ASM matrices, Schubert and Grothendieck Polynomials

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Permutations can be viewed in many different ways. I shall consider today only two interpretations :

♠ as sequences of integers, as configurations of points in the plane (non attacking rooks), as flags of vector spaces,

♠ as elements of the symmetric group, interpreted as a Coxeter group generated by simple transpositions.

The first point of view leads to order properties, inversion diagrams, Bruhat order, lattice operations on permutations, &c.

The second forces to use the group algebra, or its deformations (Hecke algebra), Yang-Baxter relations, &c.

I shall connect these two points of view using polynomials.

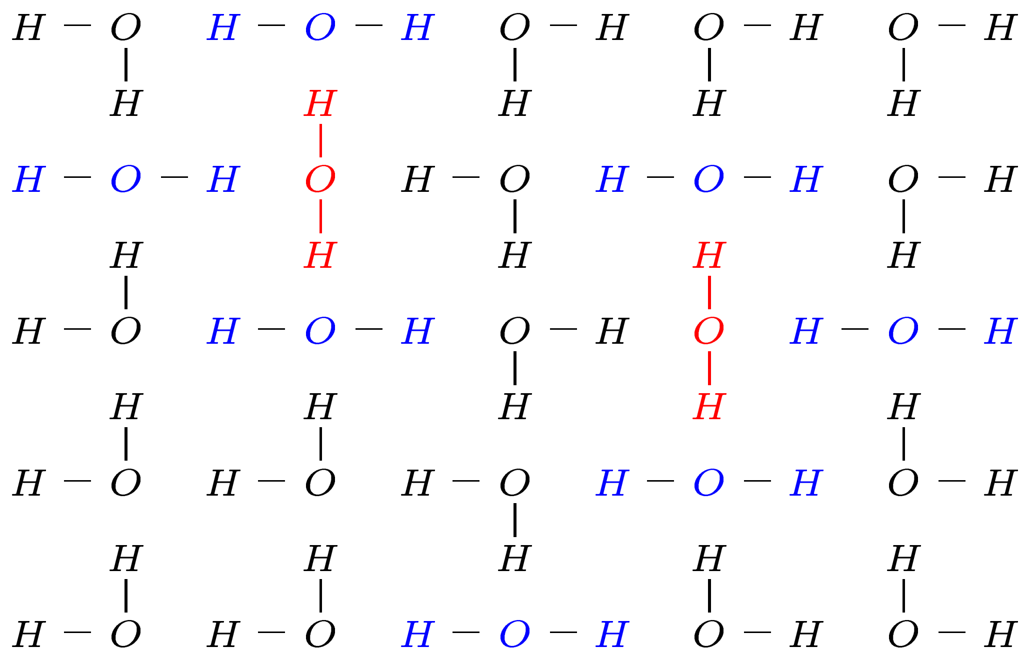
Let us begin with a two-dimensional object adapted to the season:

square ice .

For a mathematician, it is made of 6 types of molecules:

$$
H - O - H \qquad
\begin{array}{c} H \\ | \\ O \\ | \\ H \end{array} \qquad
\begin{array}{cc} H - & O \\ & | \\ & H \end{array} \qquad
\begin{array}{cc} & H \\ & | \\ H - & O \end{array} \qquad
\begin{array}{cc} O & - H \\ | & \\ H & \end{array} \qquad
\begin{array}{cc} H & \\ | & \\ O & - H \end{array}
$$

Here is an example of an ice configuration :



Ice configurations are in bijection with *alternating-sign matrices* (ASM in short), replacing horizontal molecules by 1, vertical molecules by -1, and the others by 0. Such matrices of $0, 1, -1$ are characterized by the property that non zero entries alternate in each row and column, always starting and finishing with a 1.

Continuing with the same example, we get the following ASM :

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} .$$

No information has been lost. One recovers the Ice from the ASM by a construction due to Rothe (1800!). A matrix has four corners, and four inversion diagrams: Given a 0-entry in an ASM, ignore all the other zeroes. Then the current 0 is next to a 1 in its column and its row. Replace now this 0 by a box that will be attributed to one of the diagrams, depending on the orientation :

$$\begin{array}{ll} 0 & \rightarrow 1 \\ \downarrow & \\ 1 & \end{array} \quad \text{gives} \quad \begin{array}{ll} \boxed{\text{NW}} & \rightarrow 1 \\ \downarrow & \\ 1 & \end{array} , \qquad \begin{array}{ll} 1 & \\ \uparrow & \\ 0 & \rightarrow 1 \end{array} \quad \text{gives} \quad \begin{array}{ll} 1 & \\ \uparrow & \\ \boxed{\text{SW}} & \rightarrow 1 \end{array} ,$$

$$\begin{array}{rl} 1 \leftarrow & 0 \\ & \downarrow \\ & 1 \end{array} \quad \text{gives} \quad \begin{array}{rl} 1 \leftarrow & \boxed{\text{NE}} \\ & \downarrow \\ & 1 \end{array} , \qquad \begin{array}{rl} & 1 \\ & \uparrow \\ 1 \leftarrow & 0 \end{array} \quad \text{gives} \quad \begin{array}{rl} & 1 \\ & \uparrow \\ 1 \leftarrow & \boxed{\text{SE}} \end{array} .$$

The preceding ASM gives the four diagrams :

$$\begin{bmatrix} \square & 1 & \cdot & \cdot & \cdot \\ 1 & -1 & \square & 1 & \cdot \\ \cdot & 1 & \cdot & -1 & 1 \\ \cdot & \cdot & \square & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}, \quad \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \\ \square & 1 & \cdot & -1 & 1 \\ \square & \square & \cdot & 1 & \cdot \\ \square & \square & 1 & \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} \cdot & 1 & \square & \square & \square \\ 1 & -1 & \cdot & 1 & \square \\ \cdot & 1 & \square & -1 & 1 \\ \cdot & \cdot & \cdot & 1 & \cdot \\ \cdot & \cdot & 1 & \cdot & \cdot \end{bmatrix}, \quad \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & -1 & 1 \\ \cdot & \cdot & \cdot & 1 & \blacksquare \\ \cdot & \cdot & 1 & \blacksquare & \blacksquare \end{bmatrix}.$$

As is frequent in combinatorics, we need another object in bijection with ice configurations and ASM, triangles (Young tableaux with weakly decreasing diagonals) :

Read the successive rows of an ASM, from right to left, a 1 in column i meaning that the letter i appears in the tableau, a -1 meaning that it disappears.

This gives a sequence of sets of numbers, or, equivalently, a sequence of columns of a triangle.

$$\begin{bmatrix} \cdot & 4 & \cdot & \cdot & \cdot \\ 5 & \widehat{4} & \cdot & 2 & \cdot \\ \cdot & 4 & \cdot & \widehat{2} & 1 \\ \cdot & \cdot & \cdot & 2 & \cdot \\ \cdot & \cdot & 3 & \cdot & \cdot \end{bmatrix} \longleftrightarrow \{4\},\ \{2,5\},\ \{1,4,5\},\ \{1,2,4,5\},\ \{1,2,3,4,5\}$$

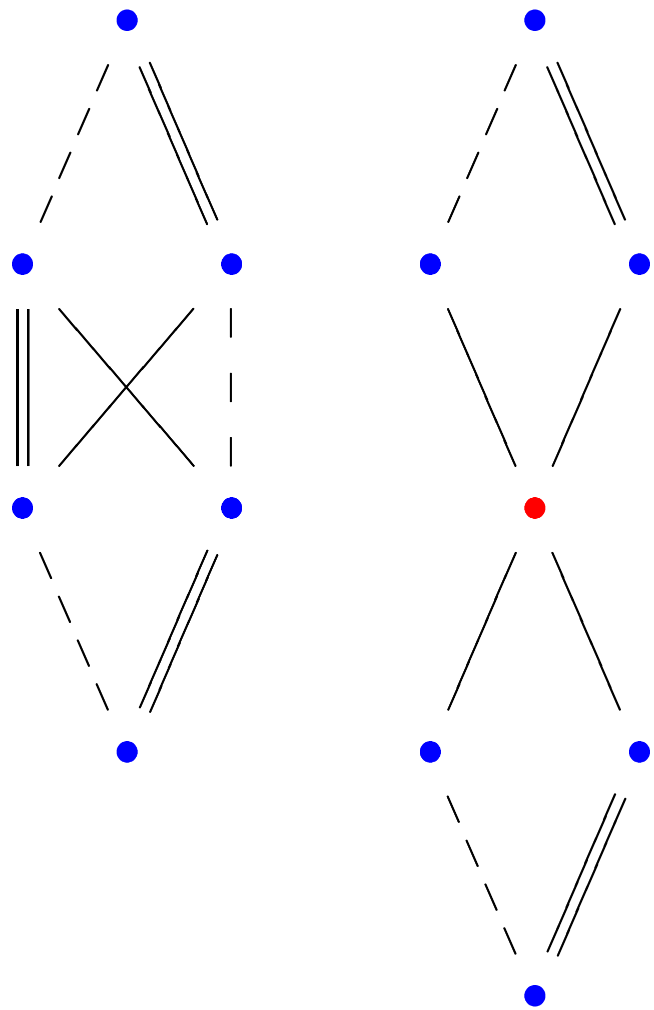
$$\begin{array}{|c|c|c|c|c|} \hline 5 \\ \cline{1-1} 4 & 5 \\ \cline{1-2} 3 & 4 & 5 \\ \cline{1-3} 2 & 2 & 4 & 5 \\ \hline 1 & 1 & 1 & 2 & 4 \\ \hline \end{array} \longleftrightarrow \begin{bmatrix} \cdot & 1 & \cdot & \cdot & \cdot \\ 1 & -1 & \cdot & 1 & \cdot \\ \cdot & 1 & \cdot & -1 & 1 \\ \cdot & \cdot & \cdot & 1 & \overset{|}{-\!\circ} \\ \cdot & \cdot & 1 & \overset{|}{-\!\circ} & \overset{|}{-\!\circ} \end{bmatrix} .$$

These three types of objects are in bijection, but different properties can be read on each of them. The most important property is a lattice structure. Given two (planar) list of numbers of the same shape, one can take their supremum componentwise. We notice that the supremum of two triangles is still a triangle.

Starting with permutations, which correspond to flags of sets, one obtains all the ASM.

Therefore, the set of ASM is the envelopping lattice of the symmetric group. This is the last work I did with M.P. Schützenberger.

Ehresmann order



Enveloping Lattice

In the usual theory of flag manifolds, we have to decompose the set of matrices into cells, each of which contains only one permutation matrix.

Invertible matrix M $\rightarrow$ flag of vector spaces =

$$< v_1 > \hookrightarrow < v_1, v_2 > \hookrightarrow < v_1, v_2, v_3 > \cdots$$

(v_i = columns of M)

$$M \;\sim\; \text{permutation}$$

iff the flag of M has the same intersection dimensions with the coordinate flag

$$< e_1 > \hookrightarrow < e_1, e_2 > \hookrightarrow < e_1, e_2, e_3 > \cdots$$

In the case of ASM, to get cells, instead of considering flags of vector spaces, we have to eliminate -1 entries.

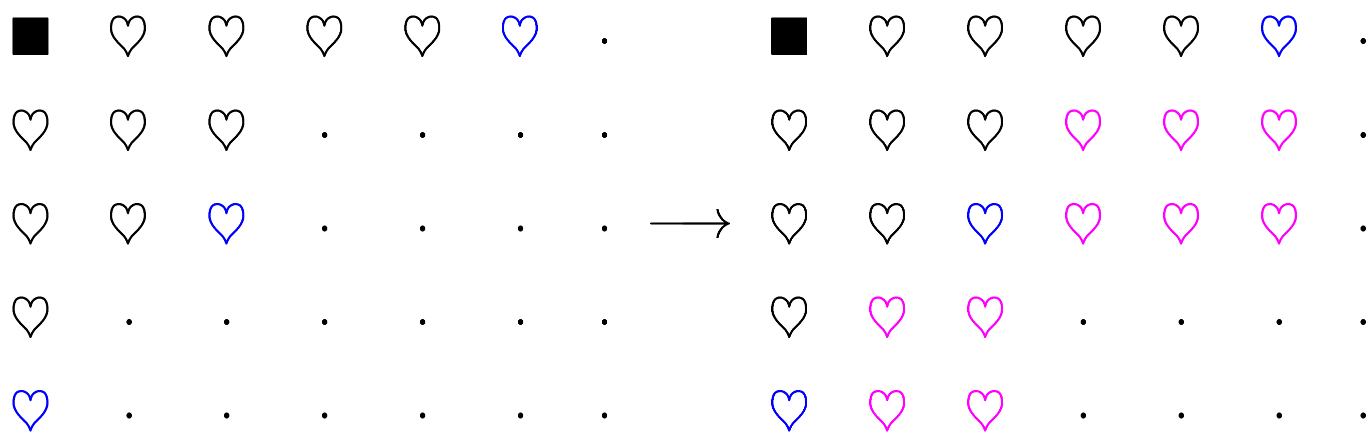
Given any entry $\blacksquare$ in a matrix, it has 1-neighbours :

maximal	$\blacksquare \quad 0 \quad 0 \quad \cdots \quad 0$
rectangle full	$0 \quad 0 \quad 0 \quad \cdots \quad 0$
of zeroes	$\vdots \quad \vdots \quad \vdots \quad \quad \vdots$
apart from the	$0 \quad 0 \quad 0 \quad \cdots \quad 0$
two corners	$0 \quad 0 \quad 0 \quad \cdots \quad 1 \leftarrow$ neighbour

Taking the union of neighbours, one gets a Ferrers' diagram full of zeroes (represented by $\heartsuit$, to distinguish them from the outside zeroes), and 1's at the corners :

$$
\text{diagram} \qquad
\begin{array}{cccccccc}
\blacksquare & \heartsuit & \heartsuit & \heartsuit & \heartsuit & 1 & \cdot & \cdot \\
\heartsuit & \heartsuit & \heartsuit & \cdot & \cdot & \cdot & \cdot & \cdot \\
\heartsuit & \heartsuit & 1 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\heartsuit & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
1 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

Inflating a Ferrers' diagram means turning outer corners into inner corners of a bigger diagram :



We shall perform such operation for any box $\blacksquare$ occupied by a -1 such that there is no other -1 in its south-east quadrant :

$$
\begin{array}{ccccccc}
-1 & \heartsuit & \heartsuit & \heartsuit & \heartsuit & 1 & \cdot \\
\heartsuit & \heartsuit & \heartsuit & 0 & 0 & 0 & \cdot \\
\heartsuit & \heartsuit & 1 & 0 & 0 & 0 & \cdot \\
\heartsuit & 0 & 0 & \cdot & \cdot & \cdot & \cdot \\
1 & 0 & 0 & \cdot & \cdot & \cdot & \cdot
\end{array}
\longrightarrow
\begin{array}{ccccccc}
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \cdot \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & \cdot \\
\heartsuit & \heartsuit & \heartsuit & \heartsuit & \heartsuit & 1 & \cdot \\
\heartsuit & \heartsuit & \heartsuit & \cdot & \cdot & \cdot & \cdot \\
\heartsuit & \heartsuit & 1 & \cdot & \cdot & \cdot & \cdot
\end{array}
$$

$$\heartsuit = \text{inner } 0$$

This operation produces a new ASM. Repeating it, one suppresses all -1 entries, and thus obtains a permutation matrix, which is independent of the order in which the inflations have been performed.

A cell is a collection of ASM reducing to the same permutation matrix.

Any statistic on the set of ASM can now be refined into a statistic on each cell. We need only reap the cells to get functions indexed by permutations.

Simplest statistic: record the positions of the boxes of the S-E diagram, and the positions of the -1 entries :

A molecule $\overset{|}{-\mathrm{o}}$ in position i, j has *weight* $y_j/x_i - 1$,

an entry -1 has *weight* y_j/x_i .

$$\begin{bmatrix} . & 1 & . & . & . \\ . & . & . & . & 1 \\ 1 & -1 & . & 1 & \overset{|}{-\mathrm{o}} \\ . & 1 & . & \overset{|}{-\mathrm{o}} & \overset{|}{-\mathrm{o}} \\ . & . & 1 & \overset{|}{-\mathrm{o}} & \overset{|}{-\mathrm{o}} \end{bmatrix} \qquad \begin{array}{c} \frac{y_3}{x_4} \left(\frac{y_1}{x_2} - 1\right) \\ \left(\frac{y_2}{x_2} - 1\right) \left(\frac{y_1}{x_1} - 1\right) \\ \left(\frac{y_2}{x_1} - 1\right) \left(\frac{y_3}{x_1} - 1\right) \end{array}$$

Theorem. The sum of the weights of all the ASM reducing to a permutation σ is equal to the Grothendieck polynomial of index σ.

For example, the three matrices :

$$
\begin{bmatrix} . & 1 & . & . & . \\ . & . & . & . & 1 \\ 1 & -1 & . & 1 & o \\ . & . & 1 & o & o \\ . & 1 & o & o & o \end{bmatrix}
\begin{bmatrix} . & 1 & . & . & . \\ . & . & . & . & 1 \\ 1 & o & . & . & o \\ . & . & . & 1 & o \\ . & . & 1 & o & o \end{bmatrix}
\begin{bmatrix} . & 1 & . & . & . \\ . & . & . & . & 1 \\ 1 & -1 & 1 & . & o \\ . & . & . & 1 & o \\ . & 1 & o & o & o \end{bmatrix}
$$

plus the preceding one, give the polynomial

$$(1-\frac{y_1}{x_2})(1-\frac{y_1}{x_1})(1-\frac{y_2}{x_1})(1-\frac{y_3}{x_1})(1-\frac{y_1y_2y_3}{x_2x_3x_4})$$

Useful only if one has at least another definition of Grothendieck polynomials !

One which is appropriate in China is related to Yang-Baxter equation.

Let s_i be the simple transposition $(i, i{+}1)$. Instead of taking products of s_i's, one wants products of factors $1 + \alpha s_i$, using parameters α.

Problem:

$$(1 + s_1)(1 + s_2)(1 + s_1) \neq (1 + s_2)(1 + s_1)(1 + s_2)$$

does not lift the braid relation

$$s_1\, s_2\, s_1 = s_2\, s_1\, s_2$$

Solution (Yang):

$$(1+s_1)(1+2s_2)(1+s_1) = (1+s_2)(1+2s_1)(1+s_2)$$

More generally,

$$(1+\alpha s_1)\,(1+(\alpha+\beta)s_2)\,(1+\beta s_1) = \\ (1+\beta s_2)\,(1+(\alpha+\beta)s_1)\,(1+\alpha s_2)\ .$$

For the symmetric group $\mathfrak{S}_n$ one can take n parameters $x_1, \ldots, x_n$, and define elements of the group algebra of $\mathfrak{S}_n$ which are in bijection with permutations, and equal to products of factors $1+\alpha s_i$.

Expanding these Yang-Baxter elements in the basis of permutations, one gets coefficients indexed by pairs of permutations.

One can in fact deform the algebra of the symmetric group, relaxing the condition $s_i^2 = 1$ into a more general quadratic relation. Each deformation produces another basis of Yang-Baxter elements.

To get Grothendieck polynomials, one needs to use the 0-Hecke algebra, satisfying

$$T_i^2 = -T_i$$

Instead of $s_1 s_2 s_1$, one has now to use

$$\left(1 + (1 - \frac{x_2}{x_1}) T_1\right) \left(1 + (1 - \frac{x_3}{x_1}) T_2\right) \left(1 + (1 - \frac{x_3}{x_2}) T_1\right) \; .$$

Expansion of such expressions produces (Laurent) polynomials in $x_1, \ldots, x_n$ indexed by pairs of permutations.

It is standard to reconstruct polynomials in two sets of variables indexed by single permutations.

```
ACE> IdcaYang([3,2,1]);
                                                          2
(x1 - x3) A[1,3,2]   (x1 x2 - x3 x2 - x3 x1 +x3 )A[3,1,2]
------------------ + ------------------------------------
        x1                          x1 x2

       (x1 - x3) A[2, 1, 3]
     + -------------------- + A[1, 2, 3]
                x1

      2          2        2    2      2      2
  (-x1 x2 -x3 x2  +x3 x1  -x3 x1 +x3 x2 +x2 x1) A[3,2,1]
 - ----------------------------------------------------
                               2
                             x1  x2
```

```
         2
  (-x1  - x3 x2 + x1 x2 + x3 x1) A[2, 3, 1]
- -----------------------------------------
                     2
                   x1
```

$$-\frac{(-x_1^2 - x_3 x_2 + x_1 x_2 + x_3 x_1)\, A[2,3,1]}{x_1^2}$$

Your computer has made the calculation. It remains to you to recognize the polynomials given by alternating sign matrices !