Old and New Results for Generalized Frobenius Partition Functions

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Acknowledgements

Thanks to Bill Chen and Arthur Yang for their invitation to speak as well as their excellent work in preparation for this conference.

Thanks to each of you for attending this talk.

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Goals For This Talk

My goals for this talk include the following:

▶ Share some introductory thoughts and definitions
▶ Share a wide variety of Ramanujan-like congruence properties satisfied by different families of generalized Frobenius partition functions
▶ Discuss in detail the proofs of a few new and unexpected congruences modulo 5
▶ Close with some brief ideas for future work in this area
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Old and New Results for Generalized Frobenius Partition Functions

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Goals For This Talk

Introductory Thoughts and Definitions

Past Congruence Properties

Recent Work of Baruah and Sarmah

Concluding Thoughts

Introductory Thoughts and Definitions

The Ferrers graph associated with a partition $\lambda_1 + \lambda_2 + \cdots + \lambda_r$ of $n$ with $\lambda_i \geq \lambda_{i+1}$ is generally represented as a set of left–justified rows of dots where the $i$th row contains $\lambda_i$ dots. For example, the Ferrers graph of the partition $7 + 7 + 5 + 4 + 2 + 2$ is given by the following:

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Introductory Thoughts and Definitions

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\[ \lambda_1 + \lambda_2 + \cdots + \lambda_r \]

of \( n \) with \( \lambda_i \geq \lambda_{i+1} \) is generally represented as a set of left-justified rows of dots where the \( i^{th} \) row contains \( \lambda_i \) dots.
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  • • • • • • •
  • • • • •
  • • • •
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  • •
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Introductory Thoughts and Definitions

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Given the Ferrers graph of a partition, delete the main diagonal which possesses $r$ dots. The remaining rows of dots to the right of the diagonal are enumerated to provide one strictly decreasing sequence of $r$ nonnegative integers (the $r^{th}$ row might be empty, producing a value of 0). The remaining dots below the main diagonal are enumerated by columns to provide a second strictly decreasing sequence of $r$ nonnegative integers. The resulting two sequences are then written in the form of a two–rowed array.
Introductory Thoughts and Definitions

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Let’s look again at our example.
Introductory Thoughts and Definitions
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Then the partition $7 + 7 + 5 + 4 + 2 + 2$ of 27 is represented by the Frobenius symbol $(6 \ 5 \ 2 \ 0 \ 5 \ 4 \ 1 \ 0)$.
Introductory Thoughts and Definitions

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$$
\begin{pmatrix}
6 & 5 & 2 & 0 \\
5 & 4 & 1 & 0
\end{pmatrix}.
$$
Introductory Thoughts and Definitions

Quick sidenote: One advantage of the Frobenius symbol is that it gives a very handy way to see the conjugate of a partition.
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In our example, the conjugate of

\[
\begin{pmatrix}
6 & 5 & 2 & 0 \\ 5 & 4 & 1 & 0 \\
\end{pmatrix}
\]

is simply

\[
\begin{pmatrix}
5 & 4 & 1 & 0 \\ 6 & 5 & 2 & 0 \\
\end{pmatrix}
\]
Introductory Thoughts and Definitions

One might ask - are there any “obvious” generalizations of these Frobenius symbols?
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One answer to this question is “yes” based on George Andrews’ work on generalized Frobenius partitions from his AMS Memoir of 1984.
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In this talk, I would like to define three families of partition functions which count such “relatives” of the original Frobenius symbols, two of which are studied extensively in George’s Memoir.
Introductory Thoughts and Definitions

So let’s now define two of these relatives. First, consider generalized Frobenius partitions that allow up to \( k \) repetitions of an integer in any row. We shall denote the number of all such partitions by \( \phi_k(n) \).
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2
\end{pmatrix}
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2 & 0 \\
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0 & 0 \\
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\end{pmatrix}
\begin{pmatrix}
1 \\ 1 \\ 0 \\ 0
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Introductory Thoughts and Definitions

Next, we define the function \( c\phi_m(n) \) as the number of generalized Frobenius partitions of \( n \) using \( m \) colors. By this we mean that we consider \( m \) copies of the nonnegative integers written \( j_i \) where \( j \geq 0 \) and \( 1 \leq i \leq m \). We then say that \( j_i < l_k \) precisely when \( j < l \) or \( j = l \) and \( i < k \).
Moreover, \( j_i \) is equal to \( l_k \) if and only if \( j = l \) and \( i = k \).
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Then $c_{\phi_m}(n)$ counts the number of generalized Frobenius partitions of $n$ under the conditions that the parts are “decreasing” (using the ordering above).
Introductory Thoughts and Definitions

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Note that $\phi_1(n) = c\phi_1(n) = p(n)$ for all $n$, so these two functions provide natural generalizations of $p(n)$ when viewed in the context of Frobenius symbols.
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Simply put for the purposes of our talk today, both \( \phi_m \) and \( c\phi_m \) arose while George Andrews was working with Rodney Baxter on Regime III of the Hard Hexagon Model in statistical mechanics.
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In his Memoir, George considered these two families of functions from NUMEROUS perspectives.
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In his Memoir, George considered these two families of functions from NUMEROUS perspectives.

I want to focus our attention today on congruence properties satisfied by these functions.
Introductory Thoughts and Definitions

Before we close this section, let me introduce one additional family of generalized Frobenius partition functions initially defined by Louis Kolitsch.

Let $c_{\phi}^m(n)$ be the number of generalized Frobenius partitions of $n$ with $m$ colors whose order is $m$ under cyclic permutation of the $m$ colors. The relationship between $c_{\phi}^m(n)$ and $c_{\phi}^d(n)$ is given by the following:

$$c_{\phi}^m(n) = \sum_{d \mid (m,n)} \mu(d) c_{\phi}^{m/d}(n/d)$$
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Let $\overline{c_{\phi}}_m(n)$ be the number of generalized Frobenius partitions of $n$ with $m$ colors whose order is $m$ under cyclic permutation of the $m$ colors.

The relationship between $\overline{c_{\phi}}$ and $c_{\phi}$ is given by the following:

$$\overline{c_{\phi}}_m(n) = \sum_{d|(m,n)} \mu(d)c_{\phi} \frac{m}{d} \left( \frac{n}{d} \right)$$
We see that in the case when \( m \) is prime, this reduces to

\[
\overline{c\phi_m(n)} = \begin{cases} 
  c\phi_m(n) & \text{if } (m, n) = 1 \\
  c\phi_m(n) - p\left(\frac{n}{m}\right) & \text{if } (m, n) = m.
\end{cases}
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Introductory Thoughts and Definitions

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\end{cases}
$$

The functions $c_{\phi_m}$ prove to be extremely rich in congruence properties.
Past Congruence Properties

We begin this section by highlighting two congruence properties that George proved in Section 10 of his Memoir. I will minimize the proof techniques here (because of time).

The first congruence result proven by George in the Memoir is the following:

\[ \phi(5n + 3) \equiv 0 \pmod{5} \]

These results will be mentioned again later in the talk.
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Theorem: (Corollary 10.1, p. 28, Memoir) For all \( n \geq 0 \),
\[
\phi_2(5n + 3) \equiv c\phi_2(5n + 3) \equiv 0 \pmod{5}.
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\]

These results will be mentioned again later in the talk.
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Theorem: (Thm 10.2, p. 29, Memoir) If $m$ is prime and $m$ does not divide $n$, then $c\phi_m(n) \equiv 0 \pmod{m^2}$.
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**Theorem: (Thm 10.2, p. 29, Memoir)** If \( m \) is prime and \( m \) does not divide \( n \), then \( c\phi_m(n) \equiv 0 \pmod{m^2} \).

The proof of this result is combinatorial in nature and rather straightforward, dealing primarily with the “colorings” of the parts in question.
Past Congruence Properties

Motivated by the above Theorem, Kolitsch soon after proved the following:

\[ c_{\phi}^m(n) \equiv 0 \pmod{m^2}. \]

Note that Kolitsch's result does not depend on \( \gcd(m,n) \) and also does not require \( m \) to be prime.
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Motivated by the above Theorem, Kolitsch soon after proved the following:

Theorem: For all $n \geq 1$ and for any $m \geq 2$,

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Past Congruence Properties

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In a short note, the following congruences were proven to hold for all $n \geq 1$:

$$c\phi_5(5n) \equiv 0 \pmod{5^3},$$

$$c\phi_7(7n) \equiv 0 \pmod{7^3},$$

and

$$c\phi_{11}(11n) \equiv 0 \pmod{11^3}.$$
Past Congruence Properties

The congruences above follow immediately from Kolitsch’s result that

\[
\begin{align*}
\overline{c\phi_5}(n) & = 5 p(5n - 1), \\
\overline{c\phi_7}(n) & = 7 p(7n - 2), \quad \text{and} \\
\overline{c\phi_{11}}(n) & = 11 p(11n - 5)
\end{align*}
\]

and the congruences already known for the partition function \( p(n) \).
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and the congruences already known for the partition function \( p(n) \).

This led me to consider whether similar congruences modulo \( m^3 \) could hold. I was not disappointed!!
Past Congruence Properties

Theorem: (JAS, *Discrete Math*, 1994) For all $n \geq 1$,

$$c\phi_2(2n) \equiv 0 \pmod{2^3}, \text{ and } c\phi_3(3n) \equiv 0 \pmod{3^4}.$$
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Theorem: (JAS, *Discrete Math*, 1994) For all $n \geq 1$,

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The proof of this theorem only involves elementary techniques, relying on generating function manipulations and Jacobi’s Triple Product Identity.
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\]

The proof of this theorem only involves elementary techniques, relying on generating function manipulations and Jacobi’s Triple Product Identity.

Louis Kolitsch subsequently proved that such a result was true modulo $m^3$ for all $m \geq 2$. 
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Let me now turn a corner and mention one additional set of results (which takes us back to the functions $\phi_m$ and $c\phi_m$).
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In 1990, Kolitsch proved an infinite family of congruences satisfied by $c\phi_3$. Namely, he proved the following:

![Theorem: (Kolitsch) Fix $\alpha \geq 1$. For all $n \geq 1$, $c\phi_3(3^{\alpha}n + \lambda\alpha) \equiv 0 \pmod{3^{\alpha+2}}$ for $\alpha$ even and $\equiv 0 \pmod{3^{\alpha+1}}$ for $\alpha$ odd, where $\lambda\alpha$ is the reciprocal of 8 modulo $3\alpha$. This led me to look more carefully for congruences satisfied by $\phi_3$. I was again not disappointed.
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Theorem: (JAS, 1994) For all $n \geq 1$, $\phi_3(3n + 2) \equiv 0 \pmod{3}$. 

Remark: Note that this is like a "companion" to the $\alpha = 1$ case of Kolitsch's family of congruences above. The proof follows the same sort of technique as the one which proved $\phi_2(5n + 3) \equiv 0 \pmod{5}$ for all $n$. Here's the proof (it's really brief!).
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Old and New Results for Generalized Frobenius Partition Functions

James Sellers
Penn State University

Acknowledgements

Goals For This Talk

Introductory Thoughts and Definitions

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$$\sum_{n=0}^{\infty} \phi_3(n)q^n = \frac{1}{(q; q)^3} \sum_{m=-\infty}^{\infty} q^{3m^2} \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}$$
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In the above, \((a; q)_\infty := (1 - a)(1 - aq)(1 - aq^2)(1 - aq^3)\ldots\)
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In the above,

\[(a; q)\infty := (1 - a)(1 - aq)(1 - aq^2)(1 - aq^3)\ldots\]

The proof is complete once we show that all coefficients of \(q^{3n+2}\) in the double sum are divisible by 3.
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In order to get a contribution to $q^{3n+2}$, we must have $3m^2 + n^2 \equiv 2 \pmod{3}$ or $n^2 \equiv 2 \pmod{3}$. This is not possible since 2 is a quadratic non-residue modulo 3. The proof is complete.
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We have companion results for \( \phi_3(3n + 2) \) and \( c\phi_3(3n + 2) \) modulo 3 as well as Kolitsch’s infinite family of results for which \( c\phi_3(3n + 2) \) is the first case.
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We also know from George’s Memoir that there are companion results for $\phi_2(5n + 3)$ and $c\phi_2(5n + 3)$ modulo 5. Could there be an infinite family of congruences modulo powers of 5 for which $c\phi_2(5n + 3)$ is the first case?
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This led to a conjecture that I published in 1994:
Past Congruence Properties

Fix $\alpha \geq 1$. For all $n \geq 1$,

$$c\phi_2(5^\alpha n + \lambda_\alpha) \equiv 0 \pmod{5^\alpha}$$

where $\lambda_\alpha$ is the reciprocal of 12 modulo $5^\alpha$. 

Dennis Eichhorn and I (2002) provided a proof for the first four cases of this conjecture via modular forms (and A LOT of computations).

Paule and Radu provided a proof of the full conjecture in a paper published in 2012 in Advances in Mathematics!
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Recent Work of Baruah and Sarmah

In 2011, Baruah and Sarmah proved a number of congruence properties for $c_{φ^4}$, all with moduli which are powers of 4. This led me to re-consider the function $c_{φ^4}$ once more to see if any additional congruences might be present. I was not disappointed. My goal now is to prove an unexpected result modulo 5 satisfied by $c_{φ^4}$. 
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Theorem: (JAS, 2012) For all \( n \geq 0 \), \( c\phi_4(10n + 6) \equiv 0 \) (mod 5).
Recent Work of Baruah and Sarmah

Theorem: (JAS, 2012) For all \( n \geq 0 \), \( c_{\phi_4}(10n + 6) \equiv 0 \pmod{5} \).

Our proof is elementary, relying on Baruah and Sarmah’s results as well as work of Srinivasa Ramanujan.
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Recall Ramanujan’s functions

\[ \phi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \quad \text{and} \quad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}. \]
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Using Jacobi’s Triple Product Identity, we have the following well–known product representations for \( \phi(q) \) and \( \psi(q) \):

\[ \phi(q) = \frac{(q^2; q^2)^5}{(q; q)^2 (q^4; q^4)^2} \]

and

\[ \psi(q) = \frac{(q^2; q^2)^2}{(q; q)^2}. \]
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Baruah and Sarmah proved the following valuable representation of the generating function for $c\phi_4$. 

$$
\sum_{n=0}^{\infty} c\phi_4(n) q^n = \phi_3(q^2) + 12 q \phi_4(q^2) \psi_2(q^4)(q^2; q^2) \infty
$$

From here, we wish to 2–dissect the generating function above (because we want to study the coefficients of $q^{10n} + 6$ in the power series representation of the generating function for $c\phi_4$).
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$$\sum_{n=0}^{\infty} c\phi_4(n) q^n = \frac{\phi^3(q^2) + 12q\phi(q^2)\psi^2(q^4)}{(q; q)_{\infty}^4}$$
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To complete this task, we follow the path laid out by Baruah and Sarmah. We begin by rewriting the generating function above as

\[
\sum_{n=0}^{\infty} c\phi_4(n)q^n = \frac{\phi^3(q^2) + 12q\phi(q^2)\psi^2(q^4)}{(q; q^2)^4(q^2; q^2)^4}. 
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Then we see the following:
Recent Work of Baruah and Sarmah

\[
\sum_{n=0}^{\infty} c\phi_4(n)q^n + \sum_{n=0}^{\infty} c\phi_4(n)(-q)^n
\]
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\[
\sum_{n=0}^{\infty} c\phi_4(n)q^n + \sum_{n=0}^{\infty} c\phi_4(n)(-q)^n = \frac{\phi^3(q^2)}{(q^2; q^2)_\infty^4} \left\{ \frac{1}{(q; q^2)_\infty^4} + \frac{1}{(-q; q^2)_\infty^4} \right\} + 12q \frac{\phi(q^2)\psi^2(q^4)}{(q^2; q^2)_\infty^4} \left\{ \frac{1}{(q; q^2)_\infty^4} - \frac{1}{(-q; q^2)_\infty^4} \right\}
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As noted by Baruah and Sarmah, we can employ work of Ramanujan to obtain

\[
\left( -q ; q \right)_4 \infty + \left( q ; q^2 \right)_4 \infty = 2 \varphi_2 \left( q^2 \right) \left( q^2 ; q^2 \right)_2 \infty
\]

and

\[
\left( -q ; q \right)_4 \infty - \left( q ; q^2 \right)_4 \infty = 8 q \psi_2 \left( q^4 \right) \left( q^2 ; q^2 \right)_2 \infty.
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\[ (-q; q^2)_\infty^4 - (q; q^2)_\infty^4 = 8q \frac{\psi^2(q^4)}{(q^2; q^2)_\infty^2}. \]

These can be used in the above to obtain the following:
Recent Work of Baruah and Sarmah

\[
\sum_{n=0}^{\infty} c\phi_4(n)q^n + \sum_{n=0}^{\infty} c\phi_4(n)(-q)^n
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\[ \sum_{n=0}^{\infty} c\phi_4(n)q^n + \sum_{n=0}^{\infty} c\phi_4(n)(-q)^n = 2\frac{\phi^5(q^2)}{(q^2; q^2)_\infty^6(q^4; q^4)_\infty^4} + 96q^2\frac{\phi(q^2)\psi^4(q^4)}{(q^2; q^2)_\infty^6(q^2; q^4)_\infty^4}. \]
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Therefore,

\[ \sum_{n \geq 0} c\phi_4(2n)q^n = \frac{\phi^5(q)}{(q; q)_6^\infty (q; q^2)_4^\infty} + 48q \frac{\phi(q)\psi^4(q^2)}{(q; q)_6^\infty (q; q^2)_4^\infty}. \]
Recent Work of Baruah and Sarmah

We now utilize all of the above to obtain

$$\sum_{n \geq 0} c\phi_4(2n)q^n = \frac{(q^2; q^2)^{29}}{(q; q)_\infty^20(q^4; q^4)_\infty^{10}} + 48q\frac{(q^2; q^2)_\infty^5(q^4; q^4)_\infty^6}{(q; q)_\infty^{12}}$$

after elementary simplifications.
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We now utilize all of the above to obtain

$$\sum_{n \geq 0} c\phi_4(2n)q^n = \frac{(q^2; q^2)^{29}}{(q; q)_{\infty}^2 (q^4; q^4)_{\infty}^{10}} + 48q \frac{(q^2; q^2)_{\infty}^5 (q^4; q^4)_{\infty}^6}{(q; q)_{\infty}^{12}}$$

after elementary simplifications.

Our goal now is to prove that, when written as power series in $q$, each of the two functions on the right–hand side of the above has the property that every coefficient of a term of the form $q^{5n+3}$ is a multiple of 5. Then our theorem follows.
Recent Work of Baruah and Sarmah

With this in mind, we first note that

\[
\frac{(q^2; q^2)^{29}}{(q; q)^{20}(q^4; q^4)^{10}} \equiv \frac{(q^{10}; q^{10})^5(q^2; q^2)^4}{(q^5; q^5)^4(q^{20}; q^{20})^2} \pmod{5}.
\]
Recent Work of Baruah and Sarmah

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\[
\frac{(q^2; q^2)^{29}}{(q; q)_{20}^2(q^4; q^4)_{10}^2} \equiv \frac{(q^{10}; q^{10})^5(q^2; q^2)_{\infty}^4}{(q^5; q^5)_{\infty}^4(q^{20}; q^{20})_{\infty}^2} \pmod{5}.
\]

Now all of the functions of \(q^5\) in the right-hand side above can be ignored; in other words, if we can show that every coefficient of a term of the form \(q^{5n+3}\) is a multiple of 5 in the power series representation of the function \((q^2; q^2)_{\infty}^4\), then every coefficient of a term of the form \(q^{5n+3}\) is a multiple of 5 in the power series representation of the function

\[
\frac{(q^{10}; q^{10})^5(q^2; q^2)_{\infty}^4}{(q^5; q^5)_{\infty}^4(q^{20}; q^{20})_{\infty}^2}
\]

as well.
Recent Work of Baruah and Sarmah

So we focus our attention on \((q^2; q^2)_4^\infty\).
Recent Work of Baruah and Sarmah

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We know from Euler’s Pentagonal Number Theorem and a well–known result of Jacobi that

\[
(q^2; q^2)_\infty^4 = \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{k+m}(2k + 1)q^{k(k+1)+m(3m-1)}.
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\]

We now consider exponents of \(q\) and want to know when

\[5n + 3 = k(k + 1) + m(3m - 1)\]

has a solution for integers \(k, m,\) and \(n\).
Recent Work of Baruah and Sarmah

When we consider this mod 5, we have

\[ 3 \equiv k(k + 1) + m(3m - 1) \pmod{5}. \]
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Completing the square (by multiplying both sides by 12 and adding appropriate amounts to both sides) gives

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We know that all squares are congruent to 0, 1, or 4 modulo 5. So we consider the nine possible cases in this congruence and note rather quickly that \((2k + 1)^2\) must be congruent to 0 modulo 5 (the other possibilities lead to no solutions).
Recent Work of Baruah and Sarmah

Thus, \(2k + 1\) is divisible by 5 whenever we consider the coefficients of the terms of the form \(q^{5n+3}\). Therefore, by our work above, we know that the coefficients of the terms of the form \(q^{5n+3}\) in the first summand of our generating function must be divisible by 5.
Recent Work of Baruah and Sarmah

Thus, $2k + 1$ is divisible by 5 whenever we consider the coefficients of the terms of the form $q^{5n+3}$. Therefore, by our work above, we know that the coefficients of the terms of the form $q^{5n+3}$ in the first summand of our generating function must be divisible by 5.

We turn our attention to the second term on the right-hand side of the generating function identity above and note the following:
Recent Work of Baruah and Sarmah

\[
\begin{align*}
48q \left( \frac{(q^2; q^2)_\infty^5 (q^4; q^4)_\infty^6}{(q; q)_\infty^{12}} \right) &= \left( \frac{(q^2; q^2)_\infty^5 (q; q)_\infty^3 (q^4; q^4)_\infty^6}{(q; q)_\infty^{15}} \right) \\
\equiv 48q \left( \frac{(q^{10}; q^{10})(q^{20}; q^{20})_\infty (q; q)_\infty^3 (q^4; q^4)_\infty}{(q^5; q^5)_\infty^3} \right) \pmod{5}.
\end{align*}
\]
Recent Work of Baruah and Sarmah

\[ 48q \frac{(q^2; q^2)_\infty^5 (q^4; q^4)_\infty^6}{(q; q)_\infty^{12}} = 48q \frac{(q^2; q^2)_\infty^5 (q; q)_\infty^3 (q^4; q^4)_\infty^6}{(q; q)_\infty^{15}} \equiv 48q \frac{(q^{10}; q^{10})(q^{20}; q^{20})_\infty^3 (q; q)_\infty^3 (q^4; q^4)_\infty}{(q^5; q^5)_\infty^3} \pmod{5}. \]

As before, we can ignore those functions which are functions of \( q^5 \).
Recent Work of Baruah and Sarmah

So our goal is to focus on the coefficients of the terms of the form $q^{5n+3}$ in

$$q(q; q)^3 (q^4; q^4)_{\infty}.$$
Recent Work of Baruah and Sarmah

So our goal is to focus on the coefficients of the terms of the form \( q^{5n+3} \) in

\[
q(q; q)\frac{3}{\infty} (q^4; q^4)\infty.
\]

This is equivalent to considering the coefficients of the terms of the form \( q^{5n+2} \) in

\[
(q; q)\frac{3}{\infty} (q^4; q^4)\infty.
\]
Recent Work of Baruah and Sarmah

As above, we know that

\[
(q; q)^3(q^4; q^4)_\infty \\
\sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{k+m}(2k + 1)q^{k(k+1)/2+2m(3m-1)}.
\]
Recent Work of Baruah and Sarmah

As above, we know that

\[
(q; q)^3 \prod_{n=1}^{\infty} (q^n; q^n) \prod_{n=1}^{\infty} (q^{4n}; q^{4n})
= \sum_{m=-\infty}^{\infty} \sum_{k=0}^{\infty} (-1)^{k+m}(2k + 1)q^{k(k+1)/2+2m(3m-1)}.
\]

Thus, we want to know when

\[5n + 2 = \frac{k(k + 1)}{2} + 2m(3m - 1)\]

has a solution for integers \(k, m,\) and \(n.\)
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As above, we find that the only way that this last equation can have solutions is if $2k + 1 \equiv 0 \pmod{5}$.
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Thus, thanks to our previous work, we can conclude that all the coefficients of the terms of the form $q^{5n+3}$ in

$$48q \frac{(q^2; q^2)^5 (q^4; q^4)_6}{(q; q)_{12}}$$

are also divisible by 5.
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This completes the proof of our theorem! \qed
Recent Work of Baruah and Sarmah

I should note that my work above has been accepted and is to appear in late 2013 in a special issue of the *Journal of the Indian Mathematical Society* dedicated to the work of Srinivasa Ramanujan.
Concluding Thoughts

We close with a number of thoughts. First, it is worth noting that our theorem implies another congruence property modulo 5, this time for the function $\phi_4(n)$.

**Theorem:** For all $n \geq 0$, $\phi_4(10n+6) \equiv 0 \pmod{5}$.

Note that in his PhD thesis, Garvan proved that, for any prime $p$, $\phi_p-1(n) \equiv c\phi_p-1(n) \pmod{p}$ for any integer $n \geq 0$.

The theorem follows.
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Theorem: For all $n \geq 0$, $c\phi_4(10n + 6) \equiv 0 \pmod{5}$.

Indeed, Baruah and Sarmah prove that

$$\sum_{n=0}^{\infty} c\phi_4(2n)q^n = 64q \frac{(q^4; q^4)_\infty^6}{(q^2; q^2)_\infty^7 (q; q^2)^{12}}.$$
Concluding Thoughts

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$$\sum_{n=0}^{\infty} \overline{c\phi_4(2n)} q^n = 64q \frac{(q^4;q^4)_\infty (q^2;q^2)_5}{(q;q)_{12}} \pmod{5}.$$
Concluding Thoughts

As in the above work, we then see that

\[
\sum_{n=0}^{\infty} c\phi_4(2n)q^n = 64q \frac{(q^4; q^4)_\infty (q^2; q^2)_\infty^5}{(q; q)_{12} (q^2; q^2)_\infty} \equiv 64q \frac{(q^{20}; q^{20})_\infty (q^4; q^4)_\infty (q^{10}; q^{10})_\infty (q; q)_3^3}{(q^5; q^5)_\infty^3} \pmod{5}.
\]
Concluding Thoughts

So we only need to consider the coefficients of $q^{5n+2}$ in $(q^4; q^4)_{\infty}(q; q)_{\infty}^3$. This was done in the work above!!! So the proof of this result for $c_{\phi}^4$ modulo 5 also follows.
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Let me simply close by saying that MUCH MORE is almost certainly true in terms of congruence properties for generalized Frobenius partition functions.
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- Could there be an infinite family of congruences modulo powers of 5 for $c \phi_4$ or $\overline{c \phi_4}$?
Concluding Thoughts

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Could there be an infinite family of congruences modulo powers of 5 for $c\phi_4$ or $\overline{c\phi_4}$? (This would be reminiscent of the infinite family of results recently proven by Paule and Radu.)
Concluding Thoughts

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- Given that we have mod 5 congruences satisfied by $c\phi_2(5n + 3)$ and $c\phi_4(10n + 6)$, can we extend this pattern in a “natural way”?
Concluding Thoughts

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- Given that we have mod 5 congruences satisfied by $c\phi_2(5n + 3)$ and $c\phi_4(10n + 6)$, can we extend this pattern in a “natural way”? For example, is it the case that $c\phi_8(20n + 12) \equiv 0 \pmod{5}$ for all $n \geq 0$ as well?
Concluding Thoughts

- What about subscripts which are not primes or powers of primes? In other words, is anything true for $c\phi_6$ for example?
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- And lots of other questions.....
Breaking News!

Earlier this year, Frank Garvan and I were able to prove (in very elementary fashion) that the following holds:

**Theorem:** If $p$ is prime and $0 < r < p$, and if $c\phi m (pn + r) \equiv 0 \pmod{p}$ for all $n \geq 0$, then $c\phi pN + m (pn + r) \equiv 0 \pmod{p}$ for all $n \geq 0$ where $N$ is any nonnegative integer.
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**Corollary:** For all $N \geq 0$ and all $n \geq 0$, 

\[
c\phi_5(N+1)(5n+4) \equiv 0 \pmod{5},
\]

\[
c\phi_7(N+1)(7n+5) \equiv 0 \pmod{7},
\]

\[
c\phi_{11}(N+1)(11n+6) \equiv 0 \pmod{11}.
\]
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Corollary: For all $N \geq 0$ and all $n \geq 0$,

$$c\phi_{5N+1}(5n + 4) \equiv 0 \pmod{5},$$
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Corollary: For all \( N \geq 0 \) and all \( n \geq 0 \),

\[
\phi_{5N+1}(5n + 4) \equiv 0 \pmod{5},
\]

\[
\phi_{7N+1}(7n + 5) \equiv 0 \pmod{7},
\]

and
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Thanks to well–known results for generalized Frobenius partitions with a small number of colors, we can then state corollaries such as the following:

Corollary: For all $N \geq 0$ and all $n \geq 0$,

$$c_{5N+1}(5n + 4) \equiv 0 \pmod{5},$$

$$c_{7N+1}(7n + 5) \equiv 0 \pmod{7},$$

and

$$c_{11N+1}(11n + 6) \equiv 0 \pmod{11}.$$
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Corollary: For all $N \geq 0$ and all $n \geq 0$, 

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Breaking News!

And just for fun:

Corollary: For all \( N \geq 0 \) and all \( n \geq 0 \),

\[ c_{\phi}^{1155N+1002} (1155n+908) \equiv 0 \pmod{1155} \]

thanks to the Chinese Remainder Theorem.
Breaking News!

And just for fun:

\[c_{\phi_1}^{1155 N + 1002} (1155 n + 908) \equiv 0 \pmod{1155}\]

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Corollary: For all $N \geq 0$ and all $n \geq 0$,

$$c\phi_{1155N+1002}(1155n + 908) \equiv 0 \pmod{1155}$$

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Old and New Results for Generalized Frobenius Partition Functions

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