Notes on Hadamard Conjugation

$ 0 Introduction

Still to be written

$ 1 Analytic Preliminaries: Some Locally Invertible Polynomial Maps

1.0 Notational conventions

For any set C, we will denote the $\mathbb{R}$-vectorspace freely generated by the elements in the set C by $\mathbb{R}[C]$ so that we can consider C as being a subset of $\mathbb{R}[C]$ and, actually, comprising the canonical basis of that space. We also denote by 0_C the zero vector in $\mathbb{R}[C]$.

For any vector p in $\mathbb{R}[C]$ and any element $c \in C$, we will denote by $p(c)$ the coefficient of the basis element c in the canonical expansion

$$p = \sum_{c \in C} p(c)\, c \tag{1}$$

of p relative to that natural basis C of $\mathbb{R}[C]$[1], and its trace $\sum_{c \in C} p(c)$ by

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[1] Clearly, associating to each $p \in \mathbb{R}[C]$ the map $C \to \mathbb{R} : c \mapsto p(c)$ defines the well known canonical isomorphism from $\mathbb{R}[C]$ onto the $\mathbb{R}$-vectorspace $\mathbb{R}^{(C)}$ of all maps p from C into $\mathbb{R}$ whose support $\{c \in C : p(c) \neq 0\}$ is finite — and it also allows us, by the way, to identify the subset $\{p \in \mathbb{R}[C] : p(c) \geq 0$ for all $c \in C$ and $\sum_{c \in C} p(c) = 1\}$ of $\mathbb{R}[C]$ with the set of all probability distributions of finite support that can be defined on C.

$\mathrm{tr}(p)$ (note that even if C is infinite, only finitely many terms in this sum are distinct from 0, so it is always well defined).

In particular, the number $c'(c)$ is defined for any two elements $c, c' \in C$ provided we consider c as a member of the canonical basis C of, and c' as just a vector in $\mathbb{R}[C]$, and this number coincides, of course, with the value of the Kronecker symbol $\delta_{c,c'}$ for the two elements c, c' in C.

Next, we define the affine subspace $\mathbb{R}[C|\rho]$ of $\mathbb{R}[C]$ for any $\rho \in \mathbb{R}$ by

$$\mathbb{R}[C|\rho] := \{p \in \mathbb{R}[C] : \mathrm{tr}(p) = \rho\} \tag{2}$$

and note

(i) that this affine subspace is a linear subspace of $\mathbb{R}[C]$ if and only if $\rho = 0$ holds in which case we have $1 + \dim \mathbb{R}[C|0] = \dim \mathbb{R}[C] = \#C$,

(ii) that $\mathbb{R}[C|0]$ can be identified, in a canonical fashion, with the tangent space $T_{\mathbb{R}[C|\rho]}(p)$ of any point p in $\mathbb{R}[C|\rho]$, for any $\rho \in \mathbb{R}$,

(iii) that the canonical embedding of C into $\mathbb{R}[C]$ actually maps C into — and, thus, identifies C with — a subset of $\mathbb{R}[C|1]$,

(iv) and that, for every $c_0 \in C$, the subset $C - c_0 := \{c - c_0 : c \in C - \{c_0\}\}$ of $\mathbb{R}[C|0]$ forms a basis of $\mathbb{R}[C|0]$.

Furthermore, given a finite family $\mathcal{C} := (C_e)_{e \in E}$ of sets C_e $(e \in E)$, parametrized by a finite index set E, we denote by

$$\prod \mathcal{C} := \prod_{e \in E} C_e \tag{3}$$

the cartesian product of the sets in that family $\mathcal{C}$, by

$$\mathbb{R}[\mathcal{C}] := \oplus_{e \in E} \mathbb{R}[C_e] \tag{4}$$

the direct sum of the family $(\mathbb{R}[C_e])_{e \in E}$ of the vectorspaces $\mathbb{R}[C_e]$ associated with the sets in $\mathcal{C}$, and by

$$\mathbb{R}[\mathcal{C}|\rho] := \{\oplus_{e \in E} p^e \in \oplus_{e \in E} \mathbb{R}[C_e] : p^e \in \mathbb{R}[C_e|\rho] \text{ for all } e \in E\} \tag{5}$$

the cartesian product of the subsets $\mathbb{R}[C_e|\rho] \subset \mathbb{R}[C_e]$ over all $e \in E$, considered as a subset of the direct sum $\mathbb{R}[\mathcal{C}] = \oplus_{e \in E} \mathbb{R}[C_e]$.

And we note that, given any subset C' of C, we may consider the associated monomial map

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$$\psi_{C'} : \mathbb{R}[C] \to \mathbb{R} : p \mapsto \prod_{c \in C'} p(c) \tag{6}$$

which, by definition, coincides with the product, $\prod_{c' \in C'} \psi_{\{c'\}}$ of the 'linear' maps $\psi_{\{c'\}}$. Obviously, its partial derivative $\frac{\partial \psi}{\partial c}(p)$ relative to the direction given by a basis element $c \in C$ at any point $p \in \mathbb{R}[C]$ coincides with $\prod_{c' \in C' - \{c\}} p(c')$ in case $c \in C'$ and vanishes otherwise in view of

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$$\begin{aligned} \frac{\partial \psi_{C'}}{\partial c}(p) &= \sum_{c' \in C'} \frac{\partial \psi_{\{c'\}}}{\partial c}(p) \prod_{c'' \in C' - \{c'\}} p(c'') \\ &= \sum_{c' \in C'} \delta_{c',c} \prod_{c'' \in C' - \{c'\}} p(c'') \\ &= \begin{cases} \prod\limits_{c' \in C' - \{c\}} p(c') & \text{if } c \in C', \\ 0 & \text{else .} \end{cases} \end{aligned} \tag{7}$$

Remark *Note that all of this, except the discussion regarding the partial derivatives of specific monomial maps, works as well if $\mathbb{R}$ is replaced by any field — or even by any commutative ring R once we avoid calling the resulting spaces 'vectorspaces' in case R is not a field, but just (free) R-modules, though we may and will still call any element in such a space or, as well, in a direct sum of such spaces, a 'point'.*

1.1 The Basic Set Up

In this paper, we always will consider

- a finite family $\mathcal{A} := (A_e)_{e \in E}$ of finite sets A_e $(e \in E)$, parametrized by a finite index set E,
- the cartesian product
$$\prod \mathcal{A} := \prod_{e \in E} A_e \tag{8}$$
of the sets in that family $\mathcal{A}$,

- the direct sum
$$\mathbb{R}[\mathcal{A}] := \oplus_{e\in E}\mathbb{R}[A_e] \tag{9}$$
of the vectorspaces $\mathbb{R}[A_e]$, $e \in E$, associated with the sets A_e in the family $\mathcal{A}$,

- a map
$$\varphi : \prod \mathcal{A} \to A : \mathbf{a} = (a_e)_e \mapsto \varphi(\mathbf{a}) \tag{10}$$
from that cartesian product into another finite set A,

- and the associated polynomial map
$$\varphi^{\mathbb{R}} : \mathbb{R}[\mathcal{A}]{\to}\mathbb{R}[A] : \mathbf{p}{\to}\varphi^{\mathbb{R}}\mathbf{p} \tag{11}$$
of the direct sum $\mathbb{R}[\mathcal{A}]$ into the $\mathbb{R}$-vectorspace $\mathbb{R}[A]$ that maps any point $\mathbf{p} = \oplus_e p^e \in \mathbb{R}[\mathcal{A}] = \oplus_{e\in E}\mathbb{R}[A_e]$ onto the point
$$\varphi^{\mathbb{R}}\mathbf{p} := \sum_{(a_e)_e\in\prod\mathcal{A}} \prod_e p^e(a_e)\,\varphi\big((a_e)_e\big) \in \mathbb{R}[A] \tag{12}$$
(see the Appendix for a simple, explictly worked-out example) — where we make use of our convention to consider the elements $a = \varphi\big((a_e)_e\big)$ in A as points in $\mathbb{R}[A]$). Equivalently, we may define $\varphi^{\mathbb{R}}\mathbf{p}$, for any point $\mathbf{p} = \oplus_e p^e \in \mathbb{R}[\mathcal{A}]$, by
$$\mathbf{13} \qquad \varphi^{\mathbb{R}}\mathbf{p} := \sum_{\mathbf{a}\in\prod_e A_e} \mathbf{p}(\mathbf{a})\,\varphi(\mathbf{a}) = \sum_{a\in A}\big(\sum_{\mathbf{a}\in\varphi^{-1}(a)} \mathbf{p}(\mathbf{a})\big)\,a \tag{13}$$
where $\mathbf{p}(\mathbf{a})$ is defined, for every $\mathbf{p} = \oplus_e p^e \in \mathbb{R}[\mathcal{A}]$ and $\mathbf{a} = (a_e)_e \in \prod\mathcal{A}$, by
$$\mathbf{14} \qquad \mathbf{p}(\mathbf{a}) := \prod_e p^e(a_e). \tag{14}$$

1.2 A First Simple Product Formula

Note that the trace
$$\mathrm{tr}(\varphi^{\mathbb{R}}\mathbf{p}) := \sum_{a\in A}(\varphi^{\mathbb{R}}\mathbf{p})(a)$$

of the $\varphi^{\mathbb{R}}$-image $\varphi^{\mathbb{R}}\mathbf{p}$ of any point $\mathbf{p} = \oplus_e p^e$ in $\mathbb{R}[\mathcal{A}]$ always coincides with the product (over all $e \in E$) of the traces

$$\mathrm{tr}(p^e) := \sum_{a_e \in A_e} p^e(a_e)$$

of the components p^e $(e \in E)$ of $\mathbf{p}$: Indeed, we have

$$\begin{aligned} \prod_e \mathrm{tr}(p^e) &= \prod_e \Big(\sum_{a_e \in A_e} p^e(a_e) \Big) \\ &= \sum_{(a_e)_e \in \prod \mathcal{A}} \prod_e p^e(a_e) = \mathrm{tr}(\varphi^{\mathbb{R}}\mathbf{p}) \end{aligned} \tag{15}$$

for all $\mathbf{p} = \oplus_e p^e \in \mathbb{R}[\mathcal{A}]$.

In particular,

(i) $\varphi^{\mathbb{R}}$ induces — by restriction — a polynomial map $\varphi^{\mathbb{R}|1}$ from the space

$$\mathbb{R}[\mathcal{A}|1] \ = \{\oplus_e p^e : p^e \in \mathbb{R}[A_e|1] \text{ for all } e \in E\} \subset \mathbb{R}[\mathcal{A}] \tag{16}$$

into the affine subspace $\mathbb{R}[A|1]$ of the vectorspace $\mathbb{R}[A]$:

$$\varphi^{\mathbb{R}|1} : \mathbb{R}[\mathcal{A}|1] \rightarrow \mathbb{R}[A|1] : \mathbf{p} \mapsto \varphi^{\mathbb{R}}\mathbf{p}, \tag{17}$$

(ii) and it maps any point $\mathbf{p} = \oplus_e p^e \in \mathbb{R}[\mathcal{A}|1]$ for which each component p^e represents, for every $e \in E$, in fact a probability distribution defined on A_e, onto a point $\varphi^{\mathbb{R}}\mathbf{p} \in \mathbb{R}[A|1]$ that represents a probability distribution defined on A.

1.3 Possible Relevance of such Maps

Such maps seem to be of some interest in quite a number of distinct situations, their images as well as their local behaviour. They may describe global signals $\varphi^{\mathbb{R}}\mathbf{p}$ elecited by a system $\mathcal{S}$ whose 'global' states P are determined by a family $\mathbf{p} = (p^e)_e$ of 'local' states p^e of certain subsystems $\mathcal{S}_e$ (e ranging in some index set E) by some not necessarily linear interaction of those subsystems.

More specifically, we may assume that, in a probabilistic set up that will be pursued in particular in the phylogenetic applications of Hadamard conjugation discussed above / to be detailed below (cf. also [**?**]), that

- there is a family $\mathcal{A} = (A_e)_{e \in E}$ of sets A_e of 'pure' states for the subsystems $\mathcal{S}_e$ $(e \in E)$,
- a set A of 'pure' states for $\mathcal{S}$,
- the 'actual' states of the subsystems $\mathcal{S}_e$ are given, for each $e \in E$, by probability distributions p^e defined on A_e,
- and the subsystems $\mathcal{S}_e$ $(e \in E)$ conspire — in a way described by a map $\varphi : \prod \mathcal{A} \to A$ — to give rise to a pure state $\varphi\big((a_e)_e\big)$ of $\mathcal{S}$ when, for each $e \in E$, the subsystem $\mathcal{S}_e$ is in a pure state a_e, i.e,. their interaction give rise to the given map $\varphi : \prod \mathcal{A} \to A$.

The 'actual' state of $\mathcal{S}$ can then be described by a probability distribution P defined on A which is — assuming independence of the events in the various subsystems $\mathcal{S}_e$ — defined exactly by the image $\varphi^{\mathbb{R}}\mathbf{p}$ of the resulting point $\mathbf{p} := \oplus_e p^e$ in $\mathbb{R}[\mathcal{A}|1]$ relative to the map $\varphi^{\mathbb{R}} : \mathbb{R}[\mathcal{A}|1] \to \mathbb{R}[A|1]$ defined in **1.1**.

1.4 More Notational Conventions

1.4.1 Let us note first that the canonical embeddings $A_e \to \mathbb{R}[A_e|1]$ $(e \in E)$ induce a canonical embedding of the product set $\prod \mathcal{A}$ into the set $\mathbb{R}[\mathcal{A}|1]$ given by

$$\prod \mathcal{A} \to \mathbb{R}[\mathcal{A}|1] : \mathbf{a} = (a_e)_e \mapsto \oplus_e a_e \in \mathbb{R}[\mathcal{A}|1] \subset \prod_{e \in E} \mathbb{R}[A_e] \tag{18}$$

that we will use to identify each element $\mathbf{a} = (a_e)_e \in \prod \mathcal{A}$ with its canonical image $\oplus_e a_e$ in $\mathbb{R}[\mathcal{A}|1]$ and — correspondingly — the product set $\prod \mathcal{A}$ with the subset $\{\oplus_e a_e : (a_e)_e \in \prod \mathcal{A}\}$ of $\mathbb{R}[\mathcal{A}|1]$.

1.4.2

Further, denoting by $\mathbf{p}^e$ the component p^e in $\mathbb{R}[A_e]$ of any point $\mathbf{p} = \oplus_e p^e$ in $\mathbb{R}[\mathcal{A}]$, we have

(i) $\mathbf{a}^e = a_e$ for every $\mathbf{a} = (a_e)_e \in \prod \mathcal{A} \subset \mathbb{R}[\mathcal{A}|1]$,

(ii) $\mathbf{p}(\mathbf{a}) = \prod_e \mathbf{p}^e(\mathbf{a}^e)$ for all $\mathbf{p}$ in $\mathbb{R}[\mathcal{A}]$ and $\mathbf{a} \in \prod \mathcal{A}$ and, therefore,

(iii) $\varphi^{\mathbb{R}}\mathbf{p} = \sum_{\mathbf{a} \in \prod \mathcal{A}} \prod_e \mathbf{p}^e(\mathbf{a}^e)\ \varphi(\mathbf{a})$ for all $\mathbf{p} \in \mathbb{R}[\mathcal{A}]$.

In particular, we have

(iv) $\mathbf{a}(\mathbf{a}') = \prod_e \mathbf{a}^e(\mathbf{a}'^e) = \prod_e \delta_{\mathbf{a}^e,\mathbf{a}'^e} = \delta_{\mathbf{a},\mathbf{a}'}$ for all $\mathbf{a}, \mathbf{a}' \in \prod \mathcal{A}$, and

(v) $\varphi^{\mathbb{R}|1}\mathbf{a} = \varphi^{\mathbb{R}}\mathbf{a} = \sum_{\mathbf{a}' \in \prod \mathcal{A}} \mathbf{a}(\mathbf{a}')\, \varphi(\mathbf{a}') = \varphi(\mathbf{a})$ for every $\mathbf{a}$ in $\prod \mathcal{A}$.

1.4.3 Next, given some fixed element $\mathbf{a_1} = (a_e^1)^e \in \prod_e A_e$ and arbitrary elements $e \in E$ and $a_e \in A_e$,

(vi) we denote by $\mathbf{a_1} \overset{e}{\leftarrow} a_e$ the unique element in $\prod \mathcal{A}$ whose e-th component coincides with a_e while every other component coincides with that of $\mathbf{a_1}$, that is, we define $\mathbf{a_1} \overset{e}{\leftarrow} a_e$ by reqiring that

$$(\mathbf{a_1} \overset{e}{\leftarrow} a_e)^f := \begin{cases} a_f^1 & \text{if } f \neq e, \\ a_e & \text{if } f = e \end{cases} \tag{19}$$

holds for all $f \in E$,

(vii) we define the subset $\mathcal{N}(\mathbf{a_1})$ of $\prod \mathcal{A}$ by

$$\mathcal{N}(\mathbf{a_1}) := \{\mathbf{a} \in \prod \mathcal{A} : \#\{e \in E : \mathbf{a}^e \neq \mathbf{a_1}^e\} \leq 1\} \tag{20}$$

and its image set under φ by $A_\varphi(\mathbf{a_1})$:

$$A_\varphi(\mathbf{a_1}) := \varphi\big(\mathcal{N}(\mathbf{a_1})\big) := \{\varphi(\mathbf{a}) : \mathbf{a} \in \mathcal{N}(\mathbf{a_1})\}, \tag{21}$$

(viii) we note in passing that

$$\#\mathcal{N}(\mathbf{a_1}) = 1 + \sum_{e \in E} (\#A_e - 1)$$

and

$$\mathcal{N}(\mathbf{a_1}) = \{\mathbf{a_1} \overset{e}{\leftarrow} a_e : e \in E, a_e \in A_e\},$$

always holds,

(xi) we denote the point $a_e \oplus (\oplus_{f \in E - \{e\}} 0_{A_f}) \in \mathbb{R}[\mathcal{A}]$ by $[e, a_e]$:

$$[e, a_e] := a_e \oplus (\oplus_{f \in E - \{e\}} 0_{A_f}) \in \mathbb{R}[\mathcal{A}], \tag{22}$$

(x) we denote the set of all such points by $\mathcal{B}_{\mathcal{A}}$:

$$\mathcal{B}_{\mathcal{A}} := \{[e, a_e] : e \in E, a_e \in A_e\}, \tag{23}$$

and note that this set always forms a 'canonical' basis of $\mathbb{R}[\mathcal{A}]$ and that

$$\mathbf{a} = \sum_e [e, \mathbf{a}^e] \tag{24}$$

holds for every $\mathbf{a} \in \prod \mathcal{A}$ considered as an element of $\mathbb{R}[\mathcal{A}]$,

(xi) we note that the set

$$\{\mathbf{a} - \mathbf{a_1} : \mathbf{a} \in \mathcal{N}(\mathbf{a_1}) - \{\mathbf{a_1}\}\} = \{[e, a_e] - [e, a^1_e] : e \in E, a_e \in A_e - \{a^1_e\}\}$$

always forms a 'canonical' basis of the direct sum

$$\mathbb{R}[\mathcal{A}|0] := \oplus_{e \in E} \mathbb{R}[A_e|0]$$

that will soon turn up as the tangent space $T_{\mathbb{R}[\mathcal{A}|1]}(\mathbf{p})$ (relative to the space $\mathbb{R}[\mathcal{A}|1]$) of any point $\mathbf{p}$ in that space.

(xii) and — to avoid repetition later on — we note that, so far, all of these notational conventions work as well when $\mathbb{R}$ is replaced by any commutative ring R with a unit element $1_R \in R$ except that the canonical bases of the various vectorspaces we have been considering then become bases of the corresponding free R-modules.

1.4.4 These notational conventions imply also that, given any $\mathbf{a} \in \prod \mathcal{A}$ and $\mathbf{p} \in \mathbb{R}[\mathcal{A}]$, the partial derivative $\frac{\partial \psi_{\mathbf{a}}}{\partial [e, a_e]}(\mathbf{p})$ of the map

$$\mathbf{25} \qquad \psi_{\mathbf{a}} : \mathbb{R}[\mathcal{A}] \rightarrow \mathbb{R} : \mathbf{p} \rightarrow \mathbf{p}(\mathbf{a}) \tag{25}$$

in the direction of the basis vector $[e, a_e]$ at some point $\mathbf{p}$ in $\mathbb{R}[\mathcal{A}]$ is given by

$$\mathbf{26} \qquad \frac{\partial \psi_{\mathbf{a}}}{\partial [e, a_e]}(\mathbf{p}) = \delta_{\mathbf{a}^e, a_e} \prod_{f \in E - \{e\}} \mathbf{p}^f(\mathbf{a}^f) \tag{26}$$

for all $e \in E$ and $a_e \in A_e$: Indeed, identifying $\mathbb{R}[\mathcal{A}]$ with $\mathbb{R}[\mathcal{B}_{\mathcal{A}}]$ and using the notation that was introduced in (**??**) for $C := \mathcal{B}_{\mathcal{A}}$, we have

$$\psi_{\mathbf{a}} = \psi_{\{[e, \mathbf{a}^e] : e \in E\}} \tag{27}$$

for all $\mathbf{a} \in \prod \mathcal{A}$; so, (**??**) follows immediately from (**??**).

In particular, if $\mathbf{p}$ coincides with the point $\mathbf{a_1} = (a^1_e)_e \in \prod \mathcal{A}$, we have

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$$\begin{aligned} \frac{\partial \psi_{\mathbf{a}}}{\partial [e, a_e]}(\mathbf{a_1}) &= \delta_{\mathbf{a}^e, a_e} \prod_{f \in E - \{e\}} \mathbf{a_1}^f (\mathbf{a}^f) \\ &= \delta_{\mathbf{a}^e, a_e} \prod_{f \in E - \{e\}} \delta_{\mathbf{a_1}^f, \mathbf{a}^f} \\ &= \delta_{\mathbf{a_1} \overset{e}{\longleftarrow} a^e, \mathbf{a}} \end{aligned} \tag{28}$$

1.5 The Local Behaviour of $\varphi^{\mathbb{R}|1}$

In this subsection, we will analyse the local behaviour of the map $\varphi^{\mathbb{R}|1}$ in a neighbourhood of a point $\mathbf{a_1} = (a^1_e)_e \in \prod \mathcal{A}$ considered — according to our conventions — as a point in $\mathbb{R}[\mathcal{A}|1]$.

Our first basic result reads:

Theorem 1: *The polynomial map*

$$\varphi^{\mathbb{R}|1} : \mathbb{R}[\mathcal{A}|1] \rightarrow \mathbb{R}[A|1] : \mathbf{p} \mapsto \varphi^{\mathbb{R}} \mathbf{p}$$

is locally injective (or bijective) at a point $\mathbf{a_1} \in \prod \mathcal{A}\,(\subset \mathbb{R}[\mathcal{A}|1])$ if and only if the restriction $\varphi_{\mathbf{a_1}}$ of the map φ to the subset $\mathcal{N}(\mathbf{a_1})$ is an injective map into (or, respectively, a bijective map onto) A.

More precisely, given any basis element $[e, a] \in \mathcal{B}_{\mathcal{A}}$, we have

$$D_{\mathbf{a_1}} \varphi^{\mathbb{R}}([e, a]) = \varphi(\mathbf{a_1} \overset{e}{\leftarrow} a_e) \tag{29}$$

for the total derivative

$$D_{\mathbf{a_1}} \varphi^{\mathbb{R}} : \mathbb{R}[\mathcal{A}] \rightarrow \mathbb{R}[A] \tag{30}$$

of the map $\varphi^{\mathbb{R}}$ at the point $\mathbf{a_1}$.

Thus, $D_{\mathbf{a_1}} \varphi^{\mathbb{R}}$ maps the canonical basis elements $[e, a] \in \mathcal{B}_{\mathcal{A}}$ of tangent space $T_{\mathbb{R}[\mathcal{A}]}(\mathbf{a_1}) = \mathbb{R}[\mathcal{A}]$ of the point $\mathbf{a_1} \in \mathbb{R}[\mathcal{A}]$ onto the canonical basis elements of the space $\mathbb{R}[A_\varphi(\mathbf{a_1})]$ and, thus, the tangent space $T_{\mathbb{R}[\mathcal{A}|1]}(\mathbf{a_1}) = \mathbb{R}[\mathcal{A}|0]$ of the same point $\mathbf{a_1}$, considered as a point in $\mathbb{R}[\mathcal{A}|1]$, onto the space $\mathbb{R}[A_\varphi(\mathbf{a_1})|0]$ which in turn implies that the following assertions all are equivalent:

(i) *the restriction* $D_{(\mathbf{a_1}|0)}\varphi^{\mathbb{R}|1}$ *of* $D_{\mathbf{a_1}}\varphi^{\mathbb{R}}$ *to this tangent space is an injective map,*

(ii) *the induced linear map from this tangent space onto its image* $\mathbb{R}[A_\varphi(\mathbf{a_1})|0]$ *is a bijection,*

(iii) *we have* $\dim \mathbb{R}[\mathcal{A}|0] = \dim(\mathbb{R}[A_\varphi(\mathbf{a_1})|0])$*,*

(iv) *we have* $\#\mathcal{N}(\mathbf{a_1}) = \#A_\varphi(\mathbf{a_1})$

(v) *the map* $\varphi_{\mathbf{a_1}} : \mathcal{N}(\mathbf{a_1}) \rightarrow A_\varphi(\mathbf{a_1})$ *of the set* $\mathcal{N}(\mathbf{a_1})$ *onto its image set* $A_\varphi(\mathbf{a_1}) = \varphi_{\mathbf{a_1}}(\mathcal{N}(\mathbf{a_1}))$ *is a bijection,*

(vi) *the map* $\varphi_{\mathbf{a_1}} : \mathcal{N}(\mathbf{a_1}) \rightarrow A$ *is injective.*

Proof: Noting that

$$\begin{aligned} \varphi^{\mathbb{R}}\mathbf{p} &= \sum_{\mathbf{a}\in\prod\mathcal{A}} \mathbf{p}(\mathbf{a})\ \varphi(\mathbf{a}) \\ &= \sum_{\mathbf{a}\in\prod\mathcal{A}} \psi_{\mathbf{a}}(\mathbf{p})\ \varphi(\mathbf{a}) \end{aligned} \tag{31}$$

holds for all $\mathbf{p} \in \mathbb{R}[\mathcal{A}]$ for the maps $\psi_{\mathbf{a}} : \mathbb{R}[\mathcal{A}] \rightarrow \mathbb{R}$ defined in (**??**) and that

$$\big(D_{\mathbf{a_1}}\psi\big)([e, a_e]) = \frac{\partial\psi}{\partial[e, a_e]}(\mathbf{a_1}) \tag{32}$$

holds (by definition of partial and total derivatives and identifying $\mathbb{R}$ with its own tangent space at any point in $\mathbb{R}$) for every continuously differentiable map $\psi : \mathbb{R}[\mathcal{A}] \rightarrow \mathbb{R}$, we see that

$$\begin{aligned} D_{\mathbf{a_1}}\varphi^{\mathbb{R}}([e, a_e]) &= \sum_{\mathbf{a}\in\prod\mathcal{A}} \big(D_{\mathbf{a_1}}\psi_{\mathbf{a}}\big)([e, a_e])\,\varphi(\mathbf{a}) \\ &= \sum_{\mathbf{a}\in\prod\mathcal{A}} \frac{\partial\psi_{\mathbf{a}}}{\partial[e, a_e]}(\mathbf{a_1})\,\varphi(\mathbf{a}) = \varphi(\mathbf{a_1} \overset{e}{\leftarrow} a_e) \end{aligned} \tag{33}$$

holds indeed in view of (**??**).

The remaining assertions are obvious in view of the fact that

(i) any map $\phi : C \to C'$ is injective if and only if it induces a bijection from C onto its image set $\phi(C')$,

(ii) a surjective map $\phi : C \to C'$ from a finite set onto another finite set is a bijection if and only if C and C' have the same cardinality,

(iii) a linear surjective map $\phi : C \to C'$ from a finite-dimensional $\mathbb{R}$-vectorspace onto another finite-dimensional $\mathbb{R}$-vectorspace is a bijection if and only if these two vectorspaces have the same dimension,

(vi) and $\#\mathcal{N}(\mathbf{a_1}) = 1 + \dim \mathbb{R}[\mathcal{A}|0]$ and $\#A_\varphi(\mathbf{a_1}) = 1 + \dim(\mathbb{R}[A_\varphi(\mathbf{a_1})|0])$ holds (in view of **1.0** (i) and **1.4.3** (viii)).

1.6 Local Inverses of $\varphi^{\mathbb{R}|1}$

It follows from the results of the previous section that, putting

$$A' := \big(A - A_\varphi(\mathbf{a_1})\big) \cup \{\mathbf{a_1}\}, \tag{34}$$

the 'extension'

$$\Phi^{\mathbb{R}|1} : \mathbb{R}[\mathcal{A}|1] \times \mathbb{R}[A'|0] \to \mathbb{R}[A|1] : \mathbf{p} \times \mathbf{q} \mapsto \varphi^{\mathbb{R}|1}\mathbf{p} + \mathbf{q}$$

of the map $\varphi^{\mathbb{R}|1}$ has an invertible Jacobian at $\mathbf{a_1} \oplus 0_{A'}$ in case $\varphi_{\mathbf{a_1}}$ is injective in which case there must, herefore, exist

(i) a neighbourhood $\mathcal{O}_{\mathbf{a_1}}$ of $\mathbf{a_1}$ in $\mathbb{R}[\mathcal{A}|1]$,

(ii) a neighbourhood $\mathcal{O}_{0_{A'}}$ of $0_{A'}$ in $\mathbb{R}[A'|0]$,

(iii) and a neighbourhood $\mathcal{O}_{\varphi(\mathbf{a_1})}$ of $\varphi(\mathbf{a_1})$ in $\mathbb{R}[A|1]$

such that restricting $\Phi^{\mathbb{R}|1}$ to $\mathcal{O}_{\mathbf{a_1}} \times \mathcal{O}_{0_{A'}}$ defines a homeomorphism from this product onto $\mathcal{O}_{\varphi(\mathbf{a_1})}$.

In other words, we can choose these three neighbourhoods so that there exist two maps

$$\eta_1 : \mathcal{O}_{\varphi(\mathbf{a_1})} \to \mathcal{O}_{\mathbf{a_1}} \subseteq \mathbb{R}[\mathcal{A}|1] \quad \text{and} \quad \eta_2 : \mathcal{O}_{\varphi(\mathbf{a_1})} \to \mathcal{O}_{0_{A'}} \subseteq \mathbb{R}[A'|0] \tag{35}$$

such that their product

$$\eta_1 \times \eta_2 : \mathcal{O}_{\varphi(\mathbf{a_1})} \to \mathcal{O}_{\mathbf{a_1}} \times \mathcal{O}_{0_{A'}} \tag{36}$$

is the inverse of the restriction of $\Phi^{\mathbb{R}|1}$ to $\mathcal{O}_{\mathbf{a_1}} \times \mathcal{O}_{0_{A'}}$, i.e., such that

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$$\begin{aligned} P &= \Phi^{\mathbb{R}|1}\big((\eta_1 \times \eta_2)(P)\big) = \Phi^{\mathbb{R}|1}\big(\eta_1(P) \times \eta_2(P)\big) \\ &= \varphi^{\mathbb{R}|1}((\eta_1(P)) + \eta_2(P) \end{aligned} \tag{37}$$

holds for all $P \in \mathcal{O}_{\varphi(\mathbf{a_1})}$, and

$$\begin{aligned} \mathbf{p} \times \mathbf{q} &= (\eta_1 \times \eta_2)\big(\Phi^{\mathbb{R}|1}(\mathbf{p} \times \mathbf{q})\big) = (\eta_1 \times \eta_2)(\varphi^{\mathbb{R}|1}\mathbf{p} + \mathbf{q}) \\ &= \eta_1(\varphi^{\mathbb{R}|1}\mathbf{p} + \mathbf{q}) \times \eta_2(\varphi^{\mathbb{R}|1}\mathbf{p} + \mathbf{q})) \end{aligned} \tag{38}$$

or, equivalently, **39**

$$\mathbf{p} = \eta_1(\varphi^{\mathbb{R}|1}\mathbf{p} + \mathbf{q}) \quad \text{and} \quad \mathbf{q} = \eta_2(\varphi^{\mathbb{R}|1}\mathbf{p} + \mathbf{q}) \tag{39}$$

for all $\mathbf{p} \in \mathcal{O}_{\mathbf{a_1}}$ and $\mathbf{q} \in \mathcal{O}_{0_{A'}}$.

In particular, η_1 is a local left inverse of $\varphi^{\mathbb{R}|1}$ at $\mathbf{a_1}$, that is, we have

40
$$\mathbf{p} = \eta_1(\varphi^{\mathbb{R}|1}\mathbf{p}) \tag{40}$$

for all $\mathbf{p} \in \mathcal{O}_{\mathbf{a_1}}$, allowing us to 'reconstruct' the point $\mathbf{p}$ from its image $\varphi^{\mathbb{R}|1}\mathbf{p}$.
Furthermore, we can use either one, η_1 or η_2, to describe the set

41
$$\varphi^{\mathbb{R}|1}(\mathcal{O}_{\mathbf{a_1}}) := \{\varphi^{\mathbb{R}|1}\mathbf{p} : \mathbf{p} \in \mathcal{O}_{\mathbf{a_1}}\} \tag{41}$$

in terms of a map F from $\mathcal{O}_{\varphi(\mathbf{a_1})}$ into some linear space for which

$$\varphi^{\mathbb{R}|1}(\mathcal{O}_{\mathbf{a_1}}) = \{P \in \mathcal{O}_{\varphi(\mathbf{a_1})} : F(P) = 0\} \tag{42}$$

holds. Indeed, (**??**,39,40) together imply that, given any $P \in \mathcal{O}_{\varphi(\mathbf{a_1})}$, the implications

$$(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (i)$$

hold for the following assertions

(i) $P = \varphi^{\mathbb{R}|1}\mathbf{p}$ holds for some $\mathbf{p} \in \mathcal{O}_{\mathbf{a_1}}$,
(ii) one has $\varphi^{\mathbb{R}|1}(\eta_1(P)) = P$,
(iii) one has $\varphi^{\mathbb{R}|1}(\eta_1(P)) - P = 0_A$,
(iv) one has $\eta_2(P)) = 0_{A'}$,
(v) one has $P = \varphi^{\mathbb{R}|1}(\eta_1(P))$.

Indeed, if $P = \varphi^{\mathbb{R}|1}\mathbf{p}$ holds for some $\mathbf{p} \in \mathcal{O}_{\mathbf{a_1}}$, we have $\varphi^{\mathbb{R}|1}(\eta_1(P)) = \varphi^{\mathbb{R}|1}(\eta_1(\varphi^{\mathbb{R}|1}\mathbf{p})) = \varphi^{\mathbb{R}|1}\mathbf{p} = P$ and, therefore, of course also $\varphi^{\mathbb{R}|1}(\eta_1(P)) - P = 0$. In turn, this implies $\eta_2(P)) = \eta_2(\varphi^{\mathbb{R}|1}(\eta_1(P))) = \eta_2(\varphi^{\mathbb{R}|1}(\eta_1(P)) + 0_{A'}) = 0_{A'}$ which, in turn, implies

$$P = \varphi^{\mathbb{R}|1}((\eta_1(P)) + \eta_2(P) = \varphi^{\mathbb{R}|1}((\eta_1(P))$$

and, therefore, $P = \varphi^{\mathbb{R}|1}\mathbf{p}$ for some $\mathbf{p} \in \mathcal{O}_{\mathbf{a_1}}$.

Note that such maps have been discussed thoroughly in the context of the theory of *phylogenetic invariants* initiated by Jim Lake, Steven Evans, and Terry Speed (cf. [**?**]).

A related approach can also be used to check whether, given any family

$$\phi_i : \mathbb{R}[A|1] \rightarrow \mathbb{R} \ \ (i = 1, \dots, k) \tag{43}$$

of continuously differentiable real-valued maps with

$$\phi_i(\varphi^{\mathbb{R}|1}\mathbf{p}) = 0 \tag{44}$$

for all $i = 1, \dots, k$ and all $\mathbf{p}$ in some neighbourhood of $\mathbf{a_1}$ in $\mathbb{R}[\mathcal{A}|1]$, the identity

$$\varphi^{\mathbb{R}|1}(\mathcal{O}_{\mathbf{a_1}}) = \{P \in \mathcal{O}_{\varphi(\mathbf{a_1})} : \phi_i(P) = 0 \text{ for all } i = 1, \dots, k\} \tag{45}$$

holds for some neighbourhood $\mathcal{O}_{\mathbf{a_1}}$ of $\mathbf{a_1}$ in $\mathbb{R}[\mathcal{A}|1]$: Indeed, this exactly the case — even if $\varphi_{\mathbf{a_1}}$ is not injective — if and only if the dimension of the subspace of the cotangent space of $\mathbb{R}[\mathcal{A}|1]$ at the point $\mathbf{a_1} \in \mathbb{R}[\mathcal{A}|1]$ that is generated by the gradients of the maps ϕ_i $(i = 1, \dots, k)$, coincides with the codimension $\#(A - A_\varphi(\mathbf{a_1})) = \#(A - \varphi(\mathcal{N}(\mathbf{a_1})))$ of the space $\mathbb{R}[A_\varphi(\mathbf{a_1})|0]$ in $\mathbb{R}[A|0]$.

Note finally that the standard proof of the Inverse Function Theorem is sufficiently ‘constructive’ to provide explicit descriptions of appropriate neighbourhoods $\mathcal{O}_{\mathbf{a_1}}$, $\mathcal{O}_{0_{A'}}$, and $\mathcal{O}_{\varphi(\mathbf{a_1})}$ for which $\Phi^{\mathbb{R}|1}$ provides a diffeomorphism between $\mathcal{O}_{\mathbf{a_1}} \times \mathcal{O}_{0_{A'}}$, and $\mathcal{O}_{\varphi(\mathbf{a_1})}$, and to compute arbitrarily good approximations of $\eta_1(P)$ and $\eta_2(P)$ for every explicitly specified point $P \in \mathcal{O}_{\varphi(\mathbf{a_1})}$.

Moreover, one could as well use the fact that $\Phi^{\mathbb{R}|1}$ is a polynomial to compute recursively the coefficients of the power-series expansion of its inverse in an appropriate neighbourhood of $\varphi(\mathbf{a_1})$ and hope that this power series converges fast enough so that its first terms provide a sufficiently good approximation of the inverse map in that neighbourhood.

Yet, as we will see below, the co-called *Hadamard conjugation* introduced in [?] allows us to construct — for appropriately specified maps $\varphi : \prod \mathcal{A} \to A$ — a purely algebraic expression (in terms of sums of roots of products of sums, but nothing worse) for a local left inverse

$$HC : \mathcal{O}_{\varphi(\mathbf{a_1})} \to \mathbb{R}[\mathcal{A}|1] \tag{46}$$

of $\varphi^{\mathbb{R}|1}$ that is defined on some appropriately specified neighbourhood $\mathcal{O}_{\varphi(\mathbf{a_1})}$ of $\varphi(\mathbf{a_1})$ in $\mathbb{R}[A|1]$, i.e., a map $HC : \mathcal{O}_{\varphi(\mathbf{a_1})} \to \mathbb{R}[\mathcal{A}|1]$ such that

$$HC(\varphi^{\mathbb{R}|1}\mathbf{p}) = \mathbf{p} \tag{47}$$

holds for all points $\mathbf{p}$ in some neighbourhood of $\mathbf{a_1}$ in $\mathbb{R}[\mathcal{A}|1]$ (also to be specified appropriately) .

$ 2 Algebraic Preliminaries: Some Useful Identities

2.0 Some Simplifying Identifications

Continuing with the notations introduced above, we consider as before a finite family $\mathcal{A} = (A_e)_{e\in E}$ of finite sets together with a specified element $\mathbf{a_1} \in \prod \mathcal{A}$ and a map φ from their product $\prod \mathcal{A}$ into a finite set A. Further, we will assume that the associated map $\varphi_{\mathbf{a_1}} : \mathcal{N}(\mathbf{a_1}) \rightarrow A$ is injective, and we will use this map to identify, for every $e \in E$, each element $a_e \in A_e$ with the image $\varphi(\mathbf{a_1} \overset{e}{\leftarrow} a_e) \in A$ of the associated element $\mathbf{a_1} \overset{e}{\leftarrow} a_e \in \prod \mathcal{A}$ — this way identifying the set A_e with the corresponding subset of A and the subset $A_\varphi(\mathbf{a_1})$ with the union $\bigcup_{e\in E} A_e$ of these subsets of A. Further, denoting the element $\varphi(\mathbf{a_1}) \in A$ (for reasons that will become obvious later on) by 1_A, we see that $1_A \in A_e$ holds for all $e \in E$ and $A_e \cap A_f = \{1_A\}$ for any two distinct elements $e, f \in E$. We will also write A^* for $A - \{1_A\}$ and A^*_e for $A_e - \{1_A\}$, $(e \in E)$ so that $\#A^* = \#A - 1$ holds for all $e \in E$ and $A^*_e \cap A^*_f = \emptyset$ for any two distinct elements $e, f \in E$.

In other words, we will now assume that we are given

- a finite set A together with a specified element $1_A \in A$,
- a family $\mathcal{A} := (A_e)_{e\in E}$ of subsets A_e of A, parametrized by a finite index set E of cardinality at least 2, with $1_A \in A_e$ for all $e \in E$ and

  $$\mathbf{48} \qquad A^*_e \cap A^*_f = \emptyset \tag{48}$$

  for any two distinct elements $e, f \in E$,
- and a map

  $$\varphi : \prod \mathcal{A} \rightarrow A : \mathbf{a} = (a_e)_e \mapsto \varphi(\mathbf{a}) \tag{49}$$

  from $\prod \mathcal{A} = \prod_{e\in E} A_e$ into A such that, with

  $$\mathbf{1}_{\mathcal{A}} := (1_A)_{e\in E} \tag{50}$$

  — this element replacing from now on and in all subsequent discussions the element $\mathbf{a_1}$ considered in the previous section — we have

  $$\mathbf{51} \qquad \varphi(\mathbf{1}_{\mathcal{A}} \overset{e}{\leftarrow} a_e) = a_e \tag{51}$$

  for every element $e \in E$ and $a_e \in A_e$.

Note that (**??**) and (**??**) together imply that the induced map

$$\varphi_{\mathbf{1}_{\mathcal{A}}} : \mathcal{N}(\mathbf{1}_{\mathcal{A}}) \rightarrow A_{\varphi}(\mathbf{1}_{\mathcal{A}}) = \bigcup_{e \in E} A_e \tag{52}$$

is necessarily bijective — the inverse being given by mapping, for each $e \in E$, any element $a_e \in A_e$ onto the element $\mathbf{1}_{\mathcal{A}} \overset{e}{\leftarrow} a_e \in (\mathcal{N}_{\mathbf{1}_{\mathcal{A}}}) \subseteq \prod \mathcal{A}$. So, we can apply the results obtained in the previous section and see that there must exist a neighbourhood $\mathcal{O}_{\varphi(\mathbf{1}_{\mathcal{A}})}$ of $\mathbf{1}_{\mathcal{A}}$ in $\mathbb{R}[A|1]$ and a map $\eta_1 : \mathcal{O}_{\varphi(\mathbf{1}_{\mathcal{A}})} \rightarrow \mathbb{R}[\mathcal{A}|1]$ such that $\varphi^{\mathbb{R}|1}(\mathbf{p}) \in \mathcal{O}_{\varphi(\mathbf{1}_{\mathcal{A}})}$ and $\eta_1(\varphi^{\mathbb{R}|1}(\mathbf{p}) = \mathbf{p}$ holds for all points $\mathbf{p}$ in some sufficiently small neighbourhood of $\mathbf{1}_{\mathcal{A}}$ in $\mathbb{R}[\mathcal{A}|1]$.

2.1 Replacing $\mathbb{R}$ by an Arbitrary Commutative Ring R

To stress the algebraic nature of the arguments that are to follow now, we assume that we are given a commutative ring R whose group of units we denote by $R^{\times}$. As before, we denote the components $p^e \in R[A_e]$ of a point $\mathbf{p} = \oplus_e p^e \in R[\mathcal{A}]$ $(= \oplus_e R[A_e])$ by $\mathbf{p}^e$, and we recall that the elements $p^e(a_e) = \mathbf{p}^e(a_e)$ are defined, for any such point $\mathbf{p} \in R[\mathcal{A}]$ and all $a_e \in A_e$, to be the coefficients $p^e(a_e) \in R$ of the canonical basis elements $a_e \in R[A_e]$ of the free R-module $R[A_e]$ in the canonical expansion

$$\mathbf{p}^e = \sum_{a_e \in A_e} p^e(a_e)\, a_e \tag{53}$$

of $\mathbf{p}^e$. We also recall that the term $\mathbf{p}(\mathbf{a})$ is defined, for any $\mathbf{p} \in R[\mathcal{A}]$ and every element $\mathbf{a} \in \prod \mathcal{A}$, by

$$\mathbf{p}(\mathbf{a}) := \prod_e \mathbf{p}^e(\mathbf{a}^e), \tag{54}$$

that the point $\varphi^R \mathbf{p} \in R[A]$ is defined, for any $\mathbf{p} \in R[\mathcal{A}]$, by

$$\varphi^R \mathbf{p} := \sum_{\mathbf{a} \in \prod \mathcal{A}} \mathbf{p}(\mathbf{a})\, \varphi(\mathbf{a}) = \sum_{a \in A} \big(\sum_{\mathbf{a} \in \varphi^{-1}(a)} \mathbf{p}(\mathbf{a}) \big)\, a \tag{55}$$

or, equivalently, by requiring that

$$\varphi^R \mathbf{p}(a) = \sum_{\mathbf{a} \in \varphi^{-1}(a)} \mathbf{p}(\mathbf{a}) \tag{56}$$

holds for all $a \in A$, and that $\varphi^R \mathbf{p} \in R[A|1]$ holds for any point $\mathbf{p} \in R[\mathcal{A}|1]$.

We want to find means to compute explicitly and without recourse to any asymptotic methods — at least for appropriately specified maps $\varphi : \prod \mathcal{A} \to A$ and all points $\mathbf{p}$ in an appropriately specified (and, hopefully, not too small) subset of $R[\mathcal{A}|1]$ — the 'input' components $\mathbf{p}^e \in \mathbb{R}[A_e|1]$ $(e \in E)$ of $\mathbf{p}$ from its 'output signal' $\varphi^{R|1}\mathbf{p} \in R[A|1]$.

2.2 R-Valued Pairings

To this end, recall first that, given any two finite sets C and D and a *pairing*

$$\langle \bullet | \circ \rangle : C \times D \to R : (c, d) \mapsto \langle c|d \rangle \tag{57}$$

of C and D with values in R, 'bilinear extension' yields, for any two points $q_C \in R[C]$ and any $q_D \in R[D]$, the element

$$\langle q_C | q_D \rangle := \sum_{c \in C, d \in D} q_C(c) \langle c|d \rangle q_D(d) \in R. \tag{58}$$

In particular, one can form the term

$$\langle q_C | d \rangle = \sum_{c \in C} q_C(c) \langle c|d \rangle \in R \tag{59}$$

for every point $q_C \in R[C]$ and every element $d \in D$, and the term

$$\langle c | q_D \rangle = \sum_{d \in D} \langle c|d \rangle q_D(d) \in R \tag{60}$$

for every $q_D \in R[D]$ and $c \in C$.

Note that this is just another way of taking account of the basic fact that any R-valued $C \times D$-matrix (which is just 'the same' as a pairing of C and D with values in R) can be used to define an R-bilinear map from $R[C] \times R[D]$ into R as well as R-linear maps

$$R[C] \to D^R : \langle q_C | \circ \rangle : D \to R : d \mapsto \langle q_C | d \rangle$$

and

$$R[D] \to C^R : \langle \bullet | q_D \rangle : C \to R : c \mapsto \langle c | q_D \rangle.$$

2.3 The Pairing We'll Need

We will now proceed as follows: We will assume that we are given another finite set B and a pairing

$$\langle \bullet | \circ \rangle : A \times B \to R : (a, b) \mapsto \langle a | b \rangle \tag{61}$$

of A and B with values in R allowing us to form the terms

$$\langle P | b \rangle := \sum_{a \in A} P(a) \langle a | b \rangle \ (b \in B) \tag{62}$$

for any point $P \in R[A]$. Restricting the given pairing, for every $e \in E$, to a pairing

$$\langle \bullet | e | \circ \rangle : A_e \times B \to R : (a_e, b) \mapsto \langle a_e | e | b \rangle := \langle a_e | b \rangle \tag{63}$$

of A_e and B, we can also form the terms

$$\langle p^e | e | b \rangle := \sum_{a_e \in A_e} p^e(a_e) \langle a_e | e | b \rangle \tag{64}$$

for any point $p^e \in R[A_e]$.

2.4 The Induced Equivalence Relations $\sim_e$ on B

Next, we observe that the pairings $\langle \bullet | e | \circ \rangle : A_e \times B \to R$ allow us also to define, for each $e \in E$, an equivalence relation $\sim_e$ on B by defining two elements $b_1, b_2 \in B$ to be $\sim_e$-equivalent if and only if $\langle a_e \, | e | \, b_1 \rangle = \langle a_e \, | e | \, b_2 \rangle$ holds for all $a_e \in A_e$:

$$b_1 \sim_e b_2 \ \Leftrightarrow \ \langle a_e \, | e | \, b_1 \rangle = \langle a_e \, | e | \, b_2 \rangle \text{ for all } a_e \in A_e. \tag{65}$$

We denote the corresponding set of $\sim_e$-equivalence classes by B/e and the $\sim_e$-equivalence class of an element $b \in B$ by $e(b)$.

Note that $\langle p^e | e | b_1 \rangle = \langle p^e | e | b_2 \rangle$ holds for any $e \in E$, any point $p^e \in R[A_e]$, and any two $\sim_e$-equivalent elements $b_1, b_2 \in B$ allowing us to define $\langle p^e | e | \overline{b} \rangle$ for any $e \in E$, $\overline{b} \in B/e$, and $p^e : A_e \to R$ by

$$\langle p^e | e | e(b) \rangle := \langle p^e | e | b \rangle \tag{66}$$

for every $b \in B$ so that, given any subset B' of B, one has

$$\mathbf{(65)} \qquad \prod_{b' \in B'} \langle p^e \, | e | \, b' \rangle = \prod_{\overline{b} \in B/e} \langle p^e \, | e | \, \overline{b} \rangle^{\#(B' \cap \overline{b})} \tag{67}$$

for every $e \in E$.

2.5 Three Crucial Results

We now define a subset $\mathcal{O}_A = \mathcal{O}_{(R|A|\langle\bullet|\circ\rangle)}$ of $R[A|1]$ by

$$\mathcal{O}_A \quad := \quad \{P \in R[A|1] : \langle P|b\rangle \in R^{\times} \text{ for all } b \in B\} \tag{68}$$

and, for each $e \in E$, a similarly defined subset $\mathcal{O}_{A_e} = \mathcal{O}_{(R|A_e|\langle\bullet|e|\circ\rangle)}$ of $R[A_e|1]$ by

$$\mathcal{O}_{A_e} \quad := \quad \{p^e \in R[A_e|1] : \langle p^e|e|b\rangle \in R^{\times} \text{ for all } b \in B\}. \tag{69}$$

Further, we put

$$\mathcal{O}_{\mathcal{A}} = \mathcal{O}_{(R|\mathcal{A}|\langle\bullet|\circ\rangle)} \; := \{\mathbf{p} \in R[\mathcal{A}|1] : \mathbf{p}^e \in \mathcal{O}_{A_e} \text{ for all } e \in E\}.$$

The three crucial observations that will enable us to construct, under appropriate conditions still to be specified, a local inverse of the map

$$\varphi^{R|1} : R[\mathcal{A}|1] \rightarrow R[A|1] : \mathbf{p} \mapsto \varphi^R \mathbf{p} \tag{70}$$

then read as follows:

Lemma 1: *If*

$$\mathbf{73} \qquad \langle\varphi(\mathbf{a})|b\rangle = \prod_{e\in E} \langle \mathbf{a}^e|e|b\rangle \tag{71}$$

holds for every $\mathbf{a} \in \prod \mathcal{A}$ *and* $b \in B$*, then we have*

$$\langle\varphi^R \mathbf{p}\,|b\rangle = \prod_{e\in E} \langle \mathbf{p}^e|e|b\rangle \tag{72}$$

for all $\mathbf{p} \in R[\mathcal{A}]$ *and* $b \in B$ *and, therefore, also*

$$\mathbf{p} \in \mathcal{O}_{\mathcal{A}} \Leftrightarrow \varphi^R \mathbf{p} \in \mathcal{O}_A \tag{73}$$

for all $\mathbf{p} \in R[\mathcal{A}]$.

Proof: Indeed, (**??**) implies that

$$\begin{aligned}
\prod_{e\in E} \langle \mathbf{p}^e|e|b\rangle &= \prod_{e\in E}\sum_{a_e\in A_e} \mathbf{p}^e(a_e)\langle a_e|e|b\rangle \\
&= \sum_{\mathbf{a}\in\prod\mathcal{A}} \prod_{e\in E} \mathbf{p}^e(\mathbf{a}^e)\langle \mathbf{a}^e|e|b\rangle \\
&= \sum_{\mathbf{a}\in\prod\mathcal{A}} \prod_{e\in E} \mathbf{p}^e(\mathbf{a}^e)\prod_{e\in E}\langle \mathbf{a}^e|e|b\rangle \\
&= \sum_{\mathbf{a}\in\prod\mathcal{A}} \mathbf{p}(\mathbf{a})\ \langle\varphi(\mathbf{a})|b\rangle \\
&= \Big\langle \sum_{\mathbf{a}\in\prod\mathcal{A}} \varphi(\mathbf{a})\mathbf{p}(\mathbf{a})\,|b\Big\rangle = \langle \varphi^R\mathbf{p}\,|b\rangle
\end{aligned}$$

holds for all $\mathbf{p}\in R[\mathcal{A}]$ and $b\in B$, as claimed.

Lemma 2: *If there exists some element 1_B in B with*

$$(\mathbf{84}) \qquad \langle a|1_B\rangle = 1 \textit{ for all } a\in A, \tag{74}$$

one has

$$\langle \mathbf{p}^e|e|b\rangle = \langle \mathbf{p}^e|e|1_B\rangle = 1 \tag{75}$$

for all $e\in E$, $b\in e(1_B)$, and $\mathbf{p}\in R[\mathcal{A}|1]$.

Proof: This is obvious.

Lemma 3: *If*

$$\mathbf{76} \qquad \#\big(e(b_1)\cap f(b)\big) = \#\big(e(b_2)\cap f(b)\big) \tag{76}$$

holds for any two distinct elements $e, f\in E$, and all $b_1, b_2, b\in B$, then we have

$$(\mathbf{77}) \qquad \prod_{b\in e(b_1)} \langle p^f|f|b\rangle = \prod_{b\in e(b_2)} \langle p^f|f|b\rangle \tag{77}$$

for any two distinct elements $e, f\in E$, all $p\in R[A_f]$, and all $b_1, b_2\in B$.

Proof: Indeed, putting $B_1 := e(b_1)$ and $B_2 := e(b_2)$, (**??**) implies that $\#(B_1 \cap \overline{b}) = \#(B_2 \cap \overline{b})$ holds for all $f \in E - \{e\}$ and $\overline{b} \in B/f$ which in turn — in view (**??**) — implies that

$$\begin{aligned}\prod_{b \in e(b_1)} \langle p^f | f | b \rangle &= \prod_{b \in B_1} \langle p^f | f | b \rangle = \prod_{\overline{b} \in B/f} \langle p^f | f | \overline{b} \rangle^{\#(B_1 \cap \overline{b})} \\ &= \prod_{\overline{b} \in B/f} \langle p^f | f | \overline{b} \rangle^{\#(B_2 \cap \overline{b})} = \prod_{b \in B_2} \langle p^f | f | b \rangle = \prod_{b \in e(b_2)} \langle p^f | f | b \rangle\end{aligned}$$

holds indeed, as claimed, for any two distinct elements $e, f \in E$ and all $p^f \in R[A_f]$ and $b_1, b_2 \in B$.

2.5.1 Note that (**??**) together with our assumption $\#E \geq 2$ implies also that, choosing some $f \in E - \{e\}$ for every $e \in E$, we have

$$\#e(b_1) = \sum_{\overline{b} \in B/f} \#\big(e(b_1) \cap \overline{b}\big) = \sum_{\overline{b} \in B/f} \#\big(e(b_2) \cap \overline{b}\big) = \#e(b_2) \tag{78}$$

for all $b_1, b_2 \in B$ and $e \in E$, i.e., all $\sim_e$-equivalence classes have the same cardinality, henceforth denoted by $|e|$, for every fixed $e \in E$. Consequently

$$|e| \, \#(B/e) = \#B \tag{79}$$

holds for all $b \in B$ and $e \in E$, and

$$\#\big(e(b) \cap f(b')\big) = \frac{|f|}{\#(B/e)} = \frac{|e|}{\#(B/f)} \tag{80}$$

for any two distinct elements $e, f \in E$ and for all $b, b' \in B$.

Of course, this is not just by chance reminiscent of the situation one encounters when one considers two subgroups U, V of a finite group G with $UV = G$ and the associated two equivalence relations $\sim_U$ and $\sim_V$ whose equivalence classes are the left cosets of U and V in G, respectively.

2.6 A Fundamental Identity

Together, these observations imply

Theorem 2: *Assume that the three assertions* (**??**), (**??**), *and* (**??**) *hold for the given pairing* $\langle\bullet|\circ\rangle : A\times B\to R$ *and consider the map*

$$HC^{\bullet} : \mathcal{O}_A \to (R^{\times})^{E\times B} \tag{81}$$

that maps any P *in* $\mathcal{O}_A$ *onto the map*

$$(\mathbf{85}) \qquad HC^{\bullet}(P) : E\times B\to R^{\times} : (e,b)\mapsto \frac{\prod_{b'\in e(b)}\langle P|b'\rangle}{\prod_{b'\in e(1_B)}\langle P|b'\rangle}. \tag{82}$$

Then

$$\varphi^{R|1}\mathbf{p}\in\mathcal{O}_A \quad \textit{and} \quad HC^{\bullet}\big(\varphi^{R|1}\mathbf{p}\big)(e,b) = \langle \mathbf{p}^e|e|b\rangle^{|e|} \tag{83}$$

holds for all $\mathbf{p}\in\mathcal{O}_{\mathcal{A}}$, $e\in E$, *and* $b\in B$.

Proof: Indeed, Lemma 1 implies that

$$\begin{aligned}
HC^{\bullet}\big(\varphi^{R|1}\mathbf{p}\big)(e,b) &= \frac{\prod\limits_{b'\in e(b)}\langle \varphi^{R|1}\mathbf{p}\,|b'\rangle}{\prod\limits_{b'\in e(1_B)}\langle \varphi^{R|1}\mathbf{p}\,|b'\rangle} \\
&= \frac{\prod\limits_{b'\in e(b)}\big(\prod\limits_{f\in E}\langle \mathbf{p}^f|f|b'\rangle\big)}{\prod\limits_{b'\in e(1_B)}\big(\prod\limits_{f\in E}\langle \mathbf{p}^f|f|b'\rangle\big)} \\
&= \prod_{f\in E}\frac{\prod\limits_{b'\in e(b)}\langle \mathbf{p}^f|f|b'\rangle}{\prod\limits_{b'\in e(1_B)}\langle \mathbf{p}^f|f|b'\rangle} \\
&= \frac{\prod\limits_{b'\in e(b)}\langle \mathbf{p}^e|e|b'\rangle}{\prod\limits_{b'\in e(1_B)}\langle \mathbf{p}^e|e|b'\rangle}\prod_{f\neq e}\frac{\prod\limits_{b'\in e(b)}\langle \mathbf{p}^f|f|b'\rangle}{\prod\limits_{b'\in e(1_B)}\langle \mathbf{p}^f|f|b'\rangle}
\end{aligned}$$

holds for all $\mathbf{p}\in\mathcal{O}_{\mathcal{A}}$, $e\in E$, and $b\in B$. In consequence, we have

$$HC^{\bullet}\big(\varphi^{R|1}\mathbf{p}\big)(e,b) = \langle \mathbf{p}^e|e|b\rangle^{|e|}$$

as claimed because

(i) $\langle \mathbf{p}^e | e | b \rangle^{|e|} \in R^{\times}$ is obviously the value of the numerator of the first fraction in the term

$$\frac{\prod_{b' \in e(b)} \langle \mathbf{p}^e | e | b' \rangle}{\prod_{b' \in e(1_B)} \langle \mathbf{p}^e | e | b' \rangle} \prod_{f \neq e} \frac{\prod_{b' \in e(b)} \langle \mathbf{p}^f | f | b' \rangle}{\prod_{b' \in e(1_B)} \langle \mathbf{p}^f | f | b' \rangle}$$

by definition of $e(b)$,

(ii) $1 = 1_R$ is that of its denominator by Lemma 2,

(iii) and the numerator and the denominator — all contained in $R^{\times}$, of course — of each of the remaining factors

$$\frac{\prod_{b' \in e(b)} \langle \mathbf{p}^f | f | b' \rangle}{\prod_{b' \in e(1_B)} \langle \mathbf{p}^f | f | b' \rangle}$$

$(f \neq e)$ coincide for each of these factors in view of Lemma 3.

Remark: *Note that, by definition,*

$$(\mathbf{87}) \qquad HC^{\bullet}(P)(e, b) = 1 \tag{84}$$

*holds for all $P \in \mathcal{O}_A$, $e \in E$, and $b \in e(1_B)$ because the enumerator and the denominator in the definition (**??**) of $HC^{\bullet}(P)(e, b)$ obviously coincide in this case.*

$ 3 Constructing an Explicit Local Inverse

3.0 Review of What We Have Achieved so Far

We go on using the notations and the assumptions of the previous sections and note that, given any $\mathbf{p}$ in $R[\mathcal{A}]$, we so far found

(i) a map Ψ_1 from $R[A]$ into R^B that maps any point $P = P_1 \in R[A]$ onto the map

$$\Psi_1(P_1) : B \to R : b \mapsto \langle P_1 | b \rangle = \sum_{a \in A} P_1(a) \langle a | b \rangle$$

and, therefore, any point P_1 in $\mathcal{O}_A \subset R[A]$ onto a map in $(R^{\times})^B\,(\subset R^B)$ — a fact that holds in particular for any point $P_1 \in R[A]$ of the form $P_1 = \varphi^R \mathbf{p}$ for some point $\mathbf{p} \in \mathcal{O}_{\mathcal{A}}$ — while the product formula

$$\Psi_1(\varphi^R \mathbf{p})(b) = \prod_{e \in E} \langle \mathbf{p}^e | e | b \rangle$$

holds for every $\mathbf{p} \in R[\mathcal{A}]$ and $b \in B$,

(ii) and a map Ψ_2 from $(R^{\times})^B$ into $(R^{\times})^{E \times B}$ that maps any P_2 in $(R^{\times})^B$ onto the map

$$\Psi_2(P_2) : E \times B \to R^{\times} : (e, b) \mapsto \frac{\prod_{b' \in e(b)} \; P_2(b')}{\prod_{b' \in e(1_B)} P_2(b')}$$

for which

$$(\Psi_2 \circ \Psi_1)(\varphi^{R|1} \mathbf{p}))(e, b) = \Psi_2(\Psi_1(\varphi^{R|1} \mathbf{p}))(e, b) = \langle \mathbf{p}^e | e | b \rangle^{|e|}$$

holds for all $\mathbf{p} \in \mathcal{O}_{\mathcal{A}}\,(\subset R[\mathcal{A}|1])$ and all $(e, b) \in E \times B$.

3.1 Introducing Two Further Assumptions Lemma 3 This suggests to go on by assuming that

(i) a family $\theta^R = (\theta^R_N)_{N \in \mathbb{N}}$ of maps

$$\theta^R_N : R^{\times} \to R^{\times} \tag{85}$$

and a family $U^R = (U_N^R)_{N\in\mathbb{N}}$ of (hopefully sufficiently large) subsets U_N^R of $R^\times$ containing 1_R have been specified such that Lemma 3

$$\mathbf{89} \qquad r = \theta_N^R(r^N) \tag{86}$$

holds for every $N \in \mathbb{N}$ and every $r \in U_N^R$ and, therefore, in particular

$$\mathbf{90} \qquad \theta_N^R(1_R) = \theta_N^R(1_R^N) = 1_R \tag{87}$$

for all $N \in \mathbb{N}$,

(ii) there exists a pairing

$$[\bullet|e|\circ] : A_e \times B : (a_e, b) \mapsto [a_e|e|b] \tag{88}$$

for every $e \in E$ such that

$$(\mathbf{92}) \qquad \sum_{b\in B} \langle a'_e|e|b\rangle \ [a_e|e|b] = \delta_{a'_e,a_e} \tag{89}$$

holds for all $a_e, a'_e \in A_e$.

3.2 Two Further Maps Ψ_3 and Ψ_4

Using these assumptions and conventions, we can now define the map

$$\Psi_3 : \, (R^\times)^{E\times B} \rightarrow (R^\times)^{E\times B} \tag{90}$$

that maps every map $P_3 : E \times B \rightarrow R^\times : (e,b) \mapsto P_3(e,b)$ onto the map

$$\Psi_3(P_3) : E \times B \rightarrow R^\times : (e,b) \mapsto \theta_{|e|}(P_3(e,b)) \tag{91}$$

so that

$$\Psi_3(\Psi_2(\Psi_1(\varphi^{R|1}\mathbf{p})))(e,b) = \langle \mathbf{p}^e|e|b\rangle$$

holds for all $(e,b) \in E \times B$ and all $\mathbf{p}$ in the subset $\mathcal{U}_\mathcal{A} = \mathcal{U}_{(R|\mathcal{A}|\langle\bullet|\circ\rangle|U^R)}$ of $\mathcal{O}_\mathcal{A}$ defined by

$$(\mathbf{97}) \qquad \mathcal{U}_\mathcal{A} := \{\mathbf{p} \in \mathcal{O}_\mathcal{A} : \langle \mathbf{p}^e|e|b\rangle \in U_{|e|}^R \text{ for all } (e,b) \in E \times B\}. \tag{92}$$

And we can define the map

$$\Psi_4 : \, R^{E\times B} \rightarrow R[\mathcal{A}] \tag{93}$$

that maps every map $P_4 : E \times B \to R : (e,b) \mapsto P_4(e,b)$ onto the point $\Psi_4(P_4) = \oplus_e \Psi_4(P_4)^e \in R[\mathcal{A}]$ for which

$$\Psi_4(P_4)^e(a_e) = \sum_{b \in B} [a_e|e|b] P_4(e,b) \tag{94}$$

for all $e \in E$ and $a_e \in A_e$. Altogether, given all the assumptions we have made so far, this leads up to the following "This is the house that Jack built"-type formula

$$\begin{aligned}
\Psi_4(\Psi_3(\Psi_2(\Psi_1(\varphi^{R|1}\mathbf{p}))))^e(a_e) &= \sum_{b\in B} \Psi_3(\Psi_2(\Psi_1(\varphi^{R|1}\mathbf{p})))(e,b)[a_e|e|b] \\
&= \sum_{b\in B} \theta_{|e|}\Big(\frac{\prod_{b'\in e(b)}\langle\varphi^{R|1}\mathbf{p}|b'\rangle}{\prod_{b'\in e(1_B)}\langle\varphi^{R|1}\mathbf{p}|b'\rangle}\Big)[a_e|e|b] \\
&= \sum_{b\in B} \theta_{|e|}\big(\langle\mathbf{p}^e|e|b\rangle^{|e|}\big)[a_e|e|b] \\
&= \sum_{b\in B} \langle\mathbf{p}^e|e|b\rangle[a_e|e|b] \\
&= \sum_{b\in B} \Big(\sum_{a'_e\in A_e} \mathbf{p}^e(a'_e)\langle a'_e|e|b\rangle\Big)[a_e|e|b] \\
&= \sum_{a'_e\in A_e} \sum_{b\in B} \mathbf{p}^e(a'_e)\langle a'_e|e|b\rangle)[a_e|e|b] \\
&= \sum_{a'_e\in A_e} \mathbf{p}^e(a'_e) \sum_{b\in B}\langle a'_e|e|b\rangle[a_e|e|b] \\
&= \sum_{a'_e\in A_e} \mathbf{p}^e(a'_e)\delta_{a'_e,a_e} = \mathbf{p}^e(a_e) \qquad (95)
\end{aligned}$$

holding true for all e in E, a_e in A_e and $\mathbf{p}$ in $\mathcal{U}_{\mathcal{A}}$.

3.3 Which elements in $R^{E\times B}$ are mapped into $R[\mathcal{A}|1]$ by Ψ_4?

Our results above imply in particular that $\Psi_4(P_4) \in R[\mathcal{A}|1]$ must hold for all elements $P_4 \in R^{E\times B}$ that are of the form $P_4 = \Psi_3(\Psi_2(\Psi_1(\varphi^{R|1}\mathbf{p})))$ for some point $\mathbf{p} \in R[\mathcal{A}|1]$ suggesting to check for which other elements P_4 in $R^{E\times B}$ this might hold.

To this end, we first recall that $\Psi_2(P_2)(e,b)=1$ holds, according to (**??**), for all $P_2 \in (R^{\times})^B$ and for every pair $(e,b)\in E\times B$ with $b\in e(1_B)$ which, in view of (**??**), implies that also $\Psi_3(\Psi_2(P_2))(e,b)=1$ must hold for all $P_2\in (R^{\times})^B$ and every pair $(e,b)\in E\times B$ as above.

Thus,

$$\begin{aligned}\sum_{a_e\in A_e}\Psi_4(P_4)^e(a_e) &= \sum_{a_e\in A_e}\big(\sum_{b\in B}[a_e|e|b]P_4(e,b)\big) \\ &= \sum_{b\in B}\big(\sum_{a_e\in A_e}[a_e|e|b]\big)P_4(e,b) \\ &= |e|\sum_{a_e\in A_e}[a_e|e|b]+\sum_{b\in B-e(1_B)}\big(\sum_{a_e\in A_e}[a_e|e|b]\big)P_4(e,b)\end{aligned} \tag{96}$$

will coincide with 1 for all $e\in E$ provided $P_4(e,b)=1$ holds for every pair $(e,b)\in E\times B$ with $b\in e(1_B)$ and we have

$$(\mathbf{102})\qquad \sum_{a_e\in A_e}[a_e|e|b]=\frac{1}{|e|}\delta_{e(b),e(1_B)}=\begin{cases}\frac{1}{|e|} & \text{if } b\in e(1_B),\\ 0 & \text{else .}\end{cases} \tag{97}$$

So, if we want to be sure that $\Psi_4(P_4)\in R[\mathcal{A}|1]$ holds for all $P_4\in R^{E\times B}$ with $P_4(e,b)=1$ for all $(e,b)\in E\times B$ with $b\in e(1_B)$, we may just require that (**??**) holds for all $e\in E$. However, we may also note first that we can always replace, for every $e\in E$, the pairing $[\bullet|e|\circ]$ by the pairing

$$[\bullet\|e\|\circ]: A_e\times B\to R:(a_e,b)\mapsto\frac{1}{|e|}\sum_{b'\in e(b)}[a_e|e|b'] \tag{98}$$

without interfering with its required properties, i.e., we may assume that $[a_e|e|b_1]=[a_e|e|b_2]$ holds, for all $e\in E$ and $a_e\in A_e$, for any two $\sim_e$-equivalent elements $b_1,b_2\in B$, in which case simple standard matrix calculations show that

$$(\mathbf{104})\qquad \sum_{a_e\in A_e}[a_e|e|b_1]\langle a_e|e|b_2\rangle=\frac{1}{|e|}\delta_{e(b_1),e(b_2)} \tag{99}$$

must hold for all $e\in E$ and $b_1,b_2\in B$ in case $\#A_e=\#B/e$ — or, equivalently, $|e|\#A_e=\#B$ — and the identities (**??**) hold for all $e\in E$ which, putting $b_1:=b$ and $b_2:=1_B$, readily implies (**??**).

3.4 The Main Result

Altogether, this establishes the following fundamental identity:

Theorem 3: *Continuing with our notations and assumptions, we have*

$$HC(\varphi^{\mathbb{R}|1}\mathbf{p}) = \mathbf{p} \tag{100}$$

for every $\mathbf{p} \in \mathcal{U}_{\mathcal{A}}$ *for the map*

$$HC : \mathcal{O}_A \rightarrow R[\mathcal{A}] \tag{101}$$

that maps every point $P \in \mathcal{O}_A$ *onto the point*

$$HC(P) := \Psi_4(\Psi_3(\Psi_2(\Psi_1(P)))). \tag{102}$$

Moreover, HC *maps* $\mathcal{O}_A$ *into* $R[\mathcal{A}|1]$ *whenever* (**??**) *holds for all* $e \in E$.

3.5 The Case $R := \mathbb{R}$

Note that, for $R := \mathbb{R}$, one can always choose $\theta_N^{\mathbb{R}}(r)$ to be the positive N-th root of the absolute value $|r|$ of r, for any $r \in \mathbb{R}^{\times}$, and $U_N^{\mathbb{R}}$ to be the set $\mathbb{R}_{>0}$ of positive real numbers implying that, with this choice of the family $U^{\mathbb{R}} = (U_N^{\mathbb{R}})_{N\in\mathbb{N}}$, one has **(108)**

$$\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{(\mathbb{R}|\mathcal{A}|\langle\bullet|\circ\rangle|U^{\mathbb{R}})} = \{\mathbf{p} \in \mathbb{R}[\mathcal{A}|1] : \langle \mathbf{p}^e|e|b\rangle > 0 \text{ for all } b \in B\}, \tag{103}$$

a set that contains in particular all $\mathbf{p} \in \mathbb{R}[\mathcal{A}|1]$ for which **(104)**

$$\begin{aligned} \|\mathbf{p} - \mathbf{1}_{\mathcal{A}}\| \; &:= \; \max(\sum_{a_e\in A_e^*} |\mathbf{p}^e(a_e) - (\mathbf{1}_{\mathcal{A}})^e(a_e)| : e \in E) \\ &= \; \max(\sum_{a_e\in A_e^*} |\mathbf{p}^e(a_e)| : e \in E) \end{aligned} \tag{104}$$

is smaller than the inverse of

$$\max\big(1 + |\langle\bullet|\circ\rangle|\big) := \max\big(1 + |\langle a|b\rangle| : (a, b) \in A^* \times B\big) \tag{105}$$

in case we have

$$\mathbf{99} \qquad \langle 1_A|b\rangle = 1 \text{ for all } b \in B \tag{106}$$

as this implies that **107**

$$\begin{aligned}|1-\langle\mathbf{p}^e|e|b\rangle| &= |\sum_{a_e\in A_e}\mathbf{p}^e(a_e)-\sum_{a_e\in A_e}\mathbf{p}^e(a_e)\langle a_e|e|b\rangle| \\ &= |\sum_{a_e\in A_e^*}\mathbf{p}^e(a_e)(1-\langle a_e|e|b\rangle)| \\ &\leq \|\mathbf{p}-\mathbf{1}_{\mathcal{A}}\| \max\big(1+|\langle\bullet|\circ\rangle|\big)<1 \qquad (107)\end{aligned}$$

holds for all $e\in E$ and $b\in B$.

Note that this implies in particular that $\mathbf{p}\in\mathcal{U}_{\mathcal{A}}$ holds in case one has $\#A_e=2$ and $|\langle a_e|e|b\rangle|\leq 1$ for all $e\in E, a_e\in A_e$ and $b\in B$ for every $\mathbf{p}\in\mathbb{R}[\mathcal{A}|1]$ for which $|\mathbf{p}^e(a_e)|<\frac{1}{2}$ holds, for all $e\in E$, for the unique element a_e in A_e that is distinct from 1_A (cf. [?] for more specific instances where this condition shows up).

3.6 The Case $R:=\mathbb{C}$

For $R:=\mathbb{C}$, things are not that simple. One could, after writing any $r\in\mathbb{C}^\times$ in polar coordinates as $r=|r|e^{\alpha i}$ with, say, $-\pi<\alpha\leq\pi$, put

$$\theta_N^{\mathbb{C}}(r):=|r|^{1/N}e^{\frac{\alpha}{N}i} \qquad (108)$$

for every $r\in\mathbb{C}^\times$ so that, at least, $\theta_N^{\mathbb{C}}(r)^N=r$ holds actually for all $r\in\mathbb{C}^\times$, and then define $U_N^{\mathbb{C}}$ by

$$U_N^{\mathbb{C}}:=\{\rho e^{\alpha i}:\rho\in\mathbb{R}_{>0},-\frac{\pi}{N}<\alpha<\frac{\pi}{N}\}. \qquad (109)$$

However, forgetting about maps $\theta_N^{\mathbb{C}}$ that satisfy the identity $\theta_N^{\mathbb{C}}(r)^N=r$ which is anyway not needed — and isn't even true in case $R=\mathbb{R}$ — a slightly different approach will work for any complex Banach algebra:

3.7 The Case of Complex Banach Algebras

Let R be any complex Banach algebra R with Banach norm $\|\ldots\|$ and a unit element $1=1_R$ so that $\mathbb{C}$ can be viewed as a subalgebra of R. Noting that the standard power series $\sum\limits_{i=0}^{\infty}\frac{r^i}{i!}$ and $-\sum\limits_{i=1}^{\infty}\frac{(1-r)^i}{i}$ for the exponential map and the logarithm converge for all $r \in R$, or for all $r \in R$ with $\|1-r\| < 1$, respectively, one may put

$$\exp(r) := \sum_{i=0}^{\infty} \frac{r^i}{i!} \tag{110}$$

for every $r \in R$ and

$$\log r := -\sum_{i=1}^{\infty} \frac{(1-r)^i}{i} \tag{111}$$

for every $r \in R$ with $\|1-r\| < 1$ and define θ_N^R by, say,

$$\theta_N^R(r) := \begin{cases} \exp\left(\frac{1}{N}\log r\right) & \text{if } \|1-r\| < 1, \\ 1 & \text{if } \|1-r\| \geq 1 \end{cases} \tag{112}$$

and U_N^R by

$$U_N^R := \left\{r \in R : \|1-r\| < \frac{1}{1+N\,\ln 2}\right\} \tag{113}$$

as Condition (**??**) will then indeed be satisfied for all $r \in U_N^R$ (see the Appendix).

Note that $\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{(R|\mathcal{A}|\langle\bullet|\circ\rangle|U^R)}$ coincides, with these definitions of the family $U^R = (U_N^R)_{N\in\mathbb{N}}$, with the set **(118)**

$$\{\mathbf{p} \in R[\mathcal{A}] : \|1_R - \langle\mathbf{p}^e|e|b\rangle\| \leq \frac{1}{1+|e|\,\ln 2} \text{ for all } e \in E \text{ and } b \in B\} \tag{114}$$

and contains therefore, in case (**??**) holds, all $\mathbf{p} \in R[\mathcal{A}|1]$ for which

$$\|\mathbf{p} - \mathbf{1}_{\mathcal{A}}\| := \max\Big(\sum_{a_e \in A_e^*} \|\mathbf{p}^e(a_e) - (\mathbf{1}_{\mathcal{A}})^e(a_e)\| : e \in E\Big) \tag{115}$$

is smaller than the inverse of

$$(1 + \max(|e| : e \in E)\ln 2)\,\max(1 + \|\langle\bullet|\circ\rangle\|) \tag{116}$$

in case (**??**) holds as this (cf. (**??**)) implies that

$$\|1_R - \langle\mathbf{p}^e|e|b\rangle\| \leq \|\mathbf{p} - \mathbf{1}_{\mathcal{A}}\|\,\max\left(1 + |\langle\bullet|\circ\rangle|\right) < \frac{1}{1+|e|\ln 2}$$

holds for all $e \in E$ and $b \in B$.

$ 4 Finite Abelian Groups

4.0 The Basic Set Up

In this section, we want to show that the above machinery can be applied, in particular, whenever R is any complex Banach algebra (with a unit element 1_R that allows us to consider the field $\mathbb{C}$ as a subring of R), A is a finite abelian group with unit element 1_A, the subsets A_e $(e \in E)$ are distinct *non-overlapping* subgroups of A (i.e., they are subgroups such that $A_e \cap A_f = \{1_A\}$ or, equivalenty, $A_e^* \cap A_f^* = \emptyset$ holds for any two distinct elements $e, f \in E$), and the map $\varphi : \prod \mathcal{A} \rightarrow A$ is given by group multiplication, i.e., one has

$$\varphi(\mathbf{a}) = \prod_{e \in E} \mathbf{a}^e$$

for all $\mathbf{a} \in \prod \mathcal{A}$.

In addition, it can also be applied — and provides even slightly better results regarding the size of the neighbourhood where $\varphi^{R|1}$ can be inverted — in case A is a finite elementary abelian 2-group (and the A_e are as above) and R is the field $\mathbb{R}$ of real numbers which includes, by the way, the particularly important instance where A is an elementary 2-group, E is any subset of A^* (including in particular the case $E := A^*$), and A_e is the subgroup of order 2 consisting of just the element e and the unit element 1_A.

This is, of course, no great surprise because the assumptions we introduced above where exactly modeled so as to apply in this specific case.

4.1 Verification of the Axioms

Indeed, if $A, \mathcal{A}$, and φ are as specified in **4.0**,

- one chooses, of course, B to be the dual group

$$\mathrm{Hom}(A, \mathbb{C}^*) = \mathrm{Hom}(A, \{z \in \mathbb{C} : |z| = 1\}) \tag{117}$$

of the group A and the pairing $\langle \bullet | \circ \rangle : A \times B \rightarrow \mathbb{C}$ to be the canonical pairing

$$\langle \bullet | \circ \rangle : A \times B \rightarrow \mathbb{C} : (a, b) \mapsto \langle a | b \rangle := b(a), \tag{118}$$

- one notes that

$$\langle a|1_B\rangle = \langle 1_A|b\rangle = \|\langle a|b\rangle\| = 1 \tag{119}$$

  holds for all $a \in A$ and $b \in B$, and

$$\langle \varphi(\mathbf{a})|b\rangle = \langle \prod_{e\in E} \mathbf{a}_e|b\rangle = \prod_{e\in E} \langle \mathbf{a}_e|e|b\rangle \tag{120}$$

  holds for all $\mathbf{a} \in \prod \mathcal{A}$ and $b \in B$,

- one recalls that
  - $B_e := e(1_B)$ is a subgroup of B of order $|e|$, and the factor group B/B_e is canonically isomorphic to the dual group $\mathrm{Hom}(A_e, \mathbb{C}^*)$ of the subgroup A_e of A,
  - the equivalence classes $e(b)$ are exactly the cosets of that subgroup in B and one has $\#A = \#B = |e|\,\#(B/e) = |e|\,\#A_e$ for all $e \in E$,
  - our assumption that $A_e \cap A_f = \{1_A\}$ holds for any two distinct elements $e, f \in E$ implies that $B_e B_f = B$ holds for any two such elements e, f in E[2] and that, therefore,

$$\#(e(b) \cap f(b')) = (B : B_e \cap B_f)$$

    must hold for any two distinct elements $e, f \in E$ and all $b, b' \in B$ because, writing $b^{-1}b'$ in the form $b^{-1}b' = b_1 b_2^{-1}$ with $b_1 \in B_e$ and $b_2 \in B_f$, we have

$$e(b) \cap f(b') = (bB_e) \cap (b'B_f) = (bb_1B_e) \cap (b'b_2B_f) = (bb_1)(B_e \cap B_f),$$

  - that

$$\sum_{b\in B} \langle a'_e|e|b\rangle\ [a_e|e|b] = \delta_{a'_e,a_e} \tag{121}$$

    holds for all $a_e, a'_e \in A_e$ for the pairing

$$[\bullet|e|\circ] : A_e \times B : (a_e, b) \mapsto [a_e|e|b] := \frac{1}{\#B}\,\langle a_e^{-1}|e|b\rangle \tag{122}$$

---

[2] Indeed, given any $b \in \mathrm{Hom}(A, \mathbb{C}^*)$, one can always extend the restriction of b to A_f to a homomorphism of $A_e\,A_f$ that maps any element in A_e onto 1 and this homomorphism in turn to a homomorphism $b_1 \in \mathrm{Hom}(A, \mathbb{C}^*)$ which then must be an element of B_e while $b_2 := b_1^{-1}b$ maps any element in A_f onto 1 and must, thus, be an element of B_f allowing us to write b in the form $b = b_1b_2$ with $b_1 \in B_e$ and $b_2 \in B_f$.

because $\sum_{b\in B} \langle a_e|e|b\rangle \frac{1}{\#B} \langle a_e^{-1}|e|b\rangle = \frac{1}{\#B} \sum_{b\in B} \langle 1_A|e|b\rangle = 1$ holds for all $a_e \in A_E$, and one has

$$\sum_{b\in B} \langle a'_e|e|b\rangle \frac{1}{\#B} \langle a_e^{-1}|e|b\rangle = \frac{1}{\#B} \sum_{b\in B} \langle a'_e a_e^{-1}|e|b\rangle = 0$$

for all $a_e, a'_e \in A_e$ with $a := a'_e a_e^{-1}$ in A_e^* in view of the fact that $\sum_{b\in B} \langle a|e|b\rangle = \langle a|e|b_0\rangle \sum_{b\in B} \langle a|e|b\rangle$ holds for all $b_0 \in B$ and that there exists, for every $a \in A^*$, some $b_0 \in B$ with $\langle a|b_0\rangle \neq 1$,

- and that

  $$\sum_{a_e\in A_e} [a_e|e|b_1]\langle a_e|e|b_2\rangle = \frac{1}{|e|}\delta_{e(b_1),e(b_2)}$$

  must hold (cf. **3.4**) for every $e \in E$ in view of $\#A_e = \#B/e$ which identity, however, follows also directly from the fact that, arguing just as above,

  $$\sum_{a_e\in A_e} \langle a_e|e|b\rangle = \#A_e\delta_{e(b),e(1_B)}$$

  holds for all $b \in B$ and, therefore,

  $$\begin{aligned}\sum_{a_e\in A_e} [a_e|e|b_1]\langle a_e|e|b_2\rangle &= \sum_{a_e\in A_e} \frac{1}{\#B} \langle a_e^{-1}|e|b_1\rangle \langle a_e|e|b_2\rangle \\ &= \sum_{a_e\in A_e} \frac{1}{\#B} \langle a_e|e|b_1^{-1}\rangle \langle a_e|e|b_2\rangle \\ &= \sum_{a_e\in A_e} \frac{1}{\#B} \langle a_e|e|b_1^{-1}b_2\rangle \\ &= \frac{\#A_e}{\#B}\delta_{e(b_1),e(b_2)} = \frac{1}{|e|}\delta_{e(b_1),e(b_2)}.\end{aligned}$$

4.2 Consequences in Case R is a Complex Banach Algebra

Consequently, if R is any complex Banach algebra and we are given a point $\mathbf{p} \in R[\mathcal{A}|1]$ with $\|1 - \langle \mathbf{p}^e|e|b\rangle\| \leq \frac{1}{1+|e|\ln 2}$ for all $b \in B$ and $e \in E$, we have

$$\mathbf{p} \;=\; \Psi_4(\Psi_3(\Psi_2(\Psi_1(\varphi^{R|1}\mathbf{p}))))$$

and therefore **(127)**

$$\mathbf{p}^e(a_e) = \sum_{b\in B} \theta_{|e|}\Big(\frac{\prod_{b'\in e(b)}\langle P|b'\rangle}{\prod_{b'\in e(1_B)}\langle P|b'\rangle}\Big)[a_e|e|b] \tag{123}$$

for all $e \in E$ and $a_e \in A_e$, for the map $P := \varphi^{R|1}\mathbf{p}$. That is, the map

$$HC : \mathcal{O}_A = \mathcal{O}_{(R|A|\langle\bullet|\circ\rangle)} \to R[\mathcal{A}|1] : P \mapsto \Psi_4(\Psi_3(\Psi_2(\Psi_1(P)))) \tag{124}$$

provides a local left inverse of the map $\varphi^{R|1} : R[\mathcal{A}|1]\to R[A|1]$ in the neighbourhood $\mathcal{U}_{\mathcal{A}}$ of the 'origin' $\mathbf{1}_{\mathcal{A}}$ of $R[\mathcal{A}|1]$ that consists (cf. (**??**)) of all $\mathbf{p}$ in $R[\mathcal{A}|1]$ with $\|1 - \langle \mathbf{p}^e|e|b\rangle\| < \frac{1}{1+|e|\,\ln 2}$ for all $e \in E$ and $b \in B$ and hence contains, in particular, all $\mathbf{p} \in R[\mathcal{A}|1]$ with $\|\mathbf{p} - \mathbf{1}_{\mathcal{A}}\| < \frac{1}{2(1+|e|\ln 2)}$ for all $e \in E$.

So, this works in particular for $R := \mathbb{C}$, the field of complex numbers itself that, by definition, contains all character values $\langle a|b\rangle$, for all characters $b \in B$, of the elements $a \in A$.

Note also that, by restriction, HC also provides a local left inverse $HC\big|_{\mathbb{R}}$, defined on $\mathcal{O}_{(\mathbb{R}|A|\langle\bullet|\circ\rangle)} \subset \mathcal{O}_{(R|A|\langle\bullet|\circ\rangle)}$, of the restriction $\varphi^{\mathbb{R}|1} : \mathbb{R}[\mathcal{A}|1]\to\mathbb{R}[A|1]$ of $\varphi^{R|1}$ to $\mathbb{R}[\mathcal{A}|1] \subset R[\mathcal{A}|1]$ for which $HC(\varphi^{\mathbb{R}|1}\mathbf{p}) = \mathbf{p}$ holds for all $\mathbf{p}$ in $\mathcal{O}_{(\mathbb{R}|\mathcal{A}|\langle\bullet|\circ\rangle)}$ with $\|1 - \langle \mathbf{p}^e|e|b\rangle\| < \frac{1}{1+|e|\,\ln 2}$ for all $e \in E$ and $b \in B$.

However, this restriction to $\mathbb{R}$ (that provides a local left inverse of $\varphi^{\mathbb{R}|1}$) can apparently not be defined properly without recourse to the complex number field $\mathbb{C}$ unless $\langle a|b\rangle \in \mathbb{R}$ holds for all $a \in A$ and $b \in B$, i.e., unless A is an elementary abelian 2-group.

4.3 Consequences in Case A is a Finite Elementary Abelian 2-Group and R is the Field $\mathbb{R}$ of Real Numbers

However, in the particular case where A is a finite elementary abelian 2-group, the character values $\langle a|b\rangle$ are either $+1$ or -1 for all $a \in A$ and $b \in B$, so we have $\langle \mathbf{p}^e|e|b\rangle \in \mathbb{R}$ for all $\mathbf{p} \in \mathbb{R}[\mathcal{A}|1]$, and $\langle P|b\rangle \in \mathbb{R}$ for all $P \in \mathbb{R}[A|1]$, for every $b \in B$. Consequently, we may apply the results of the previous sections for $R := \mathbb{R}$ and, thus, find maps

$$\Psi_1 : \mathbb{R}[A]\to\mathbb{R}^B, \quad \Psi_2 : (\mathbb{R}^\times)^B\,(\subset \mathbb{R}^B)\to(\mathbb{R}^\times)^{E\times B},$$

$$\Psi_3 : \,(\mathbb{R}^\times)^{E\times B}\to\,(\mathbb{R}^\times)^{E\times B}, \quad \Psi_4 : \,(\mathbb{R}^\times)^{E\times B}\to\,\mathbb{R}[\mathcal{A}|1],$$

defined by

$$\Psi_1(P) : B \rightarrow \mathbb{R} : b \mapsto \sum_{a \in A} P(a)\langle a|b\rangle \quad (P \in \mathbb{R}[A]),$$

$$\Psi_2(P_2) : E \times B \rightarrow \mathbb{R} : (e,b) \mapsto \frac{\prod_{b' \in e(b)} P_2(b')}{\prod_{b' \in e(1_B)} P_2(b')} \quad (P_2 \in (\mathbb{R}^{\times})^B),$$

$$\Psi_3(P_3) : E \times B \rightarrow \mathbb{R} : (e,b) \mapsto |P_3(e,b)|^{1/|e|} \quad (P_3 \in (\mathbb{R}^{\times})^B),$$

and

$$\Psi_4(P_4) := \oplus_e \sum_{a_e \in A_e} \big(\sum_{b \in B}[a_e|e|b] P_4(e,b)\big) a_e \quad (P_4 \in (\mathbb{R}^{\times})^{E \times B})$$

such that

$$HC(\varphi^{\mathbb{R}|1}\mathbf{p}) = \mathbf{p}$$

holds for the composition

$$HC = \Psi_4 \circ \Psi_3 \circ \Psi_2 \circ \Psi_1 : \mathcal{O}_{(\mathbb{R}|A|\langle \bullet|\circ\rangle)} \rightarrow \mathbb{R}[\mathcal{A}|1].$$

of these maps for all $\mathbf{p}$ in the neighbourhood $\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{(\mathbb{R}|\mathcal{A}|\langle\bullet|\circ\rangle|U^{\mathbb{R}})}$ of $\mathbf{1}_{\mathcal{A}}$ in $\mathbb{R}[\mathcal{A}|1]$ defined by

$$\mathcal{U}_{\mathcal{A}} = \{\mathbf{p} \in \mathcal{O}_{(\mathbb{R}|A|\langle\bullet|\circ\rangle)} : \langle \mathbf{p}^e|e|b\rangle > 0 \text{ for all } (e,b) \in E \times B\}.$$

Note also that in case E is a subset of A^* and $A_e = \{1_A, e\}$ holds for every $e \in E$, we have

$$\langle \mathbf{p}^e|e|b\rangle = \mathbf{p}^e(1_A)\langle 1_A|e|b\rangle + \mathbf{p}^e(e)\langle e|e|b\rangle = 1 - \mathbf{p}^e(e)(1 - \langle e|e|b\rangle)$$

for all $\mathbf{p} \in \mathbb{R}[\mathcal{A}|1]$ and $b \in B$ and, therefore,

$$\min(\langle \mathbf{p}^e|e|b\rangle : b \in B) = 1 - 2\mathbf{p}^e(e)$$

which implies that $\langle \mathbf{p}^e|e|b\rangle > 0$ holds for all $b \in B$ for some $\mathbf{p} \in \mathbb{R}[\mathcal{A}|1]$ if and only if one has $\mathbf{p}^e(e) < \frac{1}{2}$ for all $e \in E$ (cf. also **3.5**). I.e., the neighbourhood $\mathcal{U}_{\mathcal{A}} = \mathcal{U}_{(\mathbb{R}|\mathcal{A}|\langle\bullet|\circ\rangle|U^{\mathbb{R}})}$ of $\mathbf{1}_{\mathcal{A}}$ in $\mathbb{R}[\mathcal{A}|1]$ coincides with the set **(129)**

$$\mathcal{U}_{\mathcal{A}} = \{\mathbf{p} \in \mathbb{R}[\mathcal{A}|1] : \mathbf{p}^e(e) < \frac{1}{2} \text{ for all } e \in E\} \tag{125}$$

in this particularly important special case.

4.4 Extending φ in Case A is a Finite Elementary Abelian 2-Group

Furthermore, we can always extend our setup as follows: We can replace

- E by the enlarged set $E' := E \cup (A - \cup_{e\in E} A_e)$,
- the family $\mathcal{A} = (A_e)_{e\in E}$ by its "extension" $\mathcal{A}' = (A_{e'})_{e'\in E'}$ defined by $A_{e'} := A_e$ in case $e' = e \in E$, and by $A_{e'} := \{1_A, e'\}$ in case $e' \in A - \cup_{e\in E} A_e$,
- and the map φ by its "extension" $\varphi' : \prod \mathcal{A}' \rightarrow A$ also defined by taking, for each element $\mathbf{a}' \in \prod \mathcal{A}'$, the product $\prod_{e'\in E'} (\mathbf{a}')^{e'}$ of its components $(\mathbf{a}')^{e'}$ $(e' \in E')$ as its image under φ'.

This allows us to construct a local left inverse

$$HC' : \mathcal{O}_A \rightarrow \mathbb{R}[\mathcal{A}'|1].$$

of ${\varphi'}^{\mathbb{R}}$ that "extends" HC in such a way that a map $P \in \mathbb{R}[A|1]$ is contained in the image $\varphi^{\mathbb{R}}(\mathcal{O}_{\mathcal{A}})$ of $\mathcal{O}_{\mathcal{A}}$ relative to the map $\varphi^{\mathbb{R}}$ if and only if

$$P = {\varphi'}^{\mathbb{R}}(HC'(P))$$

holds and the components $HC'(P)^{e'} \in \mathbb{R}[A_{e'}|1]$ of the image $HC'(P)$ of P in $R[\mathcal{A}'|1] = \oplus_{e'\in E'} \mathbb{R}[A_{e'}|1]$ coincide, for every $e' \in A - \cup_{e\in E} A_e$, with the 'origin' 1_A of the space $\mathbb{R}[A_{e'}|1]$.

$ 5 Hadamard Conjugation

5.0 The Basic Set Up in Phylogenetic Analysis

In phylogenetic analysis, this is applied as follows:

Assume that X is a finite set of cardinality at least 3 (representing a set of *taxa* that is to be investigated) and that an X-tree T has been given, i.e., a (graph-theoretical) tree $T = (V(T), E(T))$ with vertex set $V(T)$ and edge set $E(T)$ considered as a subset of the set $\binom{V(T)}{2}$ of all 2-subsets of $V(T)$, such that

(i) X coincides with its set of *leaves* and

(ii) the degree $\deg_T(v)$ of any vertex $v \in V(T)$ is distinct from 2.

Recall that

- $E(T)$ is (as any connected graph) uniquely determined by the induced metric
$$d_T : V_T \times V_T \to \mathbb{N}_0 := \{0, 1, 2, \dots\}$$
defined on the vertex set V_T of T by associating, to any two vertices $u, v \in V_T$, the number $d_T(u, v)$ of edges $e \in E$ on the (unique) shortest path from u to v in T, i.e., the (necessarily unique!) largest metric d defined on V_T with $d(u, v) = 1$ for any two elements $u, v \in V_T$ with $\{u, v\} \in E(T)$,

- one has $d_T(u, w) \in \{d_T(v, w) \pm 1\}$ for all u, v, w in V_T with $\{u, v\} \in E(T)$,

- every edge $e = \{u, v\} \in E(T)$ induces therefore a unique *split* (or *bipartition*)
$$S_e := \{S_e(u), S_e(v)\}$$
of X into the two non-empty disjoint subsets
$$S_e(u) := \{x \in X : d_T(x, u) < d_T(x, v)\}$$
and
$$S_e(v) := \{x \in X : d_T(x, v) < d_T(x, u)\}$$
of X,

- and it is, in view of our assumption that $\deg_T(v) \neq 2$ holds for every vertex v in an X-tree T, uniquely determined by that split $S_e = \{S_e(u), S_e(v)\}$ it induces, i.e., one has $S_e \neq S_f$ for any two distinct edges $e, f \in E(T)$.

Now, assume that a (small) positive real number p_e has been specified for each edge $e \in E(T)$ that represents the *probability* that, in a 2-state model of evolution, the states of the two (generally ancestral) taxa represented by the two nodes of e are distinct (presupposing, of course, that to talk about such a probability makes any sense — yet, if it doesn't, Hadamard conjugation might actually find out!). For every split of X, i.e., for every unordered pair $\{Y, Z\}$ of disjoint subsets Y, Z of X with $Y \cup Z = X$ (including the 'trivial' pair $\{X, \emptyset\}$), this allows us to define the resulting probability $P(\{Y, Z\})$ that, relative to that model, all the taxa in Y are in the same state, and all taxa in Z in the other.

Given these split probabilities $P(\{Y, Z\})$ for all splits $\{Y, Z\}$ of X, the problem that is solved by Hadamard conjugation is

(i) to decide which splits $\{Y, Z\}$ of X are 'T-splits', i.e., are contained in the set $\mathcal{S}(T) := \{S_e : e \in E(T)\}$, and

(ii) to compute the probabilities p_e for the edges e of T corresponding to those splits.

5.1 Relations with the Concepts Discussed in Section 1

This, however, is easily recognized to be just a particular instance of the type of problem discussed in the previous sections: Indeed, to apply the machinery introduced in Section **1**, we can proceed as follows:

(i) We may define $A = A(X)$ to be the set of all splits of X.

(ii) We may define the distinguished element $1_{A(X)}$ in $A(X)$, also denoted by 1_X, to be the trivial split $\{X, \emptyset\}$.

(iii) We may define, for each $e \in E(T)$, the set A_e to be the subset $\{1_X, S_e\}$ of $A(X)$ consisting of the trivial split $1_X = \{X, \emptyset\}$ and the split S_e, thus obtaining a family $\mathcal{A} = \mathcal{A}_T := (A_e)_{e \in E(T)}$ of sets with $A_e^* = \{S_e\}$ and, hence, $A_e^* \cap A_f^* = \emptyset$ for all $e, f \in E(T)$ with $e \neq f$ and, hence, $S_e \neq S_f$.

(iv) We may define the map $\varphi = \varphi_T : \prod \mathcal{A}_T \rightarrow A(X)$ by associating, to any element $\mathbf{a}$ in $\prod \mathcal{A}_T$, the split $\varphi_T(\mathbf{a})$ of X that results from restricting to X the split $\{U, W\}$ of $V(T)$ that one obtains by putting all vertices that are connected by an edge $e \in E(T)$ with $\mathbf{a}^e = 1_X$ into the same 'split half' (U or W) and those that are connected by an edge $e \in E(T)$ with $\mathbf{a}^e = S_e$ into distinct 'split halfs', a procedure that clearly defines a well-defined split of $V(T)$ and hence, by restriction, one of X (see Fig. 1 [Text for Fig. 1: Assume that, regarding the X-tree depicted above, one forms the element $\mathbf{a} := (a_e)_e \in \prod \mathcal{A}_T$ by defining $a_e := 1_X$ if the edge e is an unbroken line, and $a_e := S_e$ if e is a dotted line. Then, the associated split of $V(T)$ separates the 'black' vertices from the 'white' ones, and the split induced by restricting this split to $X = \{1, 2, \dots, 9\}$ is the split $\{\{1, 3, 5, 9\}, \{2, 4, 6, 7, 8\}\}$]).

(v) And we may define probability distributions p^e on the various sets $A_e = \{1_X, S_e\}$, $e \in E(T)$, by

$$p^e(S_e) := p_e \text{ and } p^e(1_X) := 1 - p_e$$

and identify those with the corresponding points

$$p^e := (1 - p_e)1_X + p_e S_e \in \mathbb{R}[A_e|1].$$

5.2 Verification of Axioms

To show that the objects defined in this way satisfy indeed the requirements needed for applying our machinery, note first that, for any element $\mathbf{a} \in \prod \mathcal{A}_T$, the split $\varphi_T(\mathbf{a})$ is the unique split of X that separates two element x, y in X if and only if there exists an odd number of splits that separate x and y in the set

$$\mathcal{S}(\mathbf{a}) := \{\mathbf{a}^e : e \in E(T)\}. \tag{126}$$

In particular, φ_T maps the distinguished element $\mathbf{1}_{\mathcal{A}_T} := \oplus_e 1_X$ in $\prod \mathcal{A}_T$ onto the trivial split 1_X, and it maps, for every $e \in E$, the element $\mathbf{1}_{\mathcal{A}_T} \overset{e}{\leftarrow} (S_e)$ in $\mathcal{N}_T := \mathcal{N}(\mathbf{1}_{\mathcal{A}_T})$ onto the split S_e induced by e.

Thus, the map from $\mathcal{N}_T$ into $A(X)$ induced by φ_T is injective in view of the fact (mentioned already in **5.1** (iii)) that any edge e in an X-tree T is uniquely determined by the split S_e it induces.

Let us now return, for any X-split $\{Y, Z\}$, to the probability $P(\{Y, Z\})$ of observing — relative to the 2-state model of evolution assumed above — the states of all taxa in Y to be distinct from the states of all taxa in Z. As above, we assume that, for any edge $e \in E(T)$, we are given a probability distribution $p^e : A_e \rightarrow \mathbb{R}$ that is defined on the set $A_e = \{1_X, S_e\}$ whose elements represent the two 'local' events '*coincidence*' and '*no coincidence*' of the states of the two taxa represented by the two nodes of e, and we identify p^e with the corresponding point

$$p^e := (1 - p_e)1_X + p_e S_e \in \mathbb{R}[A_e|1].$$

Assuming independence of the local events '*coincidence*' and '*no coincidence*' at the various edges e of T, we first note that the family $(p^e)_e$ of probability distributions — or, equivalently, the associated point $\mathbf{p} := \oplus_e p^e \in \mathbb{R}[\mathcal{A}_T|1]$ — induces a probability distribution

$$\overline{\mathbf{p}} : \prod \mathcal{A}_T \rightarrow \mathbb{R} : \mathbf{a} \mapsto \mathbf{p}(\mathbf{a}) := \prod_e p^e(\mathbf{a}^e)$$

that is defined on the product set $\prod \mathcal{A}_T$ representing the 'total state space' of our system comprising all possible events of '*coincidence*' or '*no coincidence*' at all the edges of T. Referring to this probability distribution on $\prod \mathcal{A}_T$, we clearly have

$$P(\{Y, Z\}) = \sum_{\mathbf{a} \in \varphi_T^{-1}(\{Y,Z\})} \overline{\mathbf{p}}(\mathbf{a}) = \sum_{\mathbf{a} \in \varphi_T^{-1}(\{Y,Z\})} \prod_e p^e(\mathbf{a}^e) \tag{127}$$

for the probability $P(\{Y, Z\})$ of 'observing' the split $\{Y, Z\}$ of X.

In other words, with these choices of the set $A = A(X)$, the element 1_X in $A(X)$, the family $\mathcal{A}_T = \mathcal{A}_T = (A_e)_{e \in E(T)}$ of subsets $(A_e)_{e \in E(T)}$, the map $\varphi_T : \prod \mathcal{A}_T \rightarrow A(X)$, and the point $\mathbf{p} = \oplus_e p^e \in \mathbb{R}[\mathcal{A}_T|1]$, the identity

$$P(\{Y, Z\}) =_T \sum_{\mathbf{a} \in \varphi_T^{-1}(\{Y,Z\})} \prod_e \mathbf{p}^e(\mathbf{a}^e) = \varphi_T^{\mathbb{R}|1} \mathbf{p}(\{Y, Z\})$$

implies that the point

$$P := \sum_{a \in A} P(a)\, a \in \mathbb{R}[A|1].$$

corresponding to the induced probability distribution P on $A = A(X)$, and also denoted by P, coincides with the image $\varphi_T^{\mathbb{R}|1}\mathbf{p}$ of the point $\mathbf{p} \in \mathbb{R}[\mathcal{A}_T|1]$ relative to the map

$$\varphi_T^{\mathbb{R}|1} : \mathbb{R}[\mathcal{A}_T|1] \rightarrow \mathbb{R}[A|1]$$

induced by the map φ_T.

Thus, the results of Section **1** imply that, given a probability distribution P defined on the set $A = A(X)$ of all splits of X that is sufficiently close to the 'origin' 1_X of $\mathbb{R}[A_X|1]$, one can — at least approximatively — decide whether P is induced by a family $(p^e)_{e \in E(T)}$ of probability distributions for the local events '*coincidence*' and '*no coincidence*' at the various edges e of the given X-tree T.

5.3 Extending φ_T for Checking the Appropriateness of the Model and Determing the Underlying Tree if Any

Moreover (and much more importantly), one does not even need to know the X-tree T for doing this:

(i) Put $E(X) := A(X)^* = A(X) - \{1_X\}$ and $A_S := \{S, 1_X\} \subset A(X)$ for every $S \in E(X)$, thus obtaining a family $\mathcal{A}_X = (A_S)_{S \in E(X)}$ of subsets of $A = A(X)$ with $1_X \in A_S$ for all $S \in E(X)$ and $A_S^* \cap A_{S'}^* = \emptyset$ for any two distinct splits $S, S' \in E(X)$.

(ii) Define, for every non-trivial X-split S, a probability distribution p^S on the set $A_S = \{1_X, S\}$ by putting $p^S := p^e$ in case S is the X-split S_e associated to an edge $e \in E(T)$, and by $p^S := 1_X$ (considered as an element of $\mathbb{R}[A_S|1]$) if no such edge exists.

(iii) Define $\varphi_X : \prod \mathcal{A}_X \rightarrow A(X)$ by letting $\varphi_X(\mathbf{a})$ denote, for any element $\mathbf{a} = (a_S)_S \in \prod \mathcal{A}_X$, the unique split in $A(X)$ that separates two elements x, y in X if and only if there exists an odd number of splits that separate x and y in the set $\mathcal{S}(\mathbf{a}) := \{\mathbf{a}^S : S \in E(X)\}$.

Clearly, φ_X can be viewed as an extension of φ_T relative to the canonical embedding $\prod \mathcal{A}_T \rightarrow \prod \mathcal{A}_X$ defined by associating, to each $\mathbf{a}$ in $\prod \mathcal{A}_T$, the element $(a_S)_S \in \prod \mathcal{A}_X$ for which a_S is distinct from 1_X if and only if S is of the form S_e for some edge $e \in E(T)$ with $\mathbf{a}^e = S_e$.

It is obvious that the map φ_X induces an injective map from $\mathcal{N}(\mathbf{1}_{\mathcal{A}_\mathbf{X}})$ into $A(X)$ because, given any single split S, the unique split in $A(X)$ that separates two elements x, y in X if and only if there exists an odd number of splits that separate x and y in the 1-element set $\{S\}$, is nothing but the split S itself.

Thus, we can invert numerically, in a neighbourhood of the origin, the resulting map $(\varphi_X)^{\mathbb{R}|1}$. Consequently, given a probability distribution P defined on $A(X)$ (and considered as an element of $\mathbb{R}[A(X)|1]$) that has been derived from (i) an X-tree T and (ii) probabilities p_e of change along the edges e of T in the way described above, computing its pre-image will result in a family $(p^S)_{S\in E(X)}$ of points $p^S \in \mathbb{R}[A_S|1]$ for which the value $p^S(S)$ will be distinct from 0 for some $S \in A(X)$ (if and) only if S is of the form $S = S_e$ for some edge e of T (with a positive probability p_e of change). Thus, we can recognize the splits that are associated with the relevant edges of the X-tree T from the probability distribution P in this case which, according to one of the most basic facts of phylogenetic combinatorics, is perfectly sufficient for recognizing T.

However, using the results of the previous three sections, one can do much better:

5.4 Making Use of the Group Structure of the Set of Splits

5.4.1 Note first that the set $A(X)$ can be identified with the factor group of the elementary abelian 2-group $\{\pm 1\}^X$ consisting of all maps from X into the group $\{\pm 1\}$ modulo its subgroup $\{\sigma_+, \sigma_-\}$ of order 2 consisting of the two constant maps σ_+ and σ_- from X into $\{\pm 1\}$, defined by $\sigma_+(x) := 1$ and $\sigma_-(x) := -1$ for all $x \in X$. Indeed, a canonical one-to-one correspondence between $\{\pm 1\}^X/\{\sigma_+, \sigma_-\}$ and $A(X)$ can be defined by associating to any coset $\{\sigma\sigma_+, \sigma\sigma_-\} \in \{\pm 1\}^X/\{\sigma_+, \sigma_-\}$ of a map $\sigma \in \{\pm 1\}^X$ the split $\{\sigma^{-1}(+1), \sigma^{-1}(-1)\}$.

5.4.2 Correspondingly, we can — and will — view the set $A(X)$ as an elementary abelian 2-group whose multiplication is defined, for any two splits

$\{Y_1, Z_1\}, \{Y_2, Z_2\}$ in $A(X)$, by

$$\begin{aligned} \{Y_1, Z_1\} \vartriangle \{Y_2, Z_2\} & := & \{Y_1 \vartriangle Y_2, Y_1 \vartriangle Z_2\} \\ & \big(= & \{Z_1 \vartriangle Y_2, Z_1 \vartriangle Z_2\} \\ & = & \{Y_1 \vartriangle Y_2, Z_1 \vartriangle Y_2\} \\ & = & \{Y_1 \vartriangle Z_2, Z_1 \vartriangle Z_2\}\big) \end{aligned}$$

where, for any two subsets Y_1 and Y_2 of X, $Y_1 \vartriangle Y_2$ denotes — as usual — their symmetric difference $(Y_1 - Y_2) \cup (Y_2 - Y_1)$. In a more symmetric fashion, one can also define the $\vartriangle$-product of any two splits $\{Y_1, Z_1\}, \{Y_2, Z_2\} \in A(X)$ by

$$\{Y_1, Z_1\} \vartriangle \{Y_2, Z_2\} = \{(Y_1 \cap Y_2) \cup (Z_1 \cap Z_2), (Y_1 \cap Z_2) \cup (Z_1 \cap Y_2)\}.$$

Note that the trivial split $\{X, \emptyset\}$, already denoted by 1_X, is apparently the neutral element in $A(X)$ relative to this canonical multiplication, and that A_S is a subgroup of $A(X)$ of order 2 for every $S \in E(X)$.

5.4.3 Furthermore, continuing with the notations introduced above, it is straight forward to verify that, given any element

$$\mathbf{a} \in \prod \mathcal{A}_X = \prod_{S \in E(X)} \{S, 1_X\},$$

its image $\varphi_X(\mathbf{a})$ as described in **5.3** (iii) coincides with the $\vartriangle$-product over its components $\mathbf{a}^S$ $(S \in E(X))$. This follows, for instance, quite easily by induction on the cardinality of the set $\mathcal{S}(\mathbf{a}) = \{\mathbf{a}^S : S \in E(X)\}$ or, maybe even easier and more directly, by computing the $\vartriangle$-product of any pre-images of the splits in this set in the group $\{\pm 1\}^X$, representing — to make computations particularly simple — the trivial split by the element σ_+.

5.4.4 Thus, denoting the dual group of $A(X)$ by $B(X)$ and not introducing any new notation for their canonical pairing nor, for any $S \in E(X)$, for the induced pairings

$$\langle \bullet | S | \circ \rangle : A_S \times B(X) \to \mathbb{C} \, (\subseteq R)$$

and

$$[\bullet | S | \circ] : A_S \times B(X) \to \mathbb{C} \, (\subseteq R)$$

nor for the induced equivalence relations $\sim_S$ on $B(X)$, and noting that the order of $A(X)$ as well as that of $B(X)$ coincides with the number

$$N(X) := 2^{\#X-1}$$

while the order $|S| = \#B(X)_S = \#S(1_{B(X)})$ of the subgroup

$$B(X)_S = S(1_{B(X)}) = \{b \in B(X) : b \sim_S 1_{B(X)}\} = \{b \in B(X) : b(S) = 1\}$$

of $B(X)$ coincides with $2^{\#X-2} = \frac{N(X)}{2}$ for all $S \in E(X)$, we can apply the results of the previous three sections as follows:

First, note that (i) $\langle \mathbf{p}^S|S|b\rangle = \mathbf{p}^S(1_X) + \langle S|b\rangle\mathbf{p}^S(S) = 1 + (\langle S|b\rangle - 1)\mathbf{p}^S(S)$ holds for all $S \in E(X)$ and $b \in B(X)$ and that, consequently, $\mathcal{O}_{\mathcal{A}_X}$ coincides with the set

$$\{\mathbf{p} \in \mathbb{R}[\mathcal{A}_X|1] : \mathbf{p}^S(S) \neq \frac{1}{2} \text{ for all } (S,b) \in E(X) \times B(X) \text{ with } b(S) = -1\},$$

and that (ii) $(\varphi_X)^{\mathbb{R}|1}\mathbf{p} \in \mathcal{O}_{A(X)}$ holds for all elements $\mathbf{p}$ in the subset $\mathcal{O}_{A(X)} = \mathcal{O}_{(\mathbb{R}|A(X)|\langle\bullet|\circ\rangle)}$ of $\mathbb{R}[A(X)|1]$ defined by

$$\mathcal{O}_{A(X)} = \big\{P \in \mathbb{R}[A(X)|1] : \sum_{S \in A(X)} P(S)\,\langle S|b\rangle \neq 0 \text{ for all } b \in B(X)\big\}.$$

And then, define maps

$$HC_X : \mathcal{O}_{A(X)} \to \mathbb{R}[\mathcal{A}_X|1] : P \mapsto HC_X(P)$$

according to the formulae in Section **4.3** so that $HC_X(P)^S(a_S)$ is defined for every $P \in \mathcal{O}_{A(X)}$, $S \in E(X)$, and $a_S \in A_S$ by

$$HC_X(P)^S(a_S) = \sum_{b \in B(X)} \theta_{\#S(1_{B(X)})}\big(\frac{\prod_{b' \in S(b)}\langle P|b'\rangle}{\prod_{b' \in S(1_{B(X)})}\langle P|b'\rangle}\big)[a_S|S|b], \tag{128}$$

that is, by **(129)**

$$HC_X(P)^S(S) = \frac{1}{N(X)} \sum_{b \in B(X)} \theta_{N(X)/2}\big(\frac{\prod_{b' \sim_S b}\langle P|b'\rangle}{\prod_{b' \sim_S 1_{B(X)}}\langle P|b'\rangle}\big)\, b(S) \tag{129}$$

and

$$HC_X(P)^S(1_X) = \frac{1}{N(X)} \sum_{b\in B(X)} \theta_{N(X)/2}\big(\frac{\prod_{b'\sim_S b}\langle P|b'\rangle}{\prod_{b'\sim_S 1_{B(X)}}\langle P|b'\rangle}\big) \tag{130}$$

(which — repeating the argument from **3.3** — implies that

$$\begin{aligned} & HC_X(P)^S(S) + HC_X(P)^S(1_X) \\ = \; & \frac{1}{N(X)} \sum_{b\in B(X)} \theta_{N(X)/2}\big(\frac{\prod_{b'\sim_S b}\langle P|b'\rangle}{\prod_{b'\sim_S 1_{B(X)}}\langle P|b'\rangle}\big)\,(b(S)+1) \\ = \; & \frac{2}{N(X)} \sum_{b\in B(X)_S} \theta_{N(X)/2}\big(\frac{\prod_{b'\sim_S b}\langle P|b'\rangle}{\prod_{b'\sim_S 1_{B(X)}}\langle P|b'\rangle}\big) \\ = \; & \frac{2}{N(X)} \sum_{b\in B(X)_S} 1 = 1 \end{aligned}$$

always holds) and conclude that — in view of our previous results — the identity

$$HC_X\big((\varphi_X)^{\mathbb{R}|1}\mathbf{p}\big) = \mathbf{p}$$

must hold for all elements $\mathbf{p}$ in the subset $\mathcal{U}_{\mathcal{A}_X}$ of $\mathcal{O}_{\mathcal{A}_X}$ that — taking into account that $\#A_S = 2$ holds for all $S \in E(X)$ — coincides with the set

$$\{\mathbf{p} \in \mathbb{R}[\mathcal{A}_X|1] : \mathbf{p}^S(S) < \frac{1}{2} \text{ for all } S \in E(X)\}.$$

5.4.5 Of course, the most important aspect of this procedure is (as indicated already above, see [**?**] for details and a thorough discussion) that it allows us, given a 'phylogenetic signal' P in terms of a probability distribution defined on the set $A(X)$, to check whether or not there exists an X-tree T such that, except for some 'noise', the signal P could have been 'induced' by that tree and, if so, to reconstruct this tree from this signal.

Indeed, all one needs to do is to check whether P is contained in the subset $\mathcal{O}_{A(X)}$ of $\mathbb{R}[A(X)|1]$ defined above and, if so, whether

(i) all negative numbers of the form $HC_X(P)^S(S)$, $S \in E(X)$, are 'negligibly small' (in case there exist any such negtive numbers at all),

(ii) and the splits $S \in E(X)$ for which $HC_X(P)^S(S)$ is positive and not negligibly small form a collection of splits of the form $\{S_e : e \in E(X)\}$ for some X-tree $T = (V(T), E(T))$.

Recall also that the latter is well known to be the case if and only if any two such splits $S_1 = \{Y_1, Z_1\}, S_2 = \{Y_2, Z_2\}$ are *compatible*, i.e., if and only if one of the four intersections $Y_1 \cap Y_2, Y_1 \cap Z_2, Z_1 \cap Y_2, Z_1 \cap Z_2$ is empty; so, this condition is easily checked.

5.4.6 Remarkably, there is a rather explicit way to describe the dual group $B(X)$ of $A(X)$ that is used extensively in [**?**] and can lead to some further insights when dealing with proper biological data (cf. [**?**]). Recall first that the elements in the dual group $\mathrm{Hom}\,(\{\pm 1\}^X, \{\pm 1\})$ of the group $\{\pm 1\}^X$ can be identified with the subsets Y of X by associating, to each subset Y, the homomorphism

$$\tau_Y : \{\pm 1\}^X \to \{\pm 1\} : \sigma \mapsto \tau_Y(\sigma) := \prod_{y \in Y} \sigma(y).$$

Note also that, in view of

$$\begin{aligned} \langle \sigma | \tau_{Y_1} \rangle \langle \sigma | \tau_{Y_2} \rangle &= \prod_{y_1 \in Y_1} \sigma(y_1) \prod_{y_2 \in Y_2} \sigma(y_2) \\ &= \prod_{y \in Y_1 \triangle Y_2} \sigma(y) \\ &= \langle \sigma | \tau_{Y_1 \triangle Y_2} \rangle \quad (Y_1, Y_2 \subseteq X,\, \sigma \in \{\pm 1\}^X), \end{aligned} \tag{131}$$

the product of any two subsets Y_1, Y_2 of X relative to the group structure resulting from identifying $\mathcal{P}(X)$ with the dual group $\mathrm{Hom}\,(\{\pm 1\}^X, \{\pm 1\})$ of $\{\pm 1\}^X$, is given by their symmetric difference $Y_1 \triangle Y_2$. [3]

Further, it follows from these facts that the dual group $\mathrm{Hom}\,(A(X), \{\pm 1\})$ of $A(X)$ can be identified with that subgroup of $\mathcal{P}(X)$ that consists of all subsets Y of X for which

$$\tau_Y(\sigma_-) = \prod_{y \in Y} \sigma_-(y) = (-1)^{\# Y} = 1$$

---

[3] This is, of course, well in accordance with the group structure defined on the set $A(X)$ of splits of X in **5.4.1** allowing us to identify $A(X)$ also with the factor group of $\mathcal{P}(X)$ (considered, as above, as a group relative to the symmetric difference) relative to its 2-element subgroup $\{X, \emptyset\}$ whose cosets $\{X \triangle Y, \emptyset \triangle Y\} = \{X - Y, Y\}$ are, of course, nothing but the splits of X

holds and, hence, with the subset $\mathcal{P}_{even}(X)$ of $\mathcal{P}(X)$ consisting of all subsets of X of even cardinality.

However, using the abstract approach outlined above, this explicit description of the dual group $B(X)$ of $A(X)$ will not be needed here. All one needs to know is the fact — detailed in any standard textbook on algebra — that there exists a dual group for any finite abelian group A and that the canonical pairing of A with its dual has all the properties that were required above. Note also that in case A is an elementary abelian 2-group, standard linar algebra (applied to A considered as finite-dimensional vectorspace over the field $\mathbf{F}_2$ of cardinality 2) is sufficient to supply all the results from duality theory of finite groups one needs in the given context.

5.5 Multistate Characters

5.5.1 As was demonstrated in [**?**], Hadamard conjugation can also be used to analyse more complex models of sequence evolution, like e.g. Kimura's 3-parameter model. To present this application within the framework developed so far, we continue to consider a finite set X and the set $A(X)$ of its splits, together with the subset $E(X) = A(X) - \{1_X\}$ of $A(X)$ consisting of all non-trivial splits of X. However, we replace the group $\{\pm 1\}^X$ of maps from X into the 2-element group $\{\pm 1\}$ by the group M^X of maps from X into an arbitrary (yet fixed) finite group M (with unit element 1_M) of order $\#M$, and apply the machinery developed in Section **4** to the factor group $A(X|M)$ of this group modulo the subgroup $C(X|M)$ of M^X consisting of all constant maps from X into M.

For every non-trivial split $S = \{Y, Z\} \in E(X)$, we define $A(S|M)$ to be the factor group of the subgroup M_S^X of M^X consisting of all maps from X into M that are constant on Y and constant on Z, modulo the group $C(X|M)$ of constant maps from X into M introduced above that is obviously contained in M_S^X.

Further, we consider this factor group — in the natural way — as a subgroup of $A(X|M)$ and thus identify, for every $S \in E(X)$, the unit element $1_{A(X|M)} := C(X|M)/C(X|M)$ of $A(X|M)$ with that of $A(S|M)$. Note that, with $A(S|M)^* := A(S|M) - \{1_{A(X|M)}\}$, one has $A(S|M)^* \cap A(S'|M)^* = \{\emptyset\}$ for any two distinct splits S, S' of S.

Note also that, for each split $S = \{Y, Z\} \in E(X)$, there are two natural isomorphisms τ_Y and τ_Z from $A(S|M)$ onto M, one mapping each coset $\sigma C(X|M) \in A(S|M)$ onto the (well-defined!) element $\tau_Y(\sigma C(X|M)) := \sigma(y)\sigma(z)^{-1}$ $(y \in Y, z \in Z)$ in M, and the other mapping $\sigma C(X|M)$ onto the inverse $\tau_Z(\sigma C(X|M)) := \sigma(z)\sigma(y)^{-1}$ of this element.

5.5.2 We may now consider the

- the finite abelian group $A(X|M)$ of order
$$N_M(X) := \#M^{\#X-1}$$
with its unit element $1_{A(X|M)}$,
- the family $\mathcal{A}_{(X|M)} := (A(S|M))_{S\in E(X)}$ of subgroups of $A(X|M)$, each of order $\#M$, and
- the map $\varphi_M : \prod \mathcal{A}_{(X|M)} \to A(X|M)$ defined by associating, as before, to any element $\mathbf{a}$ in $\prod \mathcal{A}_{(X|M)}$ the product (in $A(X|M)$) over all of its components $\mathbf{a}^S \in A(S|M) \leq A(X|M)$[4].

As above, the fact that $A(S|M)^* \cap A(S'|M)^* = \emptyset$ holds for any two distinct splits S, S' of X, implies that this map induces, as required, an injective map from $\mathcal{N}(\mathbf{1}_{\mathcal{A}_{(X|M)}}) \subseteq \prod \mathcal{A}_{(X|M)}$ into $A(X|M)$.

In consequence, denoting the dual group of $A(X|M)$ by $B(X|M)$ and, as above, not introducing any new notation for their canonical pairing nor, for any S in $E(X)$, for the induced pairings $\langle \bullet|S|\circ\rangle : A(S|M) \times B(X|M) \to \mathbb{C}\,(\subseteq R)$ and $[\bullet|S|\circ] : A(S|M) \times B(X|M) \to \mathbb{C}\,(\subseteq R)$ or the induced equivalence relations $\sim_S$ on $B(X|M)$, and noting that the order $\#B(X|M)_S = \#S(1_{B(X|M)})$ of the subgroup

$$B(X|M)_S = S(1_{B(X|M)}) = \{b \in B(X|M) : b \sim_S 1_{B(X|M)}\}$$

of $B(X|M)$ coincides with $\#M^{\#X-2} = \frac{N_M(X)}{\#M}$ for all $S \in E(X)$, we can apply the results of the previous sections to construct, for any commutative complex Banach algebra R, a left local inverse of the polynomial map

$$\varphi_M^R : R[\mathcal{A}_{(X|M)}|1] \to R[A(X,M)|1]$$

induced by φ_M at the 'origin' $\mathbf{1}_{\mathcal{A}_{(X|M)}}$ of $R[\mathcal{A}_{(X|M)}|1]$ as follows:

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[4] Here, we follow the convention of writing $U \leq G$ rather than $U \subseteq G$ in case G is a group and U is a subset of G that forms a subgroup of G.

First, one recalls that the subset $\mathcal{O}_{A(X|M)} = \mathcal{O}_{(R|A(X|M)|\langle\bullet|\circ\rangle)}$ of $R[A(X|M)|1]$ is defined by

$$\mathcal{O}_{A(X|M)} = \Big\{P \in R[A(X|M)|1] : \sum_{S\in A(X|M)} P(S)\,\langle S|b\rangle \in R^{\times} \text{ for all } b \in B(X|M)\Big\}$$

and that $(\varphi_X)^{R|1}\mathbf{p} \in \mathcal{O}_{A(X|M)}$ holds for all elements $\mathbf{p}$ in

$$\mathcal{O}_{\mathcal{A}_{(X|M)}} = \{\mathbf{p} \in \mathbb{R}[\mathcal{A}_{(X|M)}|1] : \langle \mathbf{p}^S|S|b\rangle \in R^{\times} \text{ for all } (S,b) \in E(X){\times}B(X|M)\}.$$

And then, one defines maps

$$HC_{(X|M)} : \mathcal{O}_{A(X|M)} {\rightarrow} \mathbb{R}[\mathcal{A}_{(X|M)}|1] : P \mapsto HC_{(X|M)}(P)$$

according to the formulae in Section **4.3** so that $HC_{(X|M)}(P)^S(a_S)$ is defined, for every $P \in \mathcal{O}_{A(X|M)}$, $S \in E(X|M)$, and $a_S \in A(S|M)$, by

$$\begin{aligned} HC_{(X|M)}(P)^S(a_S) &= \sum_{b\in B(X|M)} \theta_{\#S(1_{B(X|M)})}\Big(\frac{\prod_{b'\in S(b)}\langle P|b'\rangle}{\prod_{b'\in S(1_{B(X|M)})}\langle P|b'\rangle}\Big)\,[a_S|S|b] \\ &= \frac{1}{N_M(X)} \sum_{b\in B(X|M)} \theta_{\frac{N_M(X)}{\#M}}\Big(\frac{\prod_{b'\sim_S b}\langle P|b'\rangle}{\prod_{b'\sim_S 1_{B(X|M)}}\langle P|b'\rangle}\Big)\langle a_S^{-1}|S|b\rangle \end{aligned}$$

and conclude that — in view of our previous results — the identity

$$HC_{(X|M)}\big((\varphi_{(X|M)})^{\mathbb{R}|1}\mathbf{p}\big) = \mathbf{p}$$

must hold for all elements $\mathbf{p}$ in the subset $\mathcal{U}_{\mathcal{A}_{(X|M)}}$ of $\mathcal{O}_{\mathcal{A}_{(X|M)}}$ that consists of all $\mathbf{p} \in \mathbb{R}[\mathcal{A}_{(X|M)}|1]$ for which $\langle \mathbf{p}|S|b\rangle \in U^R_{N_M(X)/\#M}$ holds for all $S \in E(X)$ and $b \in B(X|M)$.

5.5.3 In the context of a Kimura-type multistate model, this is applied as follows. First, we assume that we are given a set K on which the group M acts freely and transitively, i.e., for which a map

$$M \times K \rightarrow K : (m,k) \mapsto m\,k$$

is defined such that $1_M k = k$ and $(mm')\,k = m\,(m'\,k)$ holds for all $k \in K$ and $m, m' \in M$, and there exists, for any two elements $k, k' \in K$, a unique element $m = m(k,k') \in M$ with $m(k,k')\,k = k'$.

Next, we observe that this allows us to associate, to any map $\kappa : X \rightarrow K$, a unique element $a_\kappa \in A(X|M)$ that is defined as follows: One chooses an arbitrary, yet fixed element $x_0 \in X$, associates to any further element $x \in X$ the element $m(\kappa(x_0), \kappa(x))$, thus defining a map

$$\sigma_{x_0} : X \rightarrow M : x \mapsto m(\kappa(x_0), \kappa(x)),$$

and then notes that the coset $\sigma_{x_0} C_{(X|M)}$ in $A(X|M)$ defined by σ_{x_0} does only depend on κ, but not on the element $x_0 \in X$. So, we can denote it by a_κ, this way defining a canonical map $\kappa \mapsto a_\kappa$ from K^X into $A(X|M)$.

Consequently, given a family $(\kappa_i)_{i=1,\ldots,n}$ of n such maps (or, equivalently, an X-indexed family s of K-valued sequences $(s_x)_{x\in X}$ of lenght n), we can associate to it an element P_s in $\mathbb{R}[A_{(X|M)}|1]$ that associates, to every element $a \in A(X|M)$, the number of elements $i \in \{1, \ldots, n\}$ with $a = a_{\kappa_i}$, divided by n.

Let us now assume that those sequences evolved along a phylogenetic X-tree T from some primeordial sequence associated with the root of that tree according to a Kimura-type multistate model. That is, let us assume that, for each edge $e = \{u, v\}$ in that tree, a probability distribution p^e defined on $A(S_e|M)$ can be specified such that, for every coset $\sigma C(X|M) \in A(S_e|M)$, the number $p^e(\sigma C(X|M))$ gives the probability of finding the (again generally ancestral) taxa at u and v in states k_u and k_v ($k_u, k_v \in K$), respectively, such that $\sigma(y)(k_u) = \sigma(x)k_v$ holds for some (and, hence, for all) $x \in S_e(u)$ and $y \in S_e(v)$[5].

Assuming further that those probability distributions p^e are all strongly biased towards the '*no change*' option, i.e., that $p^e(1_{A(X|M)})$ is always close to 1, we can — in principle — reconstruct that tree (in terms of its splits) as well as the probability distributions p^e for all $e \in E(T)$ by (i) observing that $P_s \in \mathcal{O}_{A(X|M)}$ must hold in this case and that (ii) the component $HC_{(X|M)}(P_s)^S \in R[A(S|M)|1]$ of the image $HC_{(X|M)}(P_s)$ of P_s must be very close to the 'origin' $1_{A(X|M)}$ of $R[A(S|M)|1]$ for every split $S = \{Y, Z\}$ in $E(X)$ that is not a T-split and coincide — up to some hopefully negligible noise — with map $p^e : A(S_e|M) \rightarrow \mathbb{R}$ in case $S = S_e$ for some $e \in E(T)$.

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[5] Note that, given the two states $k_u, k_v \in K$, there is exactly one coset $\sigma C(X|M)$ in $A(S_e|M)$ with $\sigma(y)(k_u) = \sigma(x)k_v$ for some (or all) $x \in S_e(u)$ and $y \in S_e(v)$, viz., the coset that contains the map $\sigma \in M_S^X$ defined by $\sigma(x) := 1_M$ for all $x \in S_e(u)$ and $\sigma(y) := m(k_u, k_v)$ for all $y \in S_e(v)$.

Furthermore, as an additional check, we can compute the element

$$(\varphi_X)^{R|1} HC_{(X|M)}(P_s)$$

to see whether P is actually in the image $(\varphi_X)^{R|1}(\mathcal{O}_{\mathcal{A}_{(X|M)}})$ of the neighbourhood $\mathcal{O}_{\mathcal{A}_{(X|M)}}$ of the 'origin' $\mathbf{1}_{\mathcal{A}_{(X|M)}}$ of the space $R[\mathcal{A}_{(S|M)}|1]$ relative to the map $(\varphi_X)^{R|1}$.

Discussion

Still to be written !

Points to be discussed:

Generalizations?

(i) Replacing the subgroups A_e by arbitrary subsets A_e of A — of course, with $A_e \cap A_f = \{1_A\}$ for any two distinct elements $e, f \in E$, in particular, $E := A^*$ and $A_e := \{e, 1_A\}$ for all $e \in E = A^*$ in case A is not an elementary abelian 2-group.

(ii) All this for non-abelian groups

Galois-theoretic Aspects?

"In summary, our approach shows that, in case the above conditions all are fulfilled, one can explicitely compute the maps p^e from the map $p^\varphi = \varphi^{\mathbb{R}}(\oplus_e p^e)$ at least for families of maps $(p^e)_e$ whose components p^e are in the vicinity of the origin $O(A_e, 1_e)$ of the space $L(A_e, 1_e)$ for all $e \in E$ — just as expected in view of the more general results of the previous section that also did not allow us to construct a 'global' inverse of the map $\varphi^{\mathbb{R}}$, but were strictly local. And using this excplicit inverse, one can also check whether a given "signal" $P \in L(A, 1_A)$ is in the image of a neighbourhood of the origin of the space $L(\mathcal{A})$ relative to the map $\varphi^{\mathbb{R}}$.

It would be of interest from the point of view of (computational) algebraic geometry to classify all cases where a local inverse of the map $\varphi^{\mathbb{R}}$ can be defined explicitly, e.g. — in the spirit of Galois theory — by taking roots, only (in additon to sums and products), or to identify at least further cases where such an inverse can be constructed.”

More precisely, one should consider the ring

$$R_A := R[P_a : a \in A^*]$$

of polynomials in the variables $P_a,\ a \in A^*$, — or, perhaps a bit better, its extension

$$R_A^{\bullet} := R[P_a^{\pm 1} : a \in A^*]$$

of polynomials in the variables $P_a,\ a \in A^*$ and their inverses $P_a^{-1},\ a \in A^*$ — and the extension $R_{\mathcal{A}}$ one gets by taking the polynomial ring over R_A or $R_A^{\bullet}$ in the variables $p_{a_e}^e$ $, e \in E, a_e \in A_e$, modulo the ideal generated by all terms of the form

$$\sum_{a_e \in A_e} p_{a_e}^e - 1,\ e \in E,$$

and

$$P_a - \sum_{(a_e)_e \in \varphi_{-1}(a)} \prod_e p_{a_e}^e,\ a \in A^*,$$

and try to find out when this is — or can be embedded into — a Galois extension of R_A or $R_A^{\bullet}$, and determine the structure of the pertinent Galois group in this case.

Appendix

A Simple Example

To illustrate the definiton of the map $\varphi^{\mathbb{R}}$, consider the following example: Put $A := \{0,1,2,3\}, 1_{A(X)} := 0$, $E := \{1,2,3\}$, and $A_e := \{0,1\}$ as well as $1_e := 0$ for every $e \in E$ so that

$$\sum \mathcal{A} = \{(0,0,0),(1,0,0)(0,1,0)(0,0,1)\}$$

holds for the resulting family $\mathcal{A} := (A_e, 0)_{e \in E}$ of pointed sets. Further, define the map $\varphi : \prod_e A_e \to A$ by $\varphi((a_e)_e) := \max(ea_e : e \in E)$. Then, given any element $p = \oplus_e p^e \in L(\mathcal{A}) := \oplus_e L(A_e, 0)$, we may put $x_e := p^e(1)$ for every $e \in E$ (so that $p^e(0) = 1 - x_e$ must hold) and view the resulting triple (x_1, x_2, x_3) as the 'coordinates' of the point $\oplus_e p^e$ in $\oplus_e L(A_e, 0)$ in which coordinates the map $\varphi^{\mathbb{R}}$ can be described as mappping this point onto the sequence of the four polynomials

$$\begin{aligned}
&\varphi_0^{\mathbb{R}}(p) = (1 - x_1)(1 - x_2)(1 - x_3),\\
&\varphi_1^{\mathbb{R}}(p) = x_1(1 - x_2)(1 - x_3),\\
&\varphi_2^{\mathbb{R}}(p) = (1 - x_1)x_2(1 - x_3) + x_1x_2(1 - x_3),\\
&\varphi_3^{\mathbb{R}}(p) = (1 - x_1)(1 - x_2)x_3 + x_1(1 - x_2)x_3 + (1 - x_1)x_2x_3 + x_1x_2x_3.
\end{aligned}$$

Note that the sum of these polynomials is indeed equal to 1, and that the partial derivatives of all but the first of these polynomials along the three coordinate axes of $\oplus_e L(A_e, 0)$ at $(0, 0, 0)$ are

$$\begin{gathered}
(1, 0, 0)\\
(0, 1, 0)\\
(0, 0, 1)
\end{gathered}$$

in correspondence with the fact that φ induces a bijection between $\sum \mathcal{A} = \{(0, 0, 0)k_u, k_v, (1, 0, 0)(0, 1, 0)(0, 0, 1)\}$ and A.

Logarithms and Exponentials for Commutative Banach Algebras

A.0 Basic Definitions

We consider a commutative Banach algebra R with a unit element 1_R. For every $\rho \in \mathbb{R}_{>0}$ and $r_0 \in R$, let $U_{r_0}(\rho)$ denote the ball of radius ρ^{-1} around the point r_0, i.e., the set

$$U_{r_0}(\rho) := \big\{r \in R : \|r - r_0\| < \frac{1}{\rho}\big\}$$

(so that $\rho_1 \leq \rho_2 \Rightarrow U_{r_0}(\rho_2) \subseteq U_{r_0}(\rho_1)$ holds for all $r_0 \in R$ and $\rho_1, \rho_2 \in \mathbb{R}_{>0}$).

Clearly, the maps 'exp' and 'log' can be defined as usual — on the domains R and $U_1(1)$ of convergence of their standard power-series expansion — by

$$\exp : R \to R : r \mapsto \sum_{i=0}^{\infty} \frac{r^i}{i!}$$

and

$$\log : U_1(1) \to R : r \mapsto \sum_{i=1}^{\infty} \frac{(1-r)^i}{i},$$

respectively.

A.1 Estimating $\|\log r\|$ and $\|1-\exp r\|$ in terms of $\|1-r\|$ and $\|r\|$

Clearly, these definions imply that

$$\|1-\exp r\| \leq \frac{\|r\|}{1-\|r\|} = \frac{1}{\|r\|^{-1}-1}$$

holds for all $r \in U_0(1)$ and

$$\|\log r\| \leq \frac{\|1-r\|}{1-\|1-r\|} = \frac{1}{\|1-r\|^{-1}-1}$$

for all $r \in U_1(1)$, while equality $\|1-\exp r\| = \frac{\|r\|}{1-\|r\|}$ only holds in case $r = 0\,(=0_R)$, and $\|\log r\| = \frac{\|1-r\|}{1-\|1-r\|}$ only in case $r = 1$. In particular, given any positive constant $\rho > 0$, one has

$$r \in U_0(1+\rho) \Rightarrow \exp r \in U_1(\rho)$$

and

$$r \in U_1(1+\rho) \Rightarrow \log r \in U_0(\rho).$$

More specifically, one has

$$\|1-\exp r\| < 1$$

for all $r \in U_0(\frac{1}{\ln 2})$ in view of

$$\begin{aligned}
\|1-\exp r\| &= \|\sum_{i=1}^{\infty} \frac{r^i}{i!}\| \leq \sum_{i=1}^{\infty} \|\frac{r^i}{i!}\| \\
&\leq \sum_{i=1}^{\infty} \frac{\|r\|^i}{i!} = \sum_{i=0}^{\infty} \frac{\|r\|^i}{i!} - 1 \\
&= \exp(\|r\|) - 1 < \exp(\ln 2) - 1 = 2 - 1 = 1.
\end{aligned}$$

A.2 Basic Identities

Further, identities for the formal power series of log and exp imply that, as maps defined in terms of absolutely converging power series, they have the following properties:

(i) One has

$$\exp(r_1 + r_2) = \exp r_1 \exp r_2$$

for all $r_1, r_2 \in R$ and, hence,

$$\exp(Nr) = \exp(r)^N$$

for all $r \in R$ and $N \in \mathbb{N}$.

(ii) One has

$$r \in U_1(1) \Rightarrow \exp\big(\log r\big) = r$$

and

$$r \in U_0\big(\frac{1}{\ln 2}\big) \Rightarrow \exp r \in U_1(1) \Rightarrow \log\big(\exp r\big) = r.$$

(iii) In consequence, one has

$$r \in U_1\big(1 + \frac{N}{\ln 2}\big) \Rightarrow r^N \in U_1(1) \text{ and } \log(r^N) = N \log r \qquad (132)$$

for every $N \in \mathbb{N}$:

Indeed, $r \in U_1\big(1 + \frac{N}{\ln 2}\big)$ implies $\exp(\log r) = r$ as well as $\log(r) \in U_0\big(\frac{N}{\ln 2}\big)$ and, therefore, $N\log(r) \in U_0\big(\frac{1}{\ln 2}\big)$ which, in turn, implies

$$r^N = \big(\exp(\log r)\big)^N = \exp(N \log r) \in U_1(1)$$

and, therefore,

$$N \log r = \log(\exp(N \log r)) = \log(r^N).$$

(iv) Thus, defining a map θ_N^R for every $N \in \mathbb{N}$ by

$$\theta_N^R : U_1(1) \rightarrow R : r \mapsto \exp(\frac{1}{N} \log r),$$

we have

$$\theta_N^R(r^N) = \exp(\frac{1}{N}\log(r^N)) = \exp(\frac{1}{N}N\log(r) = \exp(\log r) = r$$

for all $r \in U_1\big(1 + \frac{N}{\ln 2}\big)$.

(v) More generally, we have $\prod\limits_{i=1}^{N} r_i \in U_1(1)$ and

$$\sum_{i=1}^{N} \log r_i = \log \prod_{i=1}^{N} r_i$$

for all $N \in \mathbb{N}$ and $r_1, \ldots, r_N \in U_1(1 + \frac{N}{ln2})$ as this implies $\log r_i < U_0(\frac{N}{ln2})$ for all $i = 1, \ldots, N$, hence, $r := \log r_1 + \cdots + \log r_N \in U_0(\frac{1}{ln2})$ and therefore

$$\prod_{i=1}^{N} r_i = \prod_{i=1}^{N} \exp(\log r_i) = \exp(\sum_{i=1}^{N} \log r_i) = \exp r \in U_1(1)$$

as well as

$$\log r_1 + \cdots + \log r_N = r = \log(\exp r) = \log(\prod_{i=1}^{N} r_i),$$

as claimed.

A.3 More General Exponential Terms

One can use this set up also to define exponential terms of the form r^z for all $r \in U_1(1)$ and $z \in R$ by

$$r^z := \exp(z \log r)$$

(vi) Obviously, one has

$$\|1 - r^z\| = \|1 - \exp(z \log r)\| \leq \frac{1}{\|z \log r\|^{-1} - 1} \leq \frac{\|1 - r\|\|z\|}{1 - \|1 - r\|(1 + \|z\|)}$$

for all $r, z \in R$ with $\|1-r\|(\|z\|+1) < 1$. In particular, given any $z \in R$ and any positive constant $\rho > 0$, we have

$$r \in U_1(1+(1+\rho)\|z\|) \Rightarrow r^z \in U_1(\rho).$$

(vii) Further, one has

$$\begin{aligned} r^{z_1+\cdots+z_N} &= \exp((z_1+\cdots+z_N)\log r) \\ &= \exp(z_1 \log r + \cdots + z_N \log r) \\ &= \exp(z_1 \log r) \ldots \exp(z_N \log r) \\ &= r^{z_1} \ldots r^{z_N} \end{aligned}$$

for all $r \in U_1(1)$ and $z_1, \ldots, z_N \in R$.

(viii) And one has

$$\begin{aligned} r^{(z_1 z_2)} &= \exp((z_1 z_2)\log r)) \\ &= \exp(z_2(z_1 \log r)) \\ &= \exp\big(z_2 \log\, \exp(z_1 \log r)\big) \\ &= \exp\big(z_2 \log\, r^{z_1}) \\ &= (r^{z_1})^{z_2} \end{aligned}$$

for all $z_1, z_2 \in R$ and $r \in U_1(1)$ with $z_1 \log r \in U_0(\frac{1}{\ln 2})$ and, hence, $\exp(z_1 \log r) \in U_1(1)$ and $z_1 \log r = \log\big(\exp(z_1 \log r)\big)$.

In particular, we have

$$r \in U_1(1+2\|z_1\|) \Rightarrow r^{(z_1 z_2)} = (r^{z_1})^{z_2}$$

for all z_1, z_2 in R as $r \in U_1(1+\frac{\|z_1\|}{\ln 2})$ implies $\log r \in U_0((\frac{\|z_1\|)}{\ln 2}))$ and, hence, $z_1 \log r \in U_0(\frac{1}{\ln 2})$.

(viii) In turn, this implies that

$$\prod_{i=1}^{N} r_i^z = (\prod_{i=1}^{N} r_i)^z$$

holds for all $N \in \mathbb{N}$, $z \in R$, and $r_1, \ldots, r_N \in U_1(1 + \frac{N}{ln2})$ in view of

$$\begin{aligned}\prod_{i=1}^{N} r_i^z &= \prod_{i=1}^{N} \exp(z \log r_i) = \exp\big(\sum_{i=1}^{N} z \log r_i\big) \\ &= \exp\big(z \sum_{i=1}^{N} \log r_i\big) = \exp\big(z \log \prod_{i=1}^{N} r_i\big) \\ &= \big(\prod_{i=1}^{N} r_i\big)^z.\end{aligned}$$

(ix) More generally, we have

$$\prod_{i=1}^{N} \big(\prod_{j=1}^{M} r_j^{z_{ij}}\big)^{w_i} = \prod_{j=1}^{M} r_j^{\sum_{i=1}^{N} z_{ij} w_i} \tag{133}$$

for all $N, M \in \mathbb{N}$ and $w_1, \ldots, w_N, z_{11}, \ldots, z_{NM}, r_1, \ldots, r_M \in R$ with

$$r_j \in U_1(1 + (2 + \frac{M}{ln2}) \max(\|z_{ij}\| : i = 1, \ldots, N)$$

for all $j = 1, \ldots, M$:

Indeed, these assumptions imply $r_j^{z_{ij}} \in U_1(1 + \frac{M}{ln2})$ for all $j = 1, \ldots, M$ and $i = 1, \ldots, N$ and, therefore,

$$\big(\prod_{j=1}^{M} r_j^{z_{ij}}\big)^{w_i} = \prod_{j=1}^{M} (r_j^{z_{ij}})^{w_i}$$

for all $i = 1, \ldots, N$.

And they imply $r_j \in U_1(1 + 2\|z_{ij}\|)$ and, hence, $(r_j^{z_{ij}})^{w_i} = r_j^{(z_{ij} w_i)}$ for all $i = 1, \ldots, N$ and $j = 1, \ldots, M$.

Thus, we have

$$\begin{aligned}\prod_{i=1}^{N} \big(\prod_{j=1}^{M} r_j^{z_{ij}}\big)^{w_i} &= \prod_{i=1}^{N} \big(\prod_{j=1}^{M} (r_j^{z_{ij}})^{w_i}\big) = \prod_{i=1}^{N} \prod_{j=1}^{M} r_j^{(z_{ij} w_i)} \\ &= \prod_{j=1}^{M} \big(\prod_{i=1}^{N} r_j^{(z_{ij} w_i)}\big) = \prod_{j=1}^{M} r_j^{\sum\limits_{i=1}^{N} z_{ij} w_i},\end{aligned}$$

just as claimed.

While in this paper, we only need the map θ_N^R, the more general definitions and facts collected above have been of use in other approaches to Hadamard conjugation and appear to be also of use, quite generally, in the phylogenetic analysis of rate matrices (cf. [**?**]).

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