Exponential Riordan Arrays

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Feb. 26, 2004

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Permutations, Paths, and Trees

Famous Examples

$$\text{Pascal} = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ 1 & 2 & 1 & & \\ 1 & 3 & 3 & 1 & \\ 1 & 4 & 6 & 4 & 1 \\ & \cdots & & & \end{bmatrix}$$

Pascal 1665

Chia Hsien 1050

$$S_1 = \begin{bmatrix} 1 & & & & \\ 1 & 1 & & 0 & \\ 2 & 3 & 1 & & \\ 6 & 11 & 6 & 1 & \\ 24 & 50 & 35 & 10 & 1 \\ & & \cdots & & \end{bmatrix}$$

Stirling numbers
First kind

$$S_2 = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & 0 & \\ 1 & 3 & 1 & & & \\ 1 & 7 & 6 & 1 & & \\ 1 & 15 & 25 & 10 & 1 & \\ & & \cdots & & & \end{bmatrix}.$$

Stirling Numbers, $2^{\underline{\underline{nd}}}$ kind

$$T = T(n,k)_{n,k\geq 0} = \begin{bmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 1 & 0 & 1 & & & 0 & \\ 0 & 3 & 0 & 1 & & & \\ 3 & 0 & 6 & 0 & 1 & & \\ 0 & 15 & 0 & 10 & 0 & 1 & \\ 15 & 0 & 45 & 0 & 15 & 0 & 1 \\ & & & \cdots & & & \end{bmatrix}.$$

$T(n,k)$ = the number of symmetric $n \times n$ permutation matrices with trace k

= # of solutions of $X^2 = I$ in the symmetric group S_n with k fixed points

Let $L = L(n,k)_{n,k \geq 0}$ be an infinite lower trianglular matrix.

If there exist exponential generating functions, $g(z)$ and $f(z)$, such that the $k^{\underline{th}}$ column of L has as its EGF

$$g(z)\,\frac{(f(z))^k}{k!} \quad \text{for } k=0,1,2,\ldots$$

then L is a <u>Riordan</u> <u>array</u>.

Notation: $L = (g(z), f(z))$

The 4 examples

$$\text{Pascal} = (e^z, z)$$

$$S_1 = \left(\frac{1}{1-z}, \ln\left(\frac{1}{1-z}\right)\right)$$

$$S_2 = (e^z, e^z - 1)$$

$$T = (e^{z^2/2}, z)$$

The Fundamental Theorem of Riordan Arrays (FTRA)

Let $A(z) = a_0 + a_1 z + a_2 \frac{z^2}{2!} + a_3 \frac{z^3}{3!} + \cdots$

and $B(z) = b_0 + b_1 z + b_2 \frac{z^2}{2!} + b_3 \frac{z^3}{3!} + \cdots$

and suppose $L = (g(z), f(z)) = (L(n,k))$

and $L \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \cdots \end{bmatrix}$.

Then

$$B(z) = g(z) A(f(z))$$

Idea of the Proof

$$\begin{bmatrix} | & | & | & \\ g & gf & g\frac{f^2}{2!} & \cdots \\ | & | & | & \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \cdots \end{bmatrix}$$

gives

$$g a_0 + g f a_1 + g \frac{f^2}{2!} a_2 + \cdots$$

$$= g \left[a_0 + a_1 f + a_2 \frac{f^2}{2!} + \cdots \right]$$

$$= g(z) \, A(f(z)) = B(z)$$

We now switch back and forth freely between sequences and their exponential generating functions (EGF's)

In many cases combinatorial problems are solved by composition of functions.

Typical applications

Proving identities
Inverting identities
Developing combinatorial interpretations

An example. What is the total number of fixed points of all involutions in S_n.

Note that

$$0, 1, 2, 3, \ldots \longleftrightarrow 0 + 1 \cdot z + 2\frac{z^2}{2!} + 3\frac{z^3}{3!} + \cdots$$

$$= z e^z = A(z)$$

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 1 & 0 & 1 & & \\ 0 & 3 & 0 & 1 & \\ 3 & 0 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$\left(e^{z^2/2}, z\right) z e^z = e^{z^2/2} \cdot z e^z = \sum_{n=1}^{\infty} n\, I(n-1) \frac{z^n}{n!}$$

Total number of fixed points is

$$n\,I(n-1).$$

The average number is

$$\frac{n\,I(n-1)}{I(n)} \xrightarrow[\text{some work}]{} \sqrt{n}$$

The Riordan Group

Suppose now that

$$g(z) = 1 + g_1 z + g_2 \frac{z^2}{2!} + \cdots$$

and $$f(z) = f_1 z + f_2 z^2/2! + f_3 z^3/3! + \cdots$$

$$f_1 \neq 0$$

Then $(g(z), f(z))$ is an element in the Riordan group. The multiplication is matrix multiplication and

$$(g(z), f(z))\,(h(z), \ell(z))$$
$$= (\,g(z)\,h(f(z)),\ \ell(f(z)))$$

$I = (1, z)$ is the identity matrix

$$(g(z), f(z))^{-1} = \left(\frac{1}{g(\bar{f}(z))}, \bar{f}(z)\right)$$

where $f(\bar{f}(z)) = \bar{f}(f(z)) = z$.

If L is a matrix, $\overline{L}$ is the same matrix with the top row removed.

Example: We want to solve for P where

$$T P = \overline{T}$$

$$\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ 1 & 0 & 1 & & \\ 0 & 3 & 0 & 1 & \\ 3 & 0 & 6 & 0 & 1 \\ & & \cdots & & \end{pmatrix} \begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ 0 & 2 & 0 & 1 & \\ 0 & 0 & 3 & 0 & 1 \\ 0 & 0 & 0 & 4 & 0 \\ & & \cdots & & \end{pmatrix} = \begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ 0 & 3 & 0 & 1 & & \\ 3 & 0 & 6 & 0 & 1 & \\ 0 & 15 & 0 & 10 & 0 & 1 \end{bmatrix}$$

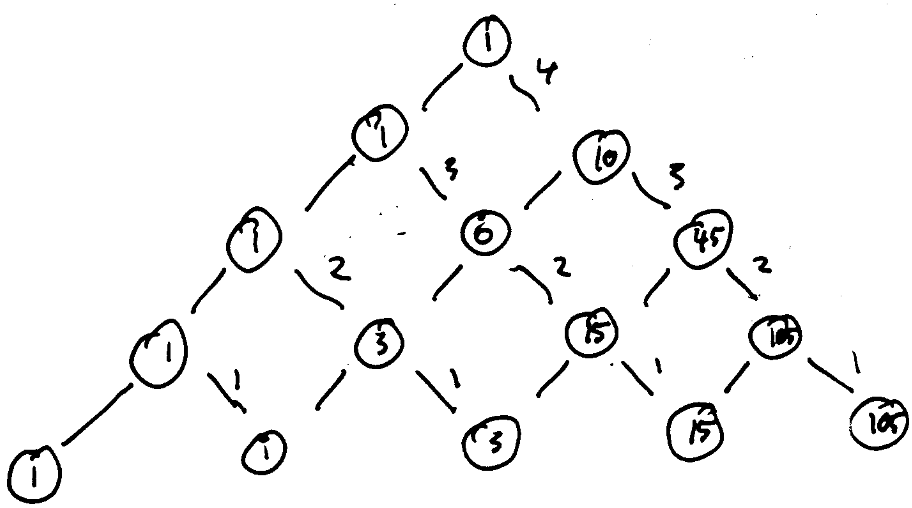
P is a <u>production matrix</u> (or Stieltjes matrix). If we know P and the top row of L is $1.0\,0\,0\cdots$ we can recover all of L.

Corollary

$$1 + z^2 + 3z^4 + 15z^6 + \cdots + (2n-1)!!\, z^{2n} + \cdots$$

$$= \cfrac{1}{1 - \cfrac{z^2}{1 - \cfrac{2z^2}{1 - \cfrac{3z^2}{1 - \cfrac{4z^2}{\cdots}}}}}$$

and as random walks



Here are some more examples of Production matrices

$$P_{Pascal} = \begin{bmatrix} 1 & 1 & & & & \\ & 1 & 1 & & 0 & \\ & & 1 & 1 & & \\ 0 & & & 1 & 1 & \\ & & & & 1 & 1_{n} \\ & & & & & \ddots \end{bmatrix}$$

$$P_{S_2} = \begin{pmatrix} 1 & 1 & & & & \\ & 2 & 1 & & & \\ & & 3 & 1 & & \\ & & & 4 & 1 & \\ & & & & 5 & 1 \\ & & & & & \ddots \end{pmatrix}$$

$$P_{S_1} = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & 0 & \\ 1 & 3 & 3 & 1 & & \\ 1 & 4 & 6 & 4 & 1 & \\ & & \cdots & & & \end{pmatrix}$$

We now define two <u>ordinary</u> generating functions.

$$R(y) = r_0 + r_1 y + r_2 y^2 + \cdots$$
$$C(y) = c_0 + c_1 y + c_2 y^2 + \cdots$$

defined by

$$\left.\begin{aligned} R(f(z)) &= f'(z) \quad \text{and} \\ C(f(z)) &= g'(z)/g(z) \end{aligned}\right\} \text{Emeric Deutsch's Differential Equations}$$

<u>Theorem</u>.

$$P = \begin{bmatrix} c_0 & r_0 & & & \\ 1!\,c_1 & \frac{1!}{1!}(c_0+r_1) & r_0 & & \\ 2!\,c_2 & \frac{2!}{1!}(c_1+r_2) & \frac{2!}{2!}(c_0+2r_1) & r_0 & \\ 3!\,c_3 & \frac{3!}{1!}(c_2+r_3) & \frac{3!}{2!}(c_1+2r_2) & \frac{3!}{3!}(c_0+3r_1) & \\ & & \cdots & & \end{bmatrix}$$

$$p_{ij} = \frac{i!}{j!}\left(c_{i-j} + j\, r_{i-j+1}\right) \qquad c_{-1} = 0$$

Description or Name	g	f	R	C
Pascal	e^z	z	1	1
Stirling_1	$\frac{1}{1-z}$	$\ln\left(\frac{1}{1-z}\right)$	e^y	e^y
Stirling_2	e^z	$e^z - 1$	$1+y$	1
Involutions	$\exp\left(\frac{z^2}{2}\right)$	z	1	y
Identity	1	z	1	0
Alternating (Zig Zag) Permutations	$\sec z + \tan z$	z	1	$\sec z$
Alternating (Zig Zag) Permutations	$\sec z$	$\tan z$	$1+y^2$	y
$(2n-1)!!$	$\frac{1}{\sqrt{1-2z}}$	$1-\sqrt{1-2z}$	$\frac{1}{1-y}$	$\frac{1}{1-y}$
Factorials	$\frac{1}{1-z}$	$\frac{z}{1-z}$	$(1+y)^2$	$1+y$
Derangements, fixed points	$\frac{\exp(-z)}{1-z}$	z	1	$\frac{z}{1-z}$
Derangements, Hankel decomposition	$\frac{\exp(-z)}{1-z}$	$\frac{z}{1-z}$	$(1+z)^2$	z

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 9 & 6 & 1 & 0 & 0 \\ 9 & 44 & 42 & 12 & 1 & 0 \\ 44 & 265 & 320 & 130 & 20 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 4 & 1 & 0 & 0 \\ 0 & 0 & 9 & 6 & 1 & 0 \\ 0 & 0 & 0 & 16 & 8 & 1 \\ 0 & 0 & 0 & 0 & 25 & 10 \end{bmatrix}$$

Since $LP = T$ written out is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 9 & 6 & 1 & 0 & 0 \\ 9 & 44 & 42 & 12 & 1 & 0 \\ 44 & 265 & 320 & 130 & 20 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 0 & 4 & 4 & 1 & 0 & 0 \\ 0 & 0 & 9 & 6 & 1 & 0 \\ 0 & 0 & 0 & 16 & 8 & 1 \\ 0 & 0 & 0 & 0 & 25 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 9 & 6 & 1 & 0 & 0 \\ 9 & 44 & 42 & 12 & 1 & 0 \\ 44 & 265 & 320 & 130 & 20 & 1 \\ 265 & 1854 & 2715 & 1420 & 315 & 30 \end{bmatrix}.$$

$$g(z) = \frac{\exp(-z)}{1-z}$$

$$f(z) = \frac{z}{1-z}$$

$$R(y) = 1$$

$$C(y) = \frac{y}{1-y}$$

Derangements, k^{th} column counts permutations with k fixed points.

Semidirect product decomposition

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 \\ 2 & 9 & 6 & 1 & 0 & 0 \\ 9 & 44 & 42 & 12 & 1 & 0 \\ 44 & 265 & 320 & 130 & 20 & 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 1 & 0 & 0 \\ 9 & 8 & 6 & 0 & 1 & 0 \\ 44 & 45 & 20 & 10 & 0 & 1 \end{bmatrix}}_{\text{in the Appell subgroup}} \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 \\ 0 & 6 & 6 & 1 & 0 & 0 \\ 0 & 24 & 36 & 12 & 1 & 0 \\ 0 & 120 & 240 & 120 & 20 & 1 \end{bmatrix}}_{\text{in the Associated Subgroup}}$$

the Appell subgroup is normal

e general rule is

$$(g(z), f(z)) = (g(z), z)(1, f(z)).$$

The elements of the form $(1, f(z))$ comprise the Associated subgroup while the elements of the form $(g(z), z)$ comprise the Appell subgroup.

Corollary

$$F(z) = \int \frac{1}{R(z)}\, dz$$

Example: For $f(z) = \tan z$ and $g(z) = \sec z$

we get

$$(\sec z, \tan z) = \begin{bmatrix} 1 \\ 0 & 1 \\ 1 & 0 & 1 \\ 0 & 5 & 0 & 1 \\ 5 & 0 & 14 & 0 & 1 \\ 0 & 61 & 0 & 30 & 0 & 1 \\ 61 & 0 & 331 & 0 & 55 & 0 & 1 \end{bmatrix}$$

and

$$P = \begin{bmatrix} 0 & 1 & & & & & \\ 1 & 0 & 1 & & 0 & & \\ & 4 & 0 & 1 & & & \\ & & 9 & 0 & 1 & & \\ 0 & & & 16 & 0 & 1 & \\ & & & & 25 & 0 & \\ & & & & & & \cdots \end{bmatrix}$$

$c_1 = 1$, $c_i = 0$ otherwise

$r_0 = r_2 = 1$, $r_i = 0$ otherwise

$$R(Y) = 1 + Y^2$$

$$F(z) = \int \frac{1}{1+Y^2}\, dY = \arctan z \quad \text{as expected.}$$

Many, but far from all, of the production matrices, P, look like Riordan arrays.
First let P^+ be P with $1\,0\,0\,0\ldots$ on top

$$P^+ = \begin{pmatrix} 1 & 0\ 0\ 0 \\ & P \\ & \end{pmatrix}$$

Is P^+ Riordan?

Yes, Identity, S_1, S_2, alternating (319-399) permutations, $(2n-1)!!$, $(2n+1)!!$, ...

No, Pascal, Involutions (T), Derangements (either kind), $(1+z, z)$, ...

<u>Theorem</u>. P^+ is a Riordan array if and only if for $L = (g(z), f(z))$ we have $g(z) = f(z)'$.

Note: The elements of the Riordan group such that $g(z) = f'(z)$ form a subgroup.

An example worked out

Professor Zeng, yesterday, started with ordered partitions. The GF is

$$\frac{1}{2-e^z} = 1 + z + 3\frac{z^2}{2!} + 13\frac{z^3}{3!} + 75\frac{z^5}{5!} + 541\frac{z^6}{6!} + \cdots$$

One approach to learning more.

A. Form the Hankel matrix.

$$H = \begin{bmatrix} 1 & 1 & 3 & 13 & 75 & 541 \\ 1 & 3 & 13 & 75 & 541 & 4683 \\ 3 & 13 & 75 & 541 & 4683 & \\ 13 & 75 & 541 & & & \\ 75 & 541 & & & & \end{bmatrix}$$

B. Row reduce = Gaussian elimination
= fundamental theorem of linear algebra

$$H = L\,D\,U$$

L is lower triangular, 1's on the main diagonal
D is diagonal, $U = L^{\text{Transpose}}$

$$L = \begin{bmatrix} 1 & & & & & \\ 1 & 1 & & & 0 & \\ 3 & 5 & 1 & & & \\ 13 & 31 & 12 & 1 & & \\ 75 & 233 & 134 & 22 & 1 & \\ & & \cdots & & & \end{bmatrix}$$

$$g(z) = 1 + z + 3\frac{z^2}{2!} + 13\frac{z^3}{3!} + 75\frac{z^4}{4!} + \cdots$$

and

$$g(z)f(z) = z + 5\frac{z^2}{2!} + 31\frac{z^3}{3!} + 233\frac{z^4}{4!} + \cdots$$

so $f(z) = z + 3\frac{z^2}{2!} + 13\frac{z^3}{3!} + 75\frac{z^4}{4!} + \cdots$

We guess

$$f(z) = \frac{1}{2-e^z} - 1 = \frac{e^z - 1}{2-e^z}$$

C. $\bar{f}(z) = \ln\left(\frac{2z+1}{z+1}\right)$

$$R(f(z)) = f'(z) = \frac{e^z}{(2-e^z)^2} = g'(z)$$

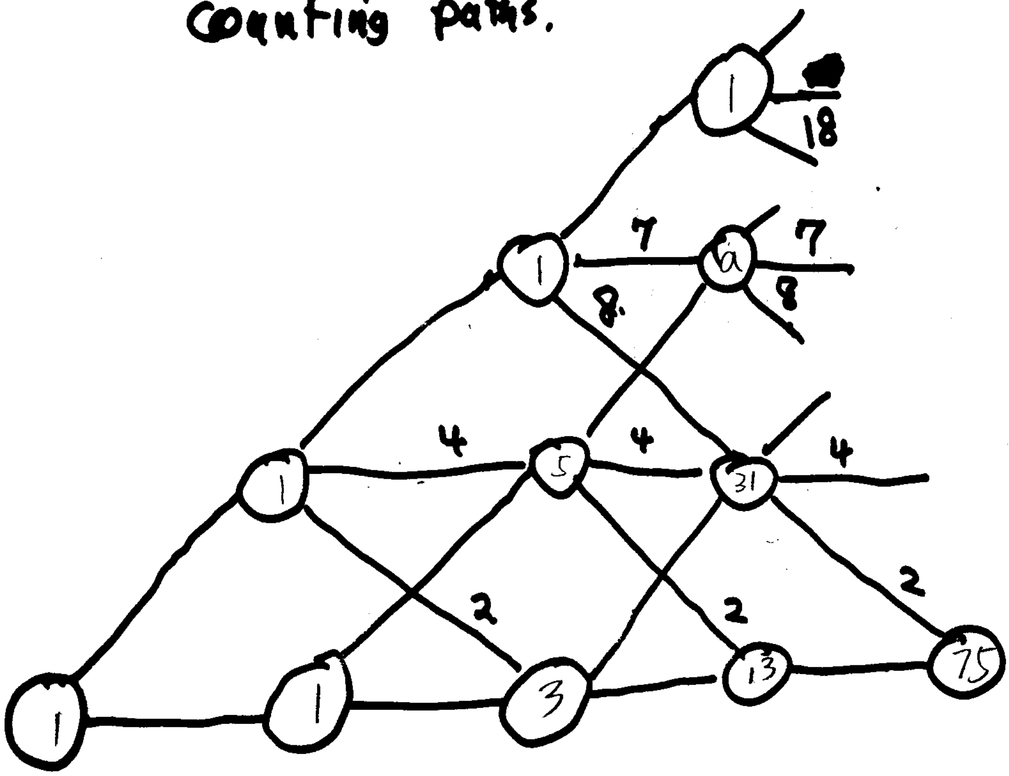
$R(z) = 1 + 3z + 2z^2$

$C(z) = 2z+1$

D.

$$P = \begin{bmatrix} 1 & 1 & & & & \\ 2 & 4 & 1 & & & 0 \\ & 8 & 7 & 1 & & \\ & & 18 & 10 & 1 & \\ 0 & & & 32 & 13 & \ddots \\ & & & & \ddots & \ddots \end{bmatrix}$$

E Combinatorial interpretation via counting paths.



The ordinary generating function in terms of continued fractions is

$$1 + z + 3z^2 + 13z^3 + 75z^4 + \cdots$$

$$= \cfrac{1}{1 - 1z - \cfrac{2z^2}{1 - 4z - \cfrac{8z^2}{1 - 7z - \cfrac{18z^2}{\cdots}}}}$$

done, but
worth 1 beer

Hint: Preferential arrangements.

Frank Schmidt
2 beers

open
1 beer and coauthorship

open

open

THANK YOU 謝謝

Homework and Open Problems

1 $$\frac{1}{\sqrt{5-4\cosh z}} = 1 + 2\frac{z^2}{2!} + 38\frac{z^4}{4!} + 1982\frac{z^6}{6!} + \cdots$$
$$=: \sum a_n z^n/n!$$

A. Interpret $\frac{1}{\sqrt{5-4\cosh z}}$ combinatorially.

B. Show that $a_{2n} \equiv 2 \pmod{9}$ for $n \geq 1$.

C. Give a combinatorial, rather than GF, proof of B.

2. Find the commutator subgroup of the Riordan group, the Appell subgroup, the Associated subgroup, the Derivative subgroup, the Hitting time subgroup, ...

3. Which q-analog gives areas under the weighted Motzkin paths?