

**Integer roots chromatic polynomials of nonchordal graphs
and the Prouhet-Tarry-Escott problem**

Santos Hernández *Florian Luca*

Dedicated to Professor Victor Neumann Lara at his sixtieth birthday

§1. The Problem and the Result. Let G be a finite simple graph without loops or multiple edges. A cycle $v_1v_2 \cdots v_rv_1$ in G is called a *chordless cycle* of G if $r \geq 4$ and no pair of non consecutive vertices of the cycle forms an edge of G . If G contains no chordless cycles, then G is called a *chordal graph*. Otherwise, it is called a *nonchordal graph*.

Recall that for a nonnegative integer x , the *chromatic polynomial* $P(G, x)$ evaluated in x of a graph G counts the number of ways of coloring the vertices of G using at most x colors and in such a way that no two vertices connected by an edge have the same color. It is known that $P(G, x)$ is a monic polynomial with integer coefficients in the variable x whose degree equals the number of vertices of G . It is also known that $P(G, x)$ has no real and negative roots. When G is a chordal graph, then all the roots of $P(G, x)$ are integers (see [7]). We denote by \mathcal{C} and $\bar{\mathcal{C}}$ the families of chordal and nonchordal graphs, respectively, and by \mathcal{L} the family of graphs whose chromatic polynomial has the property that all its roots integers. We then have that $\mathcal{C} \subseteq \mathcal{L}$. Since Read (see [6]) found a nonchordal graph G whose chromatic polynomial is $P(G, x) = x(x-1)(x-2)(x-3)^3(x-4)$, it follows that the containment $\mathcal{C} \subseteq \mathcal{L}$ is proper. Dmitriev (see [3]) constructed a family of nonchordal graphs in \mathcal{L} all of which have chordless cycles of order 4, and raised the following question.

Question. *Given any integer $p \geq 5$, does there exist a graph $G \in \mathcal{L}$ such that G has a chordless cycle of order p ?*

In the series of papers [4] and [5], Dong. et al. gave an affirmative answer to the above question for the cases $p = 4, 5, 6, 7$. In this paper, we extend the results from [4] and [5] to some larger values of p . Our result is the following.

Theorem. *The above question has an affirmative answer for $p = 8, 9, 10, 11, 13$.*

In order to prove the Theorem, we follow the construction from [5]. For positive integers $n \geq 2$ and k_1, \dots, k_n let G_{k_1, \dots, k_n} denote the graph obtained from the disjoint union of n complete graphs K_{k_1}, \dots, K_{k_n} on k_1, \dots, k_n vertices, respectively, and another vertex w , by adding edges joining each vertex in K_{k_i} with each vertex in $K_{k_{i+1}}$ for $i = 1, \dots, n-1$, and edges joining w to each vertex in K_{k_1} and K_{k_n} . From the construction, it is clear that when $n \geq 3$ the graph G_{k_1, \dots, k_n} is a nonchordal graph

2000 Mathematics Subject Classification: 05C15, 11B74, 11Y50.

Key words: Chromatic polynomials of graphs, the Prouhet-Tarry-Escott problem.

which contains a chordless cycle of order $n + 1$; e.g., any cycle of order $n + 1$ passing through w . Thus, via this construction, the authors of [5] reduced Dmitriev's question to the more particular question, namely: *Given any integer $n \geq 4$, do there exist positive integers k_1, \dots, k_n such that $G_{k_1, \dots, k_n} \in \mathcal{L}$?*

In [5], it is shown that the chromatic polynomial of G_{k_1, \dots, k_n} can be written as

$$P(G_{k_1, \dots, k_n}, x) = g(x) \cdot \left(\prod_{i=1}^n (x - k_i) - (-1)^n \prod_{i=1}^n k_i \right),$$

where $g(x)$ is a polynomial all whose roots are integers. Thus, in order to answer the last question in the affirmative, it suffices to show that for $n = 7, 8, 9, 10, 12$, there exist positive integers k_1, \dots, k_n so that all the roots of the polynomial $f(x) := \prod_{i=1}^n (x - k_i) - (-1)^n \prod_{i=1}^n k_i$ are integers.

§2. The Prouhet-Tarry-Escott Problem. This is a classical problem in diophantine equations which occurs in many guises. Given $n \geq 2$, the problem asks to find two distinct sets of integers $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ such that

$$\begin{aligned} \alpha_1 + \dots + \alpha_n &= \beta_1 + \dots + \beta_n, \\ \alpha_1^2 + \dots + \alpha_n^2 &= \beta_1^2 + \dots + \beta_n^2, \\ &\vdots \quad \quad \quad \vdots \\ \alpha_1^k + \dots + \alpha_n^k &= \beta_1^k + \dots + \beta_n^k. \end{aligned}$$

We call n the size of the solution and k the degree. We abbreviate the above system of equations by writing $[\alpha_i] =_k [\beta_i]$.

This problem has a long history. In 1750–51 Euler and Goldbach noted that

$$[a, b, c, a + b + c] =_2 [a + b, a + c, b + c].$$

A general solution of the problem in all degrees, but large sizes, came a century later, in 1851, when Prouhet found that there are n^{k+1} numbers separable into n sets, so that each pair of sets forms a solution of degree k and size n^k . Over the next 60 years, some more parametric and specific solutions of degrees two, three, four, and five were found. In the 1910's Tarry and Escott contributed at this problem and subsequently got their names were attached to it. For more details on the history of this problem we refer the reader to [1] and [2].

We start by noting that, via Newton's symmetric polynomial identities, we can formulate the Prouhet-Tarry-Escott problem in an equivalent way as follows:

Lemma. *The following are equivalent:*

$$\sum_{i=1}^n \alpha_i^j = \sum_{i=1}^n \beta_i^j, \quad \text{for } j = 1, 2, \dots, k. \quad (1)$$

$$\deg \left(\prod_{i=1}^n (x - \alpha_i) - \prod_{i=1}^n (x - \beta_i) \right) \leq n - (k + 1). \quad (2)$$

From (2) above, it follows that if $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ is a solution of degree k , then so is $\{A\alpha_1 + B, \dots, A\alpha_n + B\}$ and $\{A\beta_1 + B, \dots, A\beta_n + B\}$, where $A \neq 0$ and B are arbitrary integers. Such solutions are called equivalent. From (2), it also follows that if $N(k)$ is the least integer n such that there exists a solution of size n and degree k , then $N(k) \geq k + 1$. In what follows, we deal only with *ideal* solutions, namely those of degree k and size $k + 1$. Ideal solutions are of the particular interest since they are ‘minimal’ solutions to the problem. Observe that an ideal solution has the property that

$$\prod_{i=1}^n (x - \alpha_i) - \prod_{i=1}^n (x - \beta_i) = C \quad (3)$$

holds with some nonzero constant C . By the previous remark, and up to a linear transformation, we will assume that all the components of our ideal solutions are nonnegative, and that one of them is zero.

§3. The Connection. Here, we point out an immediate connection between the preceding two sections. Namely, that for some $n \geq 3$, there exist positive integers k_1, \dots, k_n such that $G_{k_1, \dots, k_n} \in \mathcal{L}$ if and only if the Prouhet-Tarry-Escott problem has an ideal solution of degree $n - 1$.

Suppose first that k_1, \dots, k_n are positive integers such that $f(x) := \prod_{i=1}^n (x - k_i) - (-1)^n \prod_{i=1}^n k_i$ is a polynomial whose roots are all integers, and write $f(x) = c \prod_{i=1}^n (x - \alpha_i)$ where c and $\alpha_i \in \mathbf{Z}$ for $i = 1, \dots, n$. Identifying leading coefficients, we get $c = 1$. Evaluating $f(x)$ in $x := 0$, we get $f(0) = 0$, and so we may assume that $\alpha_1 = 0$. Furthermore, since $f(x)$ is a factor of the chromatic polynomial $P(G_{k_1, \dots, k_n}, x)$ of G_{k_1, \dots, k_n} whose real roots are nonnegative, it follows that $\alpha_i \geq 0$ holds for $i = 1, \dots, n$. Thus,

$$\prod_{i=1}^n (x - \alpha_i) - \prod_{i=1}^n (x - k_i) = (-1)^n \prod_{i=1}^n k_i, \quad (4)$$

and from (3) and (4) we see that $\{\alpha_1, \dots, \alpha_n\}$ and $\{k_1, \dots, k_n\}$ is an ideal solution of the Prouhet-Tarry-Escott problem.

Conversely, let $\{\alpha_1, \dots, \alpha_n\}$ and $\{k_1, \dots, k_n\}$ be an ideal solution of the Prouhet-Tarry-Escott problem. Assume that $\alpha_1 = 0$. Then

$$\prod_{i=1}^n (x - \alpha_i) - \prod_{i=1}^n (x - k_i) = C$$

holds with some constant C . Since $\alpha_1 = 0$, we may evaluate the above relation in $x := 0$ and obtain $C = (-1)^{n+1} \prod_{i=1}^n k_i$. Therefore, the polynomial $\prod_{i=1}^n (x - k_i) -$

$(-1)^n \prod_{i=1}^n k_i$ is a polynomial whose roots are the integers $\alpha_1, \dots, \alpha_n$, and therefore $G_{k_1, \dots, k_n} \in \mathcal{L}$.

§4. The Proof of the Theorem. The proof is an immediate consequence of the known fact that the Prouhet-Tarry-Escott problem admits ideal solutions for all sizes $2 \leq n \leq 12$, except for $n = 11$ where no solution is known (see [1]). Here, we list some ideal solutions for the cases $n = 7, 8, 9, 10$ and 12 together with the names of their discoverers.

$$[0, 18, 27, 58, 64, 89, 101] =_6 [1, 13, 38, 44, 75, 84, 102] \text{ (E.B. Escott, 1910).}$$

$$[0, 4, 9, 23, 27, 41, 46, 50] =_7 [1, 2, 11, 20, 30, 39, 48, 49] \text{ (G. Tarry, 1913).}$$

$$[0, 24, 30, 83, 86, 133, 157, 181, 197] =_8 [1, 17, 41, 65, 112, 115, 168, 174, 198] \\ \text{(A. Letac, 1940's).}$$

$$[0, 3083, 3301, 11893, 23314, 24186, 35607, 44199, 44417, 47500] =_9 \\ [12, 2865, 3519, 11869, 23738, 23762, 35631, 43981, 44635, 47488] \text{ (A. Letac, 1940's)}$$

$$[0, 11, 24, 65, 90, 129, 173, 212, 237, 278, 291, 302] =_{11}$$

$$[3, 5, 30, 57, 104, 116, 186, 198, 245, 272, 297, 299]$$

(N. Kuosa, J.C. Merignac, C. Shuwen, 1999).

Concluding Remark. While it is likely that Dmitriev's question has an affirmative answer for all $p \geq 4$, in this note we merely pointed out that the construction for candidates G which might provide an affirmative answer to this question for a suitable choice of parameters k_1, \dots, k_n employed in [5], while clever, reduces to the question of the existence of ideal solutions of size n to the Prouhet-Tarry-Escott problem, which, in turn, is likely to be even more difficult to decide than Dmitriev's original question. It could be that a different construction of 'parametric' graphs than the one in [5] might yield better results towards an affirmative answer to Dmitriev's question.

Acknowledgements

The first author was partly supported by a CONACyT Doctoral Fellowship. The second author was partly supported by Grants SEP-CONACyT 37259-E and PAPIIT IN-104602 from the UNAM.

References

- [1] Borwein, P. *Computational Excursions in Analysis and Number Theory*. CMS Books in Mathematics, Springer-Verlag New York, Inc. (2002).
- [2] Borwein, P., Ingalls, C. *The Prouhet-Tarry-Escott Problem Revisited*. L'Enseignement Mathématique 40 (1994), 3-27.
- [3] Dmitriev, I.G. *Weakly cyclic graphs with integral chromatic number*. Metody Diskret. Analiz. (In Russian), 34 (1980), 3-7.
- [4] Dong, F.M., Koh, K.M. *Non-chordal graphs having integral-root chromatic polynomials*. Bull. Combin. Appl. 22 (1998), 67-77.
- [5] Dong, F. M., Teo, K.L., Koh, K.M., Hendy, M.D. *Non-chordal graphs having integral-root chromatic polynomials II*. Discrete Mathematics 245 (2002), 247-253.
- [6] Read, R.C. *Review*. Math. Rev. 50 6909, (1975).
- [7] Read, R.C., Tutte, W.T. *Chromatic polynomials*. Selected Topics in Graph Theory 3, Academic Press, New York, (1988), 15-42.

Mathematical Institute, UNAM
Ap. Postal 61-3 (Xangari), CP 58 089
Morelia, Michoacán, Mexico

shh@matmor.unam.mx
fluca@matmor.unam.mx