1)^k g(k). $$
**Sylvester’s identity**

The above approach is called **combinatorial telescoping** motivated by the idea of creative telescoping of Zeilberger (J. Symbolic Comput., 1991).

**Example (Sylvester, Amer. J. Math., 1882)**

\[
\sum_{k=0}^{\infty} (-1)^k q^{k(3k+1)/2} x^k \frac{1 - xq^{2k+1}}{(q; q)_k (xq^{k+1}; q)_\infty} = 1.
\]

Define

\[Q_{n,k} = \{(\tau, \lambda) : \tau = (k^{k+1}, k-1, \ldots, 2, 1), \lambda_i \neq 2k+1, m_{>k}(\lambda) = n-k\},\]

where \(m_{>k}(\lambda)\) denotes the number of parts of \(\lambda\) which are greater than \(k\). See the following figure for an illustration.
Sylvester’s identity

We have the following combinatorial interpretation for the $k$th summand of the left hand side of Sylvester’s identity.

$$f(k) = q^{k(3k+1)/2}x^k \frac{1 - xq^{2k+1}}{(q; q)_k(xq^{k+1}; q)_\infty} = \sum_{n \geq k} x^n \sum_{(\tau, \lambda) \in Q_{n,k}} q^{\lambda + |\mu|}.$$ 


Let

$$H_{n,k} = \{ (\tau, \lambda) \in Q_{n,k} : m_{k+1}(\lambda) \geq m_k(\lambda) \}.$$

Then for each positive integer $n$, we have a combinatorial telescopin
g

$$\phi_{n,k} : Q_{n,k} \longrightarrow \{n\} \times Q_{n,k} \cup H_{n,k} \cup H_{n,k+1}.$$
Sylvester’s identity

Let

\[ I_n(q) = \sum_{k=0}^{\infty} (-1)^k \sum_{(\tau, \lambda) \in Q_{n,k}} q^{ |\tau| + |\lambda| }. \]

Noting that \( H_{n,0} = \emptyset \) because of the definition \( m_0(\lambda) = +\infty \), the above identity implies the recurrence relation

\[ I_n(q) = q^n I_n(q), \]

which implies that \( I_n(q) = 0 \) for \( n \geq 1 \). Clearly \( I_0(q) = 1 \), and hence Sylvester’s identity holds.
**Watson’s identity**

**Example (Watson, J. London Math. Soc., 1929)**

\[
\sum_{k=0}^{\infty} (-1)^k \frac{1 - aq^{2k}}{(q; q)_k (aq^k; q)_\infty} a^{2k} q^{k(5k-1)/2} = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q; q)_n}.
\]

Set

\[P_k = \{ (\tau, \lambda, \mu) : \tau = (k^{2k}, k-1, \ldots, 2, 1), \ \lambda_{\ell(\lambda)} \geq k, \ \lambda_i \neq 2k, \ \mu_1 \leq k \},\]

where \(k^{2k}\) denotes \(2k\) occurrences of a part \(k\). It is clear to see that the \(k\)th summand of the left hand side of Watson’s identity has the following combinatorial interpretation:

\[f(k) = \frac{1 - aq^{2k}}{(q; q)_k (aq^k; q)_\infty} a^{2k} q^{k(5k-1)/2} = \sum_{(\tau, \lambda, \mu) \in P_k} a^{\ell(\lambda)+2k} q^{|\tau|+|\lambda|+|\mu|}.\]
According to the exponent of $a$ in the above definition, we divide $P_k$ into a disjoint union of subsets

$$P_{n,k} = \{ (\tau, \lambda, \mu) \in P_k : \ell(\lambda) = n - 2k \},$$

with $P_{n,0} = \{ (\emptyset, \lambda, \emptyset) \in P_0 : \ell(\lambda) = n \}$ and $P_{n,k} = \emptyset$ for $n < 2k$. The elements of $P_{n,k}$ are illustrated in the following figure.

*Figure:* The diagram for $(\tau, \lambda, \mu) \in P_{n,k}$

Let
\[ H_{n,k} = \{(\tau, \lambda, \mu) \in P_{n,k} : m_k(\lambda) + 2 > m_k(\mu)\}. \]

Then, for any positive integer \( n \) and any nonnegative integer \( k \), there is a bijection
\[
\phi_{n,k} : P_{n,k} \longrightarrow \{n\} \times P_{n,k} \cup \{2n-1\} \times P_{n-1,k} \cup H_{n,k} \cup H_{n,k+1}.
\]

Let
\[
F_n(a, q) = \sum_{k=0}^{\infty} (-1)^k \sum_{(\tau, \lambda, \mu) \in P_{n,k}} a^n q^{\mid\tau\mid + \mid\lambda\mid + \mid\mu\mid}.
\]

By the combinatorial telescoping approach, then for any positive integer \( n \), we have
\[
F_n(a, q) = q^n F_n(a, q) + aq^{2n-1} F_{n-1}(a, q).
\]
Since $F_0(a, q) = 1$, by iteration we find that

$$F_n(a, q) = \frac{aq^{2n-1}}{1 - q^n} F_{n-1}(a, q)$$

$$= \frac{aq^{2n-1}}{1 - q^n} \frac{aq^{2n-3}}{1 - q^{n-1}} F_{n-2}(a, q)$$

$$= \frac{aq^{2n-1}}{1 - q^n} \frac{aq^{2n-3}}{1 - q^{n-1}} \cdots \frac{aq}{1 - q}$$

$$= \frac{a^n q^{n^2}}{(q; q)_n}.$$

Summing over $n$, we arrive at Watson’s identity.