PATHS, PERMUTATIONS AND TREES:

FROM FIBONACCI TO CATALAN

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FIBONACCI NUMBERS

$$\begin{cases} f_n = f_{n-1} + f_{n-2} & n \geq 2 \\ f_0 = 1 \\ f_1 = 1 \end{cases}$$

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$

$$f(x) = \frac{1}{1 - x - x^2}$$

$$f_n \simeq \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^{n+1} \qquad \Phi = \frac{+1+\sqrt{5}}{2}$$

CATALAN NUMBERS

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

$$\begin{cases} C_n = \sum_{i=1}^{n-1} C_i C_{n-1-i} \\ C_0 = 1 \\ C_1 = 1 \end{cases}$$

$1, 2, 5, 14, 42, \ldots$

$$C(x) = \frac{1 - \sqrt{1-4x}}{2x}$$

WHAT IS THERE BETWEEN FIBONACCI AND CATALAN ?

f_n	1, 2, 3, 5, 8, 13, 21	FIBONACCI
t_n	1, 2, 4, 7, 13, 24, 44	"TRIBONACCI"
2^{n-1}	1, 2, 4, 8, 16, 32, 64	
P_n	1, 2, 5, 12, 29, 70, 169	PELL
f_{2n-2}	1, 2, 5, 13, 34, 89, 233	EVEN-INDEX FIBONACCI
C_n	1, 2, 5, 14, 42, 132, 429	CATALAN

PERMUTATIONS

S_n = SET OF PERMUTATIONS ON $[n] = \{1, 2, \dots n\}$

$$\pi = \pi_1 \pi_2 \cdots \pi_n \in S_n$$

$$\sigma = \sigma_1 \sigma_2 \cdots \sigma_k \in S_k$$

$\pi \in S_n(\sigma)$ IF $\nexists$ $j_1 < j_2 < \cdots < j_k$ SUCH THAT

$$\pi_{j_i} < \pi_{j_l} \iff \sigma_i < \sigma_l$$

EX.

$$7465312 \in S_7(123)$$

$$7154326 \notin S_7(123)$$

$$S_n(\sigma_1, \sigma_2, \ , \sigma_j) = S(\sigma_1) \cap S(\sigma_2) \cap \cdots \cap S(\sigma_j)$$

EX.

$$6745231 \in S_7(123, 132, 213)$$

$$6475231 \notin S_7(123, 132, 213)$$

PERMUTATIONS WITH FORBIDDEN SUBSEQUENCES

- J. Bandlow, Eric S. Egge and K. Killpatrick, A weight-preserving bijection between Schröder paths and Schröder permutations, *Ann. Combin.*, to appear.
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- O. Guibert, Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young, PH. D. Thesis, Université de Bordeaux I, 1995.
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- J. West, Generating trees and forbidden subsequences, *Discrete Math.* 157 (1996), 363-374.

- F.R.K. Chung, R.L. Graham, V.E. Hoggatt Jr. and M. Kleiman, The number of Baxter permutations, *J. Combin. Theory Ser. A* 24 (1978), 382-394.
- J. West, Generating trees and the Catalan and Schröder numbers, *Discrete Math.*, 146 (1995), 247-262.
- J. West, Generating trees and forbidden subsequences, *Discrete Math.* 157 (1996), 363-374.

$S_n(123)$

$\pi = \pi_1 \pi_2 \cdots \pi_n \quad \in S_n(123)$

S. T.

$$\pi_1 > \pi_2 > \cdots > \pi_{k-1} < \pi_k$$

$$\cdot\,\pi_1 \cdot \pi_2 \cdot \cdots \cdot \pi_{k-1} \cdot \pi_k\ \pi_{k+1} \cdots \pi_n \qquad k \text{ ACTIVE SITES}$$

$$\Downarrow$$

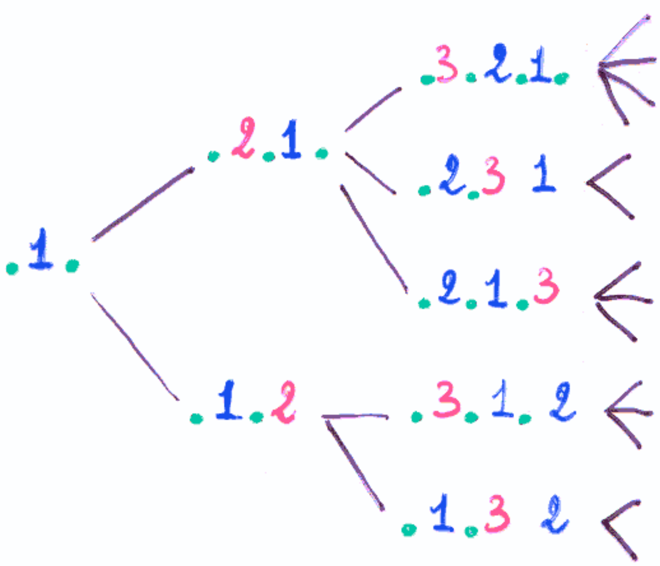
$$\cdot\,(n+1) \cdot \pi_1 \cdot \pi_2 \cdot \cdots \cdot \pi_{k-1} \cdot \pi_k\ \pi_{k+1} \cdots \pi_n \qquad k+1 \text{ A.S.}$$

$$\cdot\,\pi_1 \cdot (n+1)\ \pi_2 \cdots \pi_n \qquad 2$$

$$\cdot\,\pi_1 \cdot \pi_2 \cdot (n+1)\ \pi_3 \cdots \pi_n \qquad 3$$

$$\vdots$$

$$\cdot\,\pi_1 \cdot \pi_2 \cdot \cdots \cdot (n+1)\ \pi_k \cdots \pi_n \qquad k$$



$$\begin{cases} (2) \\ (k) \rightsquigarrow (2) \cdots (k)(k+1) \end{cases} \qquad |S_n(123)| = C_n = \frac{1}{n+1}\binom{2n}{n}$$

CATALAN

$S_n(123, 132, 213)$

ONLY THE FIRST TWO POSITIONS CAN BE ACTIVE SITES BECAUSE OF 123 AND 213

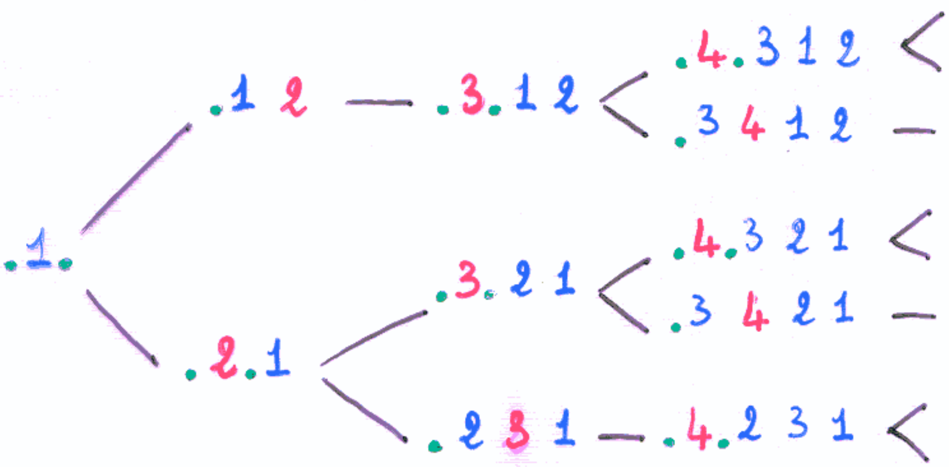
IF $\pi_1 < \pi_2$ THEN ONLY THE FIRST SITE IS ACTIVE BECAUSE OF 132

$\pi = \pi_1 \pi_2 \cdots \pi_n$

$\pi_1 < \pi_2$ $\quad .\pi_1 \pi_2 \cdots \pi_n \to .(n+1).\pi_1 \pi_2 \cdots \pi_n$ $\quad 2$

$\pi_1 > \pi_2$ $\quad .\pi_1 . \pi_2 \cdots \pi_n \to .(n+1).\pi_1 \pi_2 \cdots \pi_n$ $\quad 2$

$\quad .\pi_1 . \pi_2 \cdots \pi_n \to .\pi_1 (n+1) \pi_2 \cdots \pi_n$ $\quad 1$

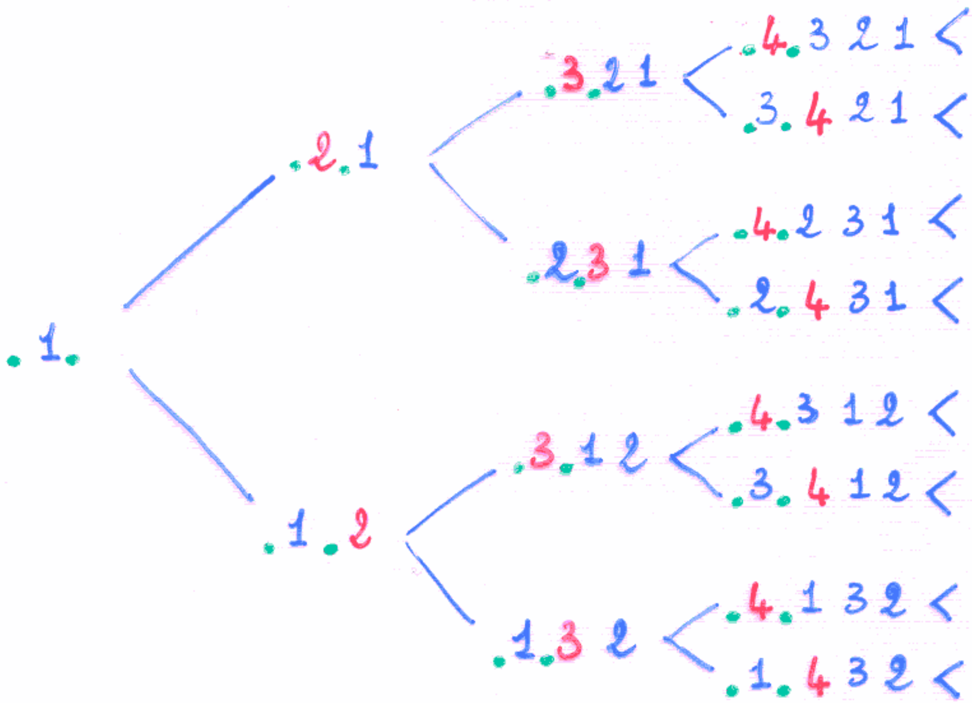


$$\begin{cases} (2) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2) \end{cases}$$

FIBONACCI

$S_n(123, 213)$

ANY PERMUTATION $\pi \in S_n(123, 213)$ HAS TWO ACTIVE SITES (THE FIRST TWO POSITIONS)

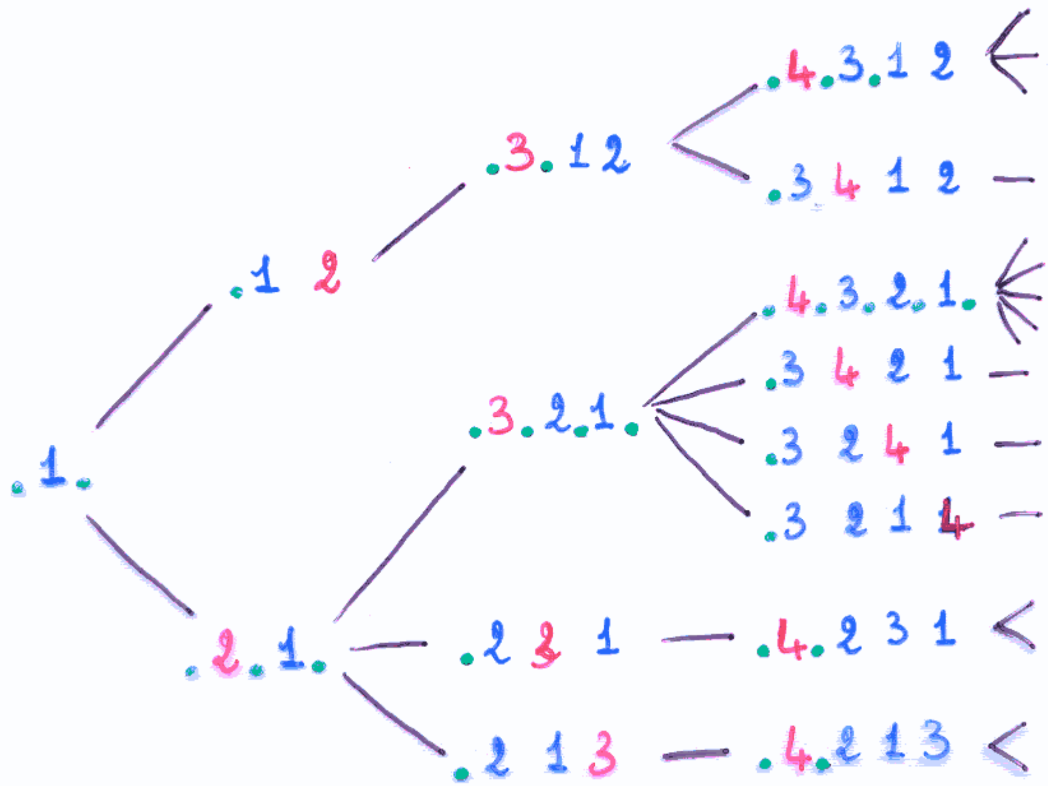


$$\begin{cases} (2) \\ (2) \rightsquigarrow (2)(2) \end{cases}$$

$$|S_n(123, 213)| = 2^{n-1}$$

$$\begin{cases} x_n = 2x_{n-1} \\ x_1 = 1 \end{cases}$$

$S_n(123, 132)$



$$\begin{cases} (2) \\ (k) \rightsquigarrow (1)^{k-1}(k+1) \end{cases}$$

$$|S_n(123, 132)| = 2^{n-1}$$

$$\begin{cases} X_n = X_{n-1} + \quad + X_1 + X_0 \\ X_0 = 1 \\ X_1 = 1 \end{cases}$$

WHAT IS THERE BETWEEN FIBONACCI AND 2^{n-1} ?

FIBONACCI	1	2	3	5	8	13	$f_n = f_{n-1} + f_{n-2}$
"TRIBONACCI"	1	2	4	7	13	24	$t_n = t_{n-1} + t_{n-2} + t_{n-2}$
"TETRANACCI"	1	2	4	8	15	29	$q_n = q_{n-1} + q_{n-2} + q_{n-3} + q_{n-4}$
⋮	1	2	4	8	16	31	
	⋮						
"K-NACCI"	1	2	4	8	16	32	$v_n = \sum_{i=1}^{k} v_{n-i}$
⋮	⋮						
2^n	1	2	4	8	16		$d_n = \sum_{i=1}^{n} d_{n-i}$
							$d_n = 2 d_{n-1}$

FIBONACCI $\quad S_n(123, 132, 213)$

$2^{n-1} \quad S_n(123, 132)$

$S_n(123, 213)$

FIBONACCI $\quad S_n(123, 132, 213)$

$2^{n-1} \quad S_n(123, 213)$

THE PATTERN 132 MUST DISAPPEAR

WE "GENERALIZE" THIS PATTERN

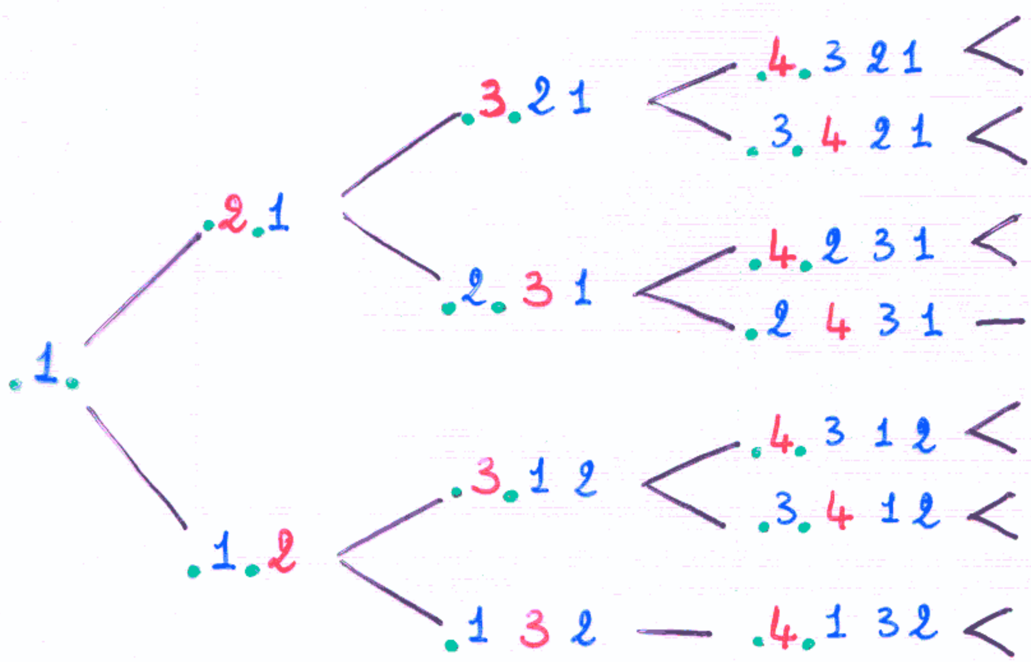
1432

15432

⋮

$1\,(k+1)\,k \cdots 2$

FIRST STEP: $\quad S_n(123, 213, 1432)$



(2_1)

$(2_1) \rightsquigarrow (2_1)(2_2)$

$(2_2) \rightsquigarrow (2_1)(1)$

1, 2, 4, 7, 13, ...

"TRIBONACCI"

$t_n = t_{n-1} + t_{n-2} + t_{n-3}$

STEP K

$$S_n(123, 213, 1(k+1)\cdots 2)$$

$$\begin{cases} (2_1) \\ (2_1) \rightsquigarrow (2_1)(2_2) \\ \vdots \\ (2_j) \rightsquigarrow (2_1)(2_{j+1}) \\ \vdots \\ (2_{k-1}) \rightsquigarrow (2_1)(1) \end{cases}$$

"K-NACCI"

$$v_n = v_{n-1} + v_{n-2} + \cdots + v_{n-k}$$

WHEN $K \to \infty$

$$\begin{cases} (2) \\ (2) \rightsquigarrow (2)(2) \end{cases}$$

$$S_n(123, 213)$$

FIBONACCI $S_n(123, 132, 213)$

2^{n-1} $S_n(123, 132)$

WE "GENERALIZE" THE PATTERN 213

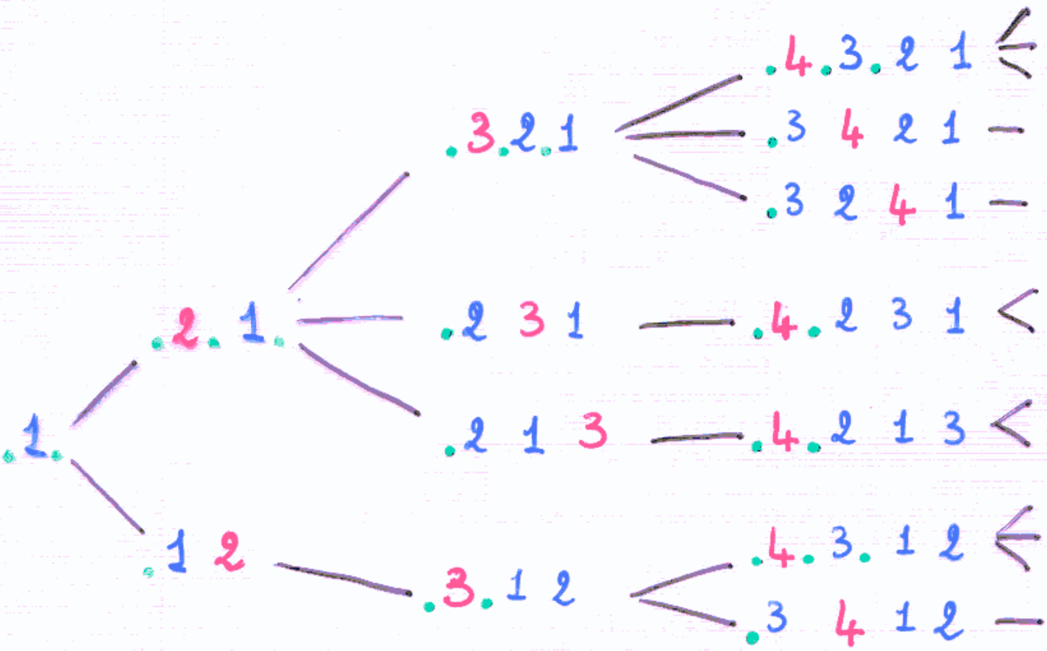
3214

43215

⋮

$k(k-1)\cdots 1(k+1)$

FIRST STEP: $S_n(123, 132, 3214)$



$$\begin{cases} (2) \\ (2) \rightsquigarrow (1)(3) \\ (3) \rightsquigarrow (1)(1)(3) \\ (1) \rightsquigarrow (2) \end{cases}$$

1, 2, 4, 7, 13

"TRIBONACCI"

STEP K

$$S_n(123, 213, k(k-1)\cdots 1(k+1))$$

$$\begin{cases} (2) \\ (1) \rightsquigarrow (2) \\ \vdots \\ (h) \rightsquigarrow (1)^{h-1}(h+1) \qquad \text{FOR } h < k \\ \vdots \\ (k) \rightsquigarrow (1)^{k-1}(k) \end{cases}$$

WHEN $k \to \infty$

$$\begin{cases} (2) \\ (k) \rightsquigarrow (1)^{k-1}(k+1) \end{cases}$$

$$S_n(123, 213)$$

FROM 2^{n-1} TO CATALAN

2^{n-1} : $S_n(123, 213)$

C_n : $S_n(123)$

WE GENERALIZE THE PATTERN 213

3214

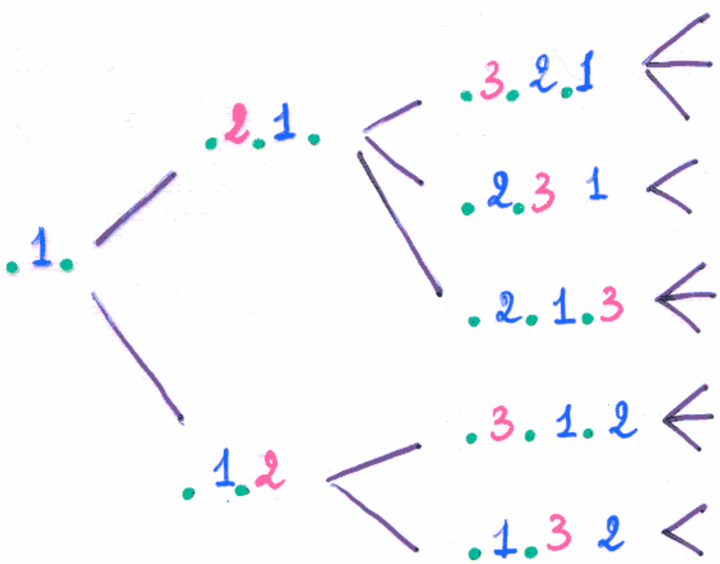
43215

$\vdots$

$k(k-1)\cdots 1(k+1)$

FIRST STEP

$S_n(123, 3214)$



$$\begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(3)(3) \end{cases}$$

1, 2, 5, 13, 34

EVEN INDEX FIBONACCI

f_{2n-2}

STEP K

$$S_n(123,\ k(k-1)\cdots 1(k+1))$$

(2)

$(2) \rightsquigarrow (2)(3)$

$\vdots$

$(h) \rightsquigarrow (2)(3)\cdots(h-1)(h)(h+1) \qquad h < k$

$\vdots$

$(k) \rightsquigarrow (2)(3)\cdots(k-1)(k)(k)$

WHEN $k \to \infty$

$$\begin{cases} (2) \\ (k) \rightsquigarrow (2)\cdots(k-1)(k)(k+1) \end{cases}$$

$$S_n(123)$$

2^{n-1}: $S_n(123, 132)$

C_n : $S_n(123)$

WE "GENERALIZE" THE PATTERN 132

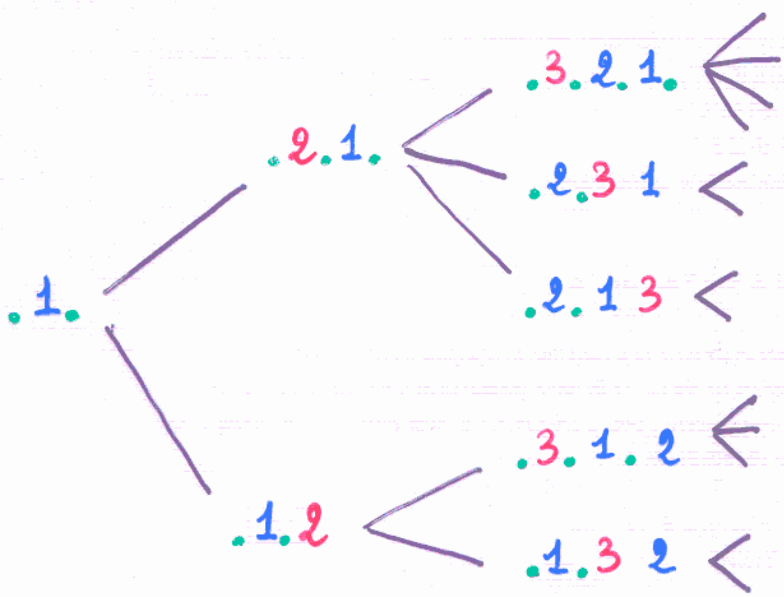
2143

32154

$\vdots$

$(k-1) \cdots 1\,(k+1)\,k$

FIRST STEP: $S_n(123, 2143)$



$$\begin{cases} (2) \\ (k) \rightsquigarrow (2)^{k-1}(k+1) \end{cases}$$

1, 2, 5, 13, 34

EVEN INDEX FIBONACCI

f_{2n-2}

STEP K

$$S_n(123, (k-1)\cdots 1\,(k+1)\,k)$$

(2)

$(h) \leadsto (2)\cdots(h)(h+1) \qquad h < k$

$(h) \leadsto (2)\cdots(k-2)(k-1)^{h-k+2}(h+1) \qquad h \geq k$

WHEN $k \to \infty$

(2)

$(k) \leadsto (2)(3)\cdots(k-1)(k)(k+1)$

$$S_n(123)$$

FROM FIBONACCI TO CATALAN DIRECTLY

FIBONACCI $S_n(123, 132, 213)$

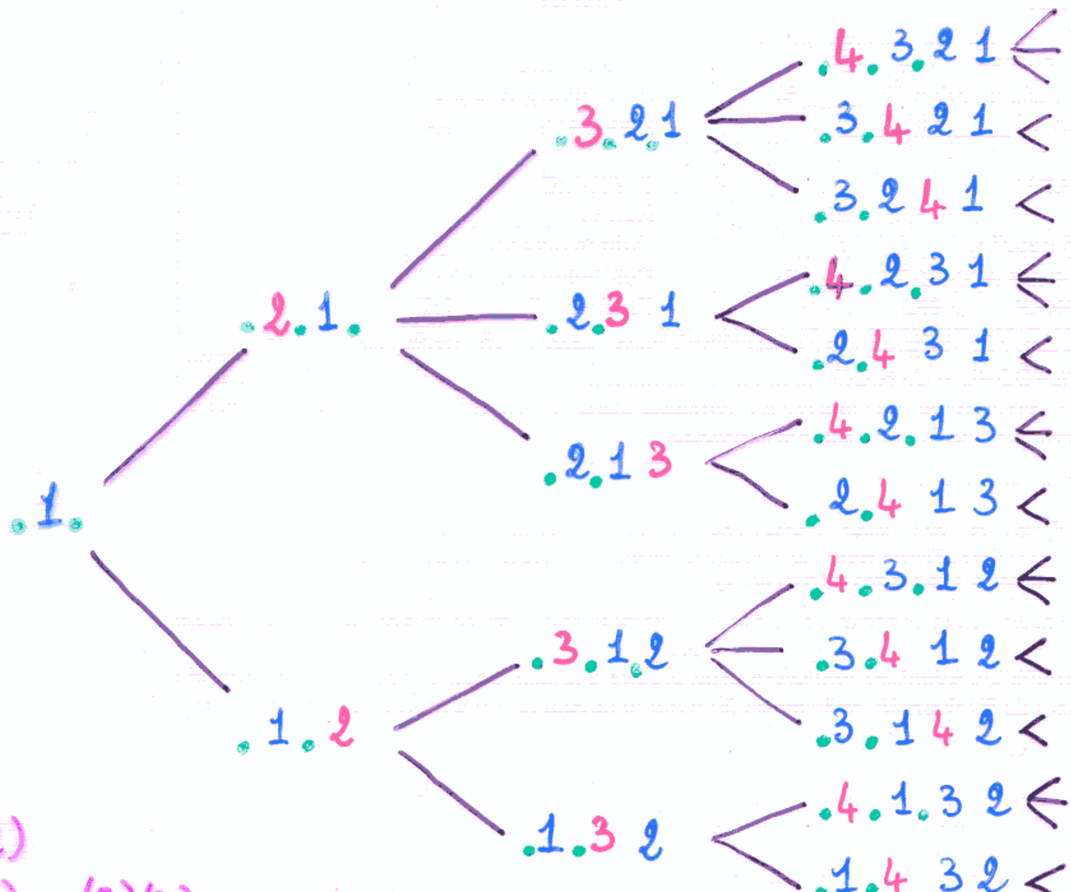
CATALAN $S_n(123)$.

WE "GENERALIZE" THE TWO PATTERN
132 AND 213

2143 3214
32154 43215

$(k-1) \cdots 1 (k+1) k$ $k (k-1) \cdots 1 (k+1)$

FIRST STEP: $S_n(123, 2143, 3214)$



(2)
(2) ⇝ (2)(3)
(3) ⇝ (2)(3)(3)

PELL NUMBERS
1, 2, 5, 12, 29, 70

$P_n = 2P_{n-1} + P_{n-2}$

STEP K

$$S_n(123,\ (k-1)\cdots 1\,(k+1)\,k,\ k\,(k-1)\cdots 1\,(k+1))$$

(2)

$\vdots$

$(h) \rightsquigarrow (2)(3)\cdots(h)(h+1) \qquad h < k$

$\vdots$

$(k) \rightsquigarrow (2)(3)\cdots(k-1)(k-1)(k)$

WHEN $k \to \infty$

(2)

$(k) \rightsquigarrow (2)(3)\cdots(k)(k+1)$

$$S_n(123)$$

PELL

$$p_n = 2p_{n-1} + p_{n-2}$$

$$S_n(123, 2143, 3214)$$

$$\begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(2)(3) \end{cases}$$

EVEN INDEX FIBONACCI

$$t_{2n-2} = 2t_{2n-4} + t_{2n-6} + \cdots + t_2 + t_0$$

$S_n(123, 3214)$

(2)

$(2) \rightsquigarrow (2)(3)$

$(3) \rightsquigarrow (2)(3)(3)$

$S_n(123, 2143)$

(2)

$(k) \rightsquigarrow (2)^{k-1}(k+1)$

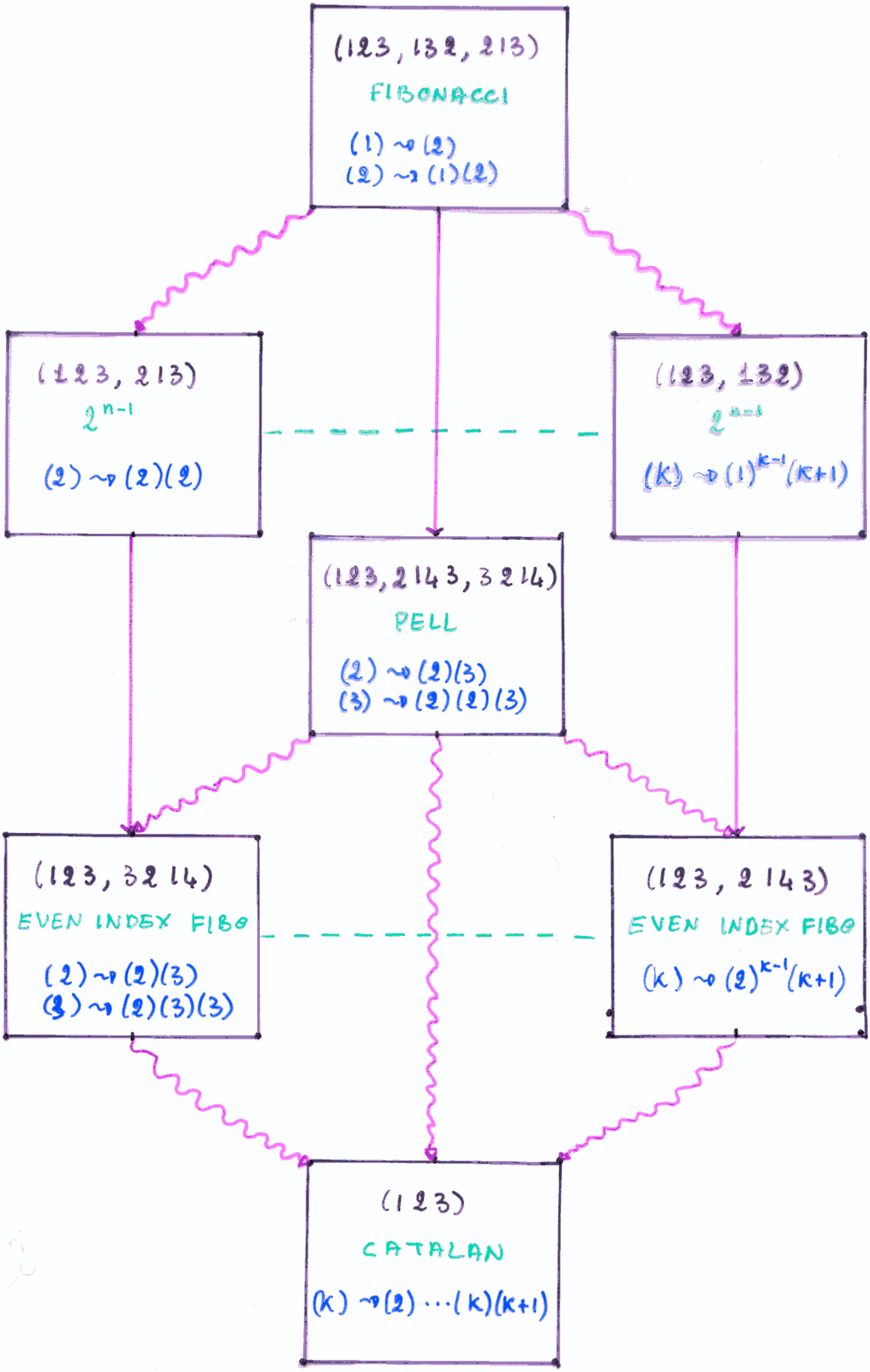
$S_n(123, 3214, 21(k+1)\cdots 3)$

$$\begin{cases} (2) \\ (2) \rightsquigarrow (2)(3_1) \\ (3_1) \rightsquigarrow (2)(3_1)(3_2) \\ \vdots \\ (3_{k-3}) \rightsquigarrow (2)(3_1)(3_{k-2}) \\ (3_{k-2}) \rightsquigarrow (2)(2)(3_1) \end{cases}$$

$S_n(123, 2143, (k-1)\cdots 1(k+1)k)$

$$\begin{cases} (2) \\ \vdots \\ (h) \rightsquigarrow (2)^{h-1}(h+1) \quad h < k \\ \vdots \\ (k) \rightsquigarrow (2)^{k-1}(k) \end{cases}$$

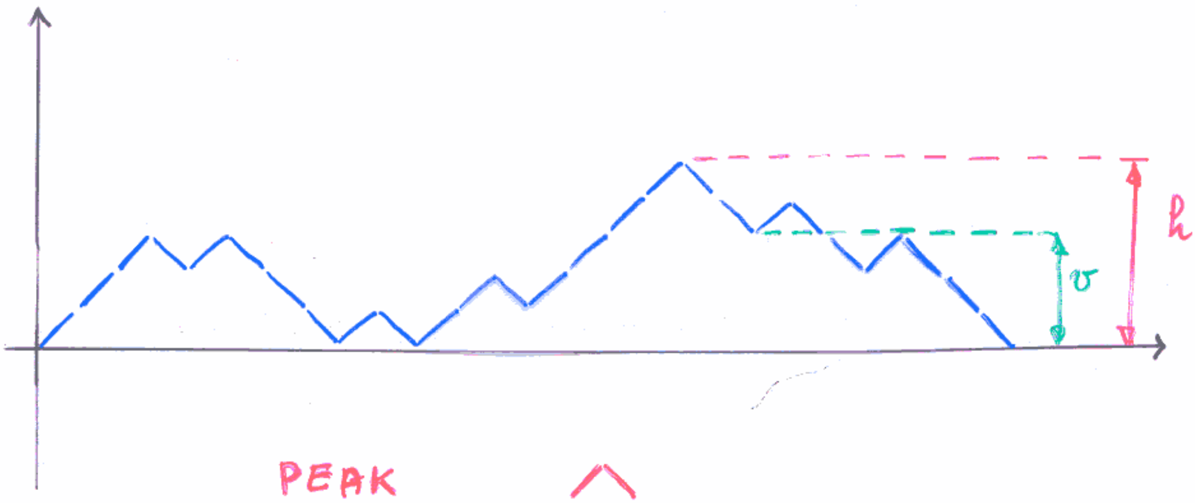
$$y_n = 2y_{n-1} + y_{n-2} + \cdots + y_{n-k+2}$$


(123, 132, 213)
FIBONACCI
(1) ⇝ (2)
(2) ⇝ (1)(2)
(123, 213)
2^{n-1}
(2) ⇝ (2)(2)
(123, 132)
2^{n-1}
$(k) \rightsquigarrow (1)^{k-1}(k+1)$
(123, 2143, 3214)
PELL
(2) ⇝ (2)(3)
(3) ⇝ (2)(2)(3)
(123, 3214)
EVEN INDEX FIBO
(2) ⇝ (2)(3)
(3) ⇝ (2)(3)(3)
(123, 2143)
EVEN INDEX FIBO
$(k) \rightsquigarrow (2)^{k-1}(k+1)$
(123)
CATALAN
(k) ⇝ (2)···(k)(k+1)

DYCK PATHS

/ RISE STEP

\ FALL STEP



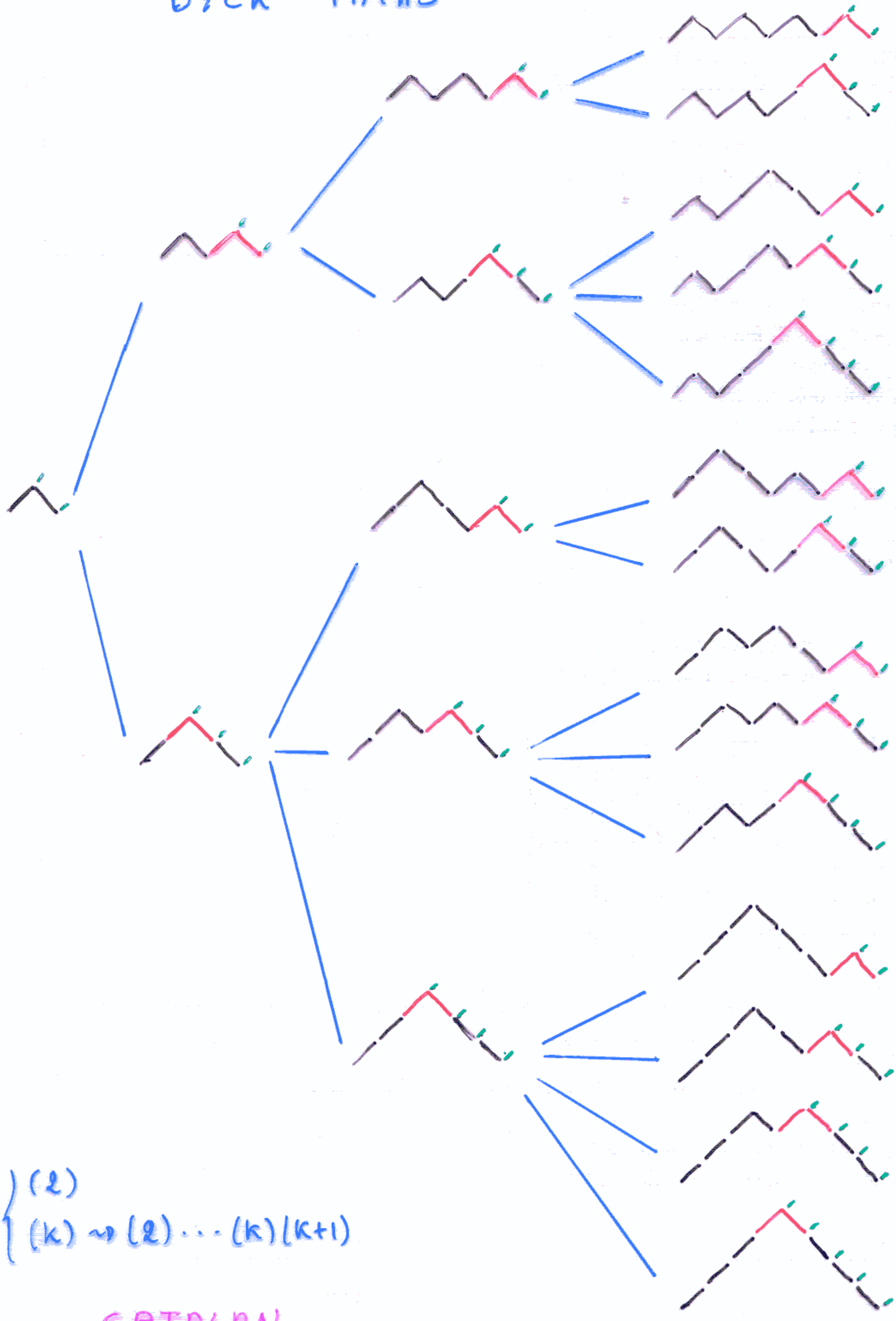
PEAK ^

VALLEY V

ECO METHOD

- E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, A methodology for plane trees enumeration, *Discrete Math.*, 180 (1998), 45-64.
- E. Barcucci, A. Del Lungo and E. Pergola, Random generation of trees and other combinatorial objects, *Theoretical Computer Science*, 218 (1999), 219-232.
- E. Barcucci, A. Del Lungo, E. Pergola and R. Pinzani, ECO: A methodology for enumeration of combinatorial objects, *J. Diff. Equ. Appl.*, 5 (1999), 435-490.

DYCK PATHS

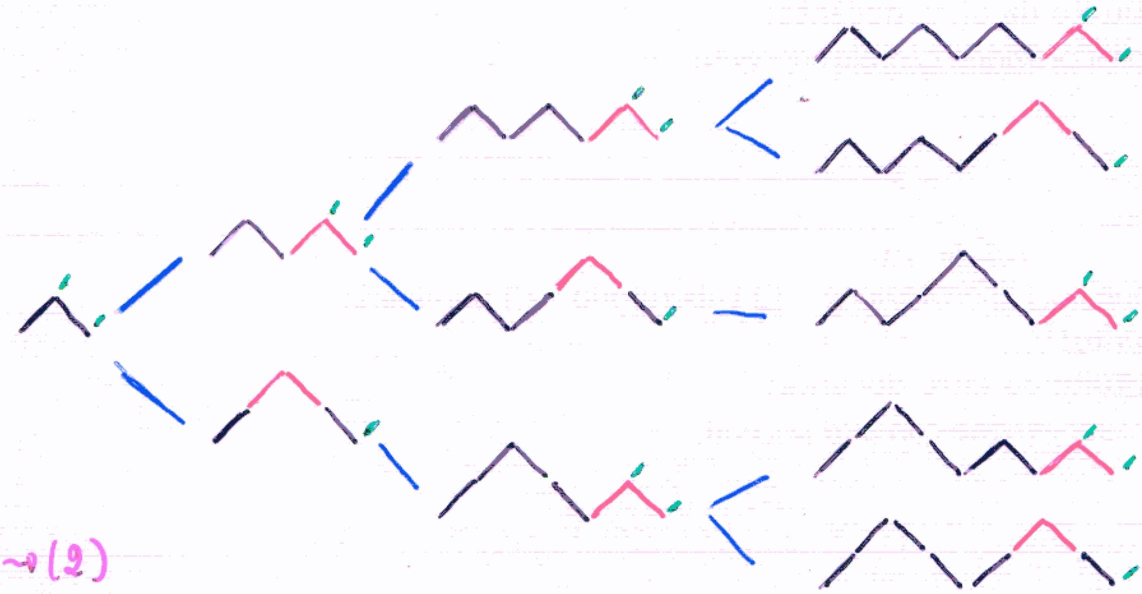


$$\begin{cases} (2) \\ (k) \rightsquigarrow (2) \cdots (k)(k+1) \end{cases}$$

CATALAN

FIBONACCI PATHS

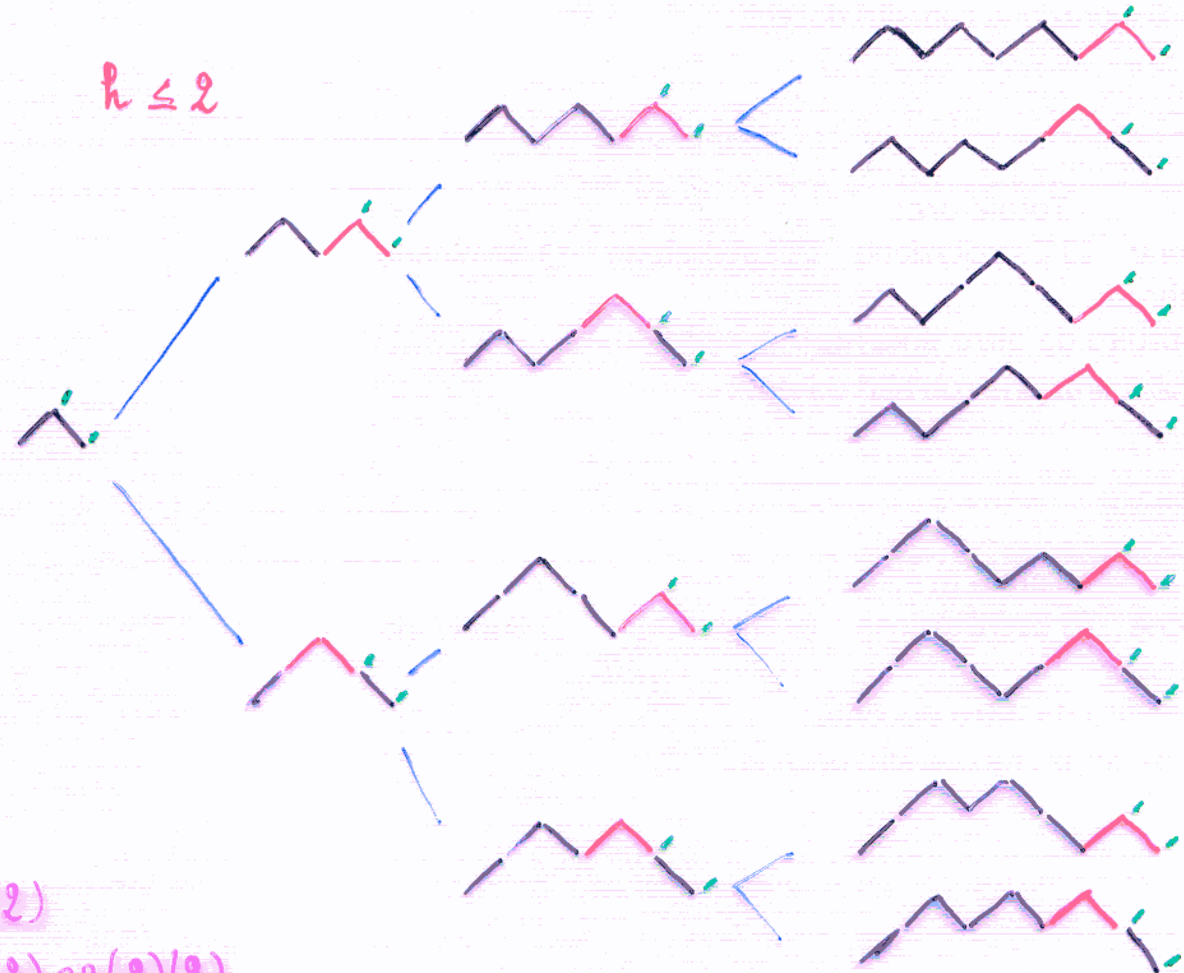
$h \le 2 \qquad v = 0$



$$\begin{cases} (2) \\ (1) \rightsquigarrow (2) \\ (2) \rightsquigarrow (1)(2) \end{cases}$$

2^{n-1} PATHS

$h \le 2$



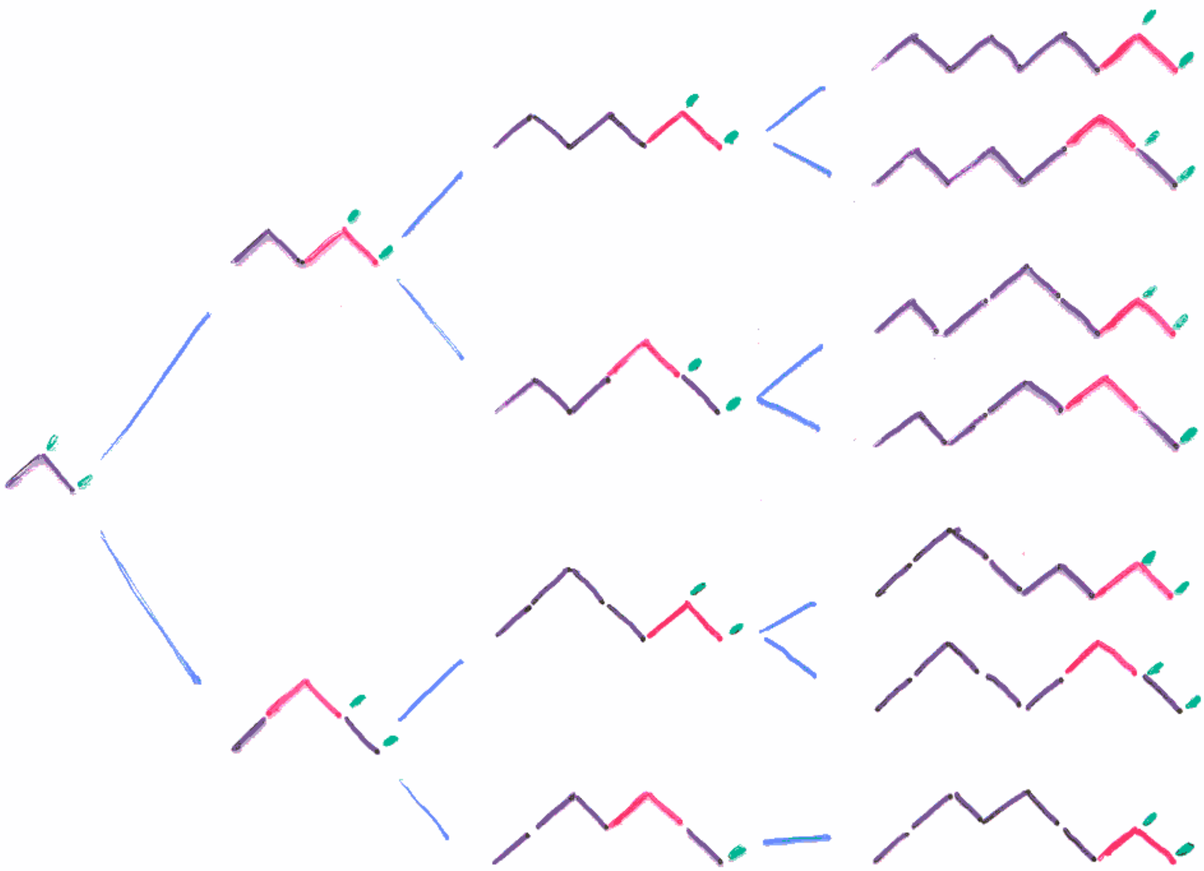
$$\begin{cases} (2) \\ (2) \rightsquigarrow (2)(2) \end{cases}$$

FROM FIBONACCI TO 2^{n-1}

- $h \leq 2$
- AT MOST K CONSECUTIVE VALLEYS AT LEVEL 1 $(K = 1, 2, 3, \ldots)$

$k = 1$: "TRIBONACCI" PATHS

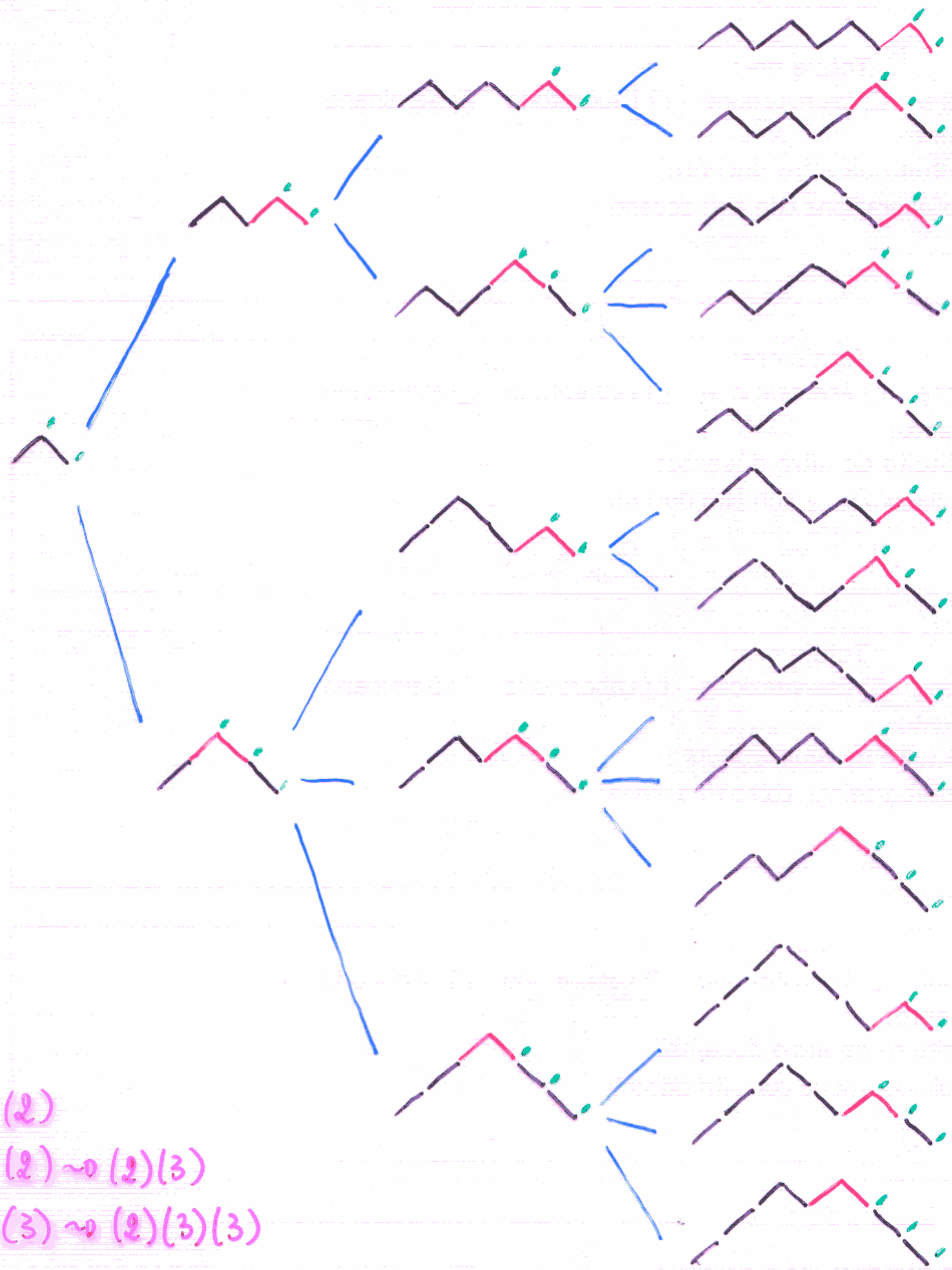
NO CONSECUTIVE VALLEYS AT LEVEL 1



FROM 2^{n-1} TO CATALAN

$h \leq K \qquad (K = 3, 4, 5, \ldots)$

$K = 3$ EVEN INDEX FIBONACCI



$$\begin{cases} (2) \\ (2) \rightsquigarrow (2)(3) \\ (3) \rightsquigarrow (2)(3)(3) \end{cases}$$

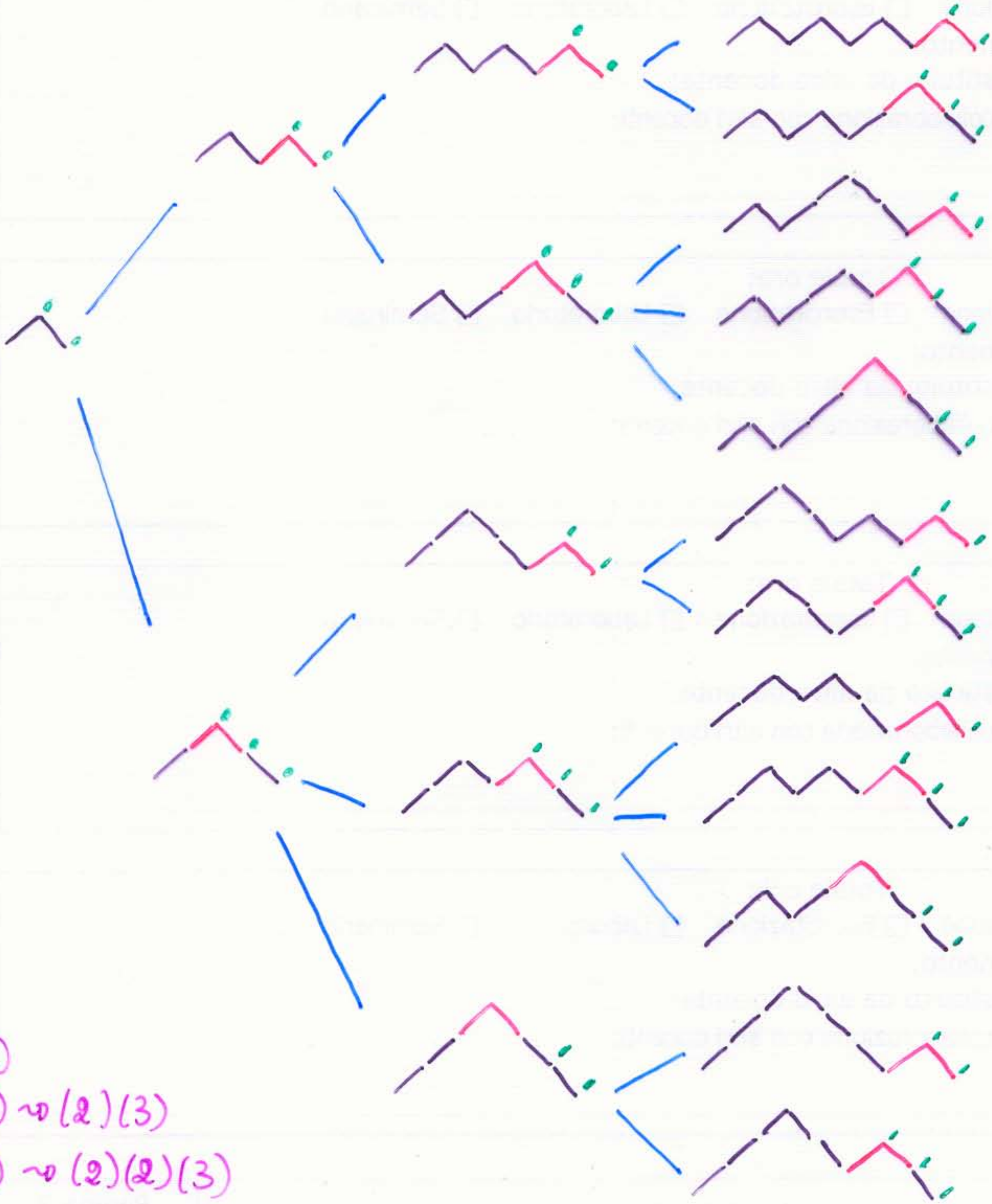
FROM FIBONACCI TO CATALAN "DIRECTLY"

$h \leq k$

$v \leq k-2$ $\quad (k=3,4,5,\dots)$

$k=3$ $\quad h \leq 3$

$v \leq 1$ $\quad$ PELL



(2)

(2) ⇝ (2)(3)

(3) ⇝ (2)(2)(3)

FROM PATHS TO TREES