Some Results on Enumeration of Schröder Paths with Flaws

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   - Large and small Schröder numbers
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Introduction

The (*large*) *Schröder numbers*

$$\{r_n\}_{n\geq 0} = \{1, 2, 6, 22, 90, 394, \dots\}$$

have been found in many combinatorial configurations, e.g.,

- unitary-binary trees with all branches of odd length,
- plane trees with leaves colored in two colors,
- Royal paths, from $(0,0)$ to $(2n,0)$ with steps $\{(1,1),(1,-1),(2,0)\}$

The *small Schröder numbers*

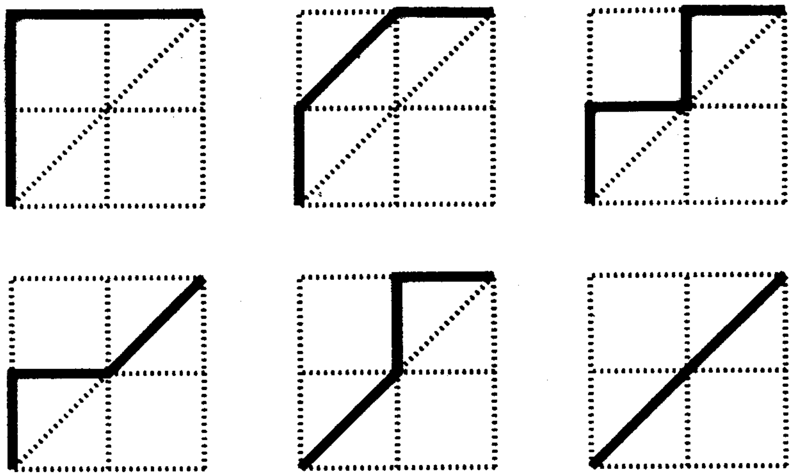
$$\{s_n\}_{n\geq 0} = \{1, 1, 3, 11, 45, 197, \dots\}$$

appears in

- arbitrary word bracketings,
- plane rooted trees without vertex of degree 1,
- convex polygon dissections

Large/small Schröder paths

The m-th Schröder number r_m counts the number of paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,0)$ to (m,m) with the north step $(0,1)$, east step $(1,0)$ and diagonal step $(1,1)$ that never pass below the line $y=x$. Such paths are called *(large) m-Schröder paths.*



the 2-Schröder paths

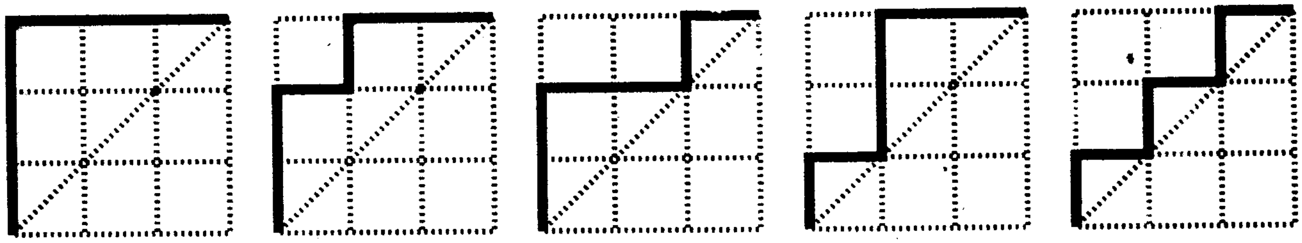
Note that

$$r_m = 2s_m \quad \text{for } m \geq 1.$$

The m-th small Schröder number s_m counts the number of m-Schröder paths without diagonal steps on the line $y=x$.

Catalan paths

The m-th Catalan number $c_m = \frac{1}{m+1}\binom{2m}{m}$ counts the number of paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,0)$ to (m,m) with the *north* step $(0,1)$ and *east* step $(1,0)$ that never pass below the line $y=x$. Such paths are called the *Catalan paths of semilength* m (or *m-Catalan paths*).



the 3-Catalan paths

The generating function of Catalan numbers $C = C(x) = \sum_{n\geq 0} c_n x^n$ satisfies the equation

$$C = 1 + xC^2.$$

Taylor expansions

By a *Taylor expansion*, the generating function is expanded in a way the remainder of which is expressed as a function of the generating function itself.

For example, by iteration, the initial expansions of C are

$$\begin{aligned} C &= 1 + xC^2 \\ &= 1 + x + x^2(C^2 + C^3) \\ &= 1 + x + 2x^2 + x^3(2C^2 + 2C^3 + C^4) \\ &= 1 + x + 2x^2 + 5x^3 \\ &\qquad + x^4(5C^2 + 5C^3 + 3C^4 + C^5). \end{aligned}$$

To see this, the second expansion of C is derived as follows:

$$\begin{aligned} C &= 1 + xC^2 \\ &= 1 + xC(1 + xC^2) \\ &= 1 + xC + x^2C^3 \\ &= 1 + x(1 + xC^2) + x^2C^3 \\ &= 1 + x + x^2(C^2 + C^3). \end{aligned}$$

Motivations

In fact, the n-th Taylor expansion of C can be expressed in the form

$$C = \sum_{i=0}^{n-1} c_i x^i + x^n F_n(C),$$

in which $x^n F_n(C)$ is called the *remainder*.

Questions:

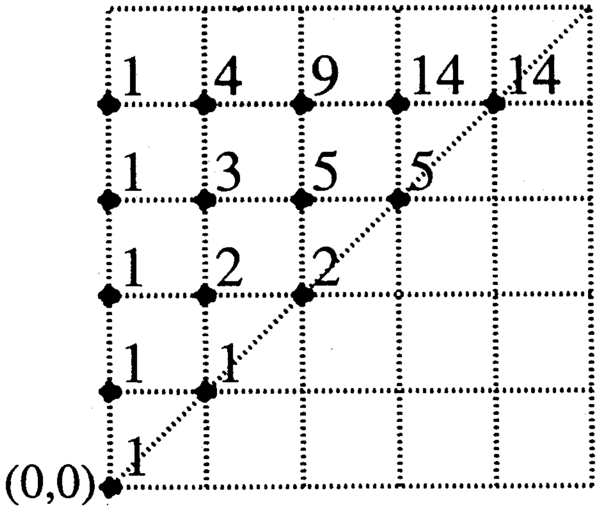
1. How to determine the coefficients of $F_n(z)$?

2. What are their combinatorial interpretations?

3. What are the expansions for other numbers, e.g., Motzkin numbers, Schröder numbers, etc.?

Observations

For $q \geq p \geq 0$, the number at entry (p, q) of the following array indicates the number $\alpha_{p,q}$ of paths from $(0, 0)$ to (p, q) with the steps $\{(0, 1), (1, 0)\}$ that never pass below the line $y = x$.



y=n-1

Remark: The entries in the $(n-1)$-th row coincide with the coefficients of the remainder $F_n(z)$ of the n-th expansions of C.

Theorem. (Eu, Liu & Yeh) *The n-th Taylor expansion of the C can be expressed in the form*

$$C = \sum_{i=0}^{n-1} c_i x^i + x^n F_n(C),$$

where $F_n(z) = \sum_{p=0}^{n-1} \frac{n-p}{n+p}\binom{n+p}{p} z^{n-p+1}$.

Remarks:

- The *ballot numbers:*

$$[x^n]C^k = \frac{k}{2n+k}\binom{2n+k}{n}.$$

- $\alpha_{p,q} = [x^p]C^{q-p+1}$.

- $F_n(z) = \sum_{p=0}^{n-1} \alpha_{p,n-1} z^{n-p+1}$.

Taylor expansions of R

The generating function of Schröder numbers $R = R(x) = \sum_{n\geq 0} r_n x^n$ satisfies the equation

$$R = 1 + xR + x\ R^2.$$

By iteration, we have

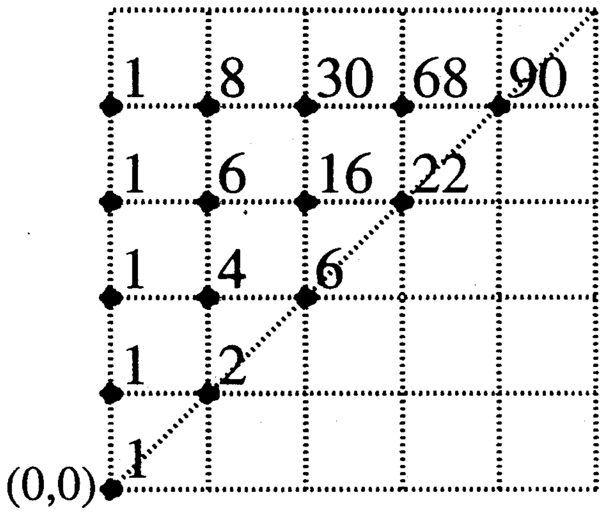
$$\begin{aligned} R &= 1 + xR + xR^2 \\ &= 1 + 2x + x^2(2R + R^2) + x^2(2R^2 + R^3) \\ &= 1 + 2x + 6x^2 + x^3(6R + 4R^2 + R^3) \\ &\qquad +x^3(6R^2 + 4R^3 + R^4). \end{aligned}$$

In general, the n-th Taylor expansion of R can be expressed in the form

$$R = \sum_{i=0}^{n-1} r_i x^i + x^n(G_n(R) + G_n(R)R).$$

For large Schröder paths

For $q \geq p \geq 0$, the number at entry (p, q) indicates the number $\beta_{p,q}$ of paths from $(0, 0)$ to (p, q) with the steps $\{(0, 1), (1, 0), (1, 1)\}$ that never pass below the line $y = x$.



- $\beta_{p,q} = [x^p]R^{q-p+1}$.

- The coefficients of $G_n(z)$ in the remainder of the n-th expansions of R can be obtained from the array.

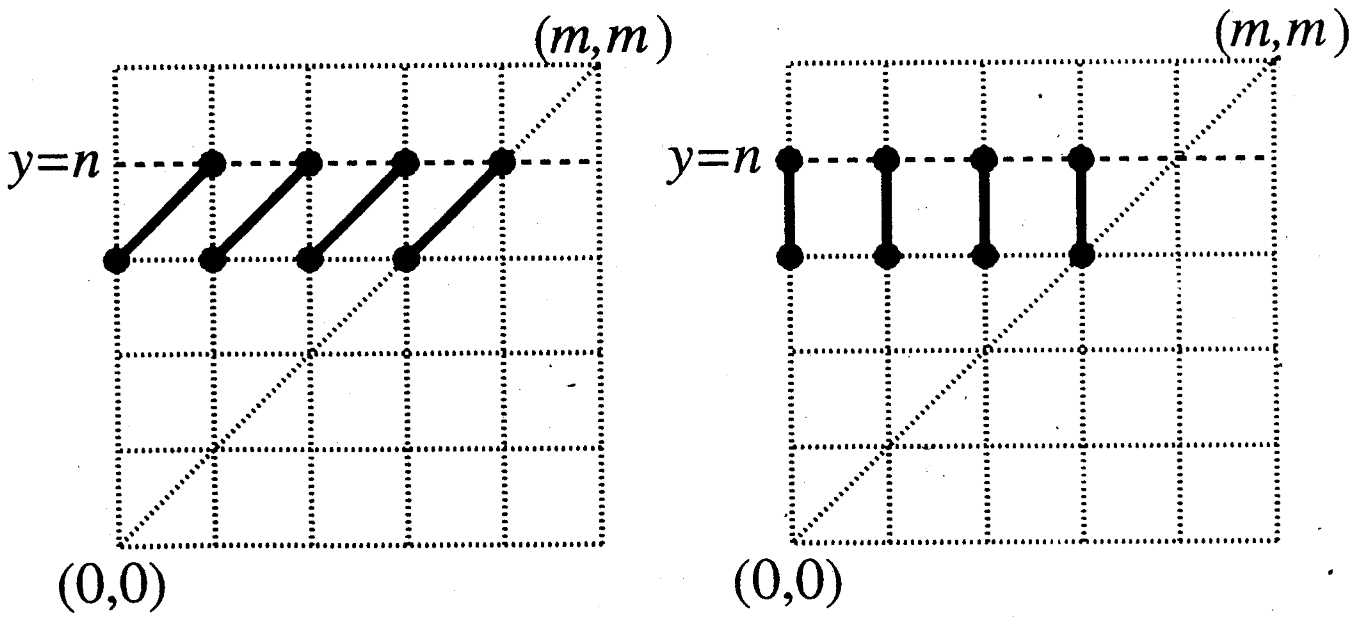
For large Schröder numbers

Theorem. *The n-th Taylor expansion of R can be expressed in the form*

$$R = \sum_{i=0}^{n-1} r_i x^i + x^n (G_n(R) + G_n(R)R),$$

where

$$G_n(z) = \sum_{k=1}^{n} g_{n,k} z^k \text{ and } g_{n,k} = [x^{n-k}]R^k.$$



$x^n G_n(R)$ $\qquad$ $x^n G_n(R)R$

For small Schröder numbers

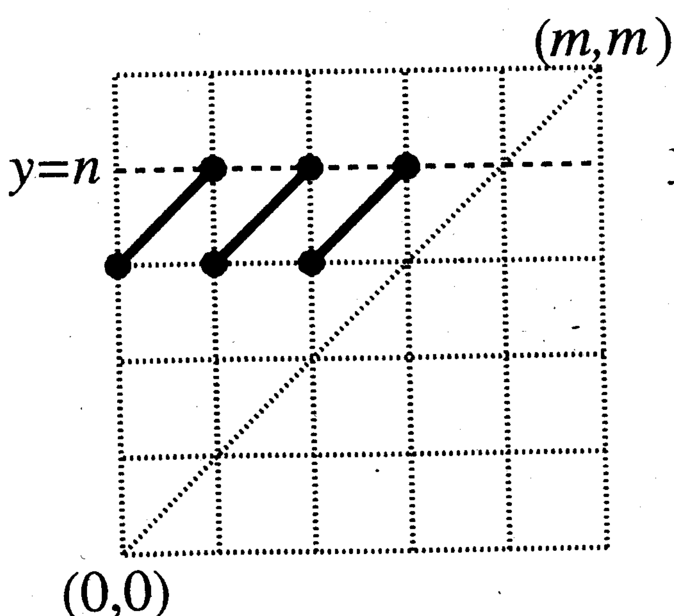
Theorem. *The n-th Taylor expansion of S can be expressed in the form*

$$S = \sum_{i=0}^{n-1} s_i x^i + x^n (H_n(R)S + H_n^*(R)S).$$

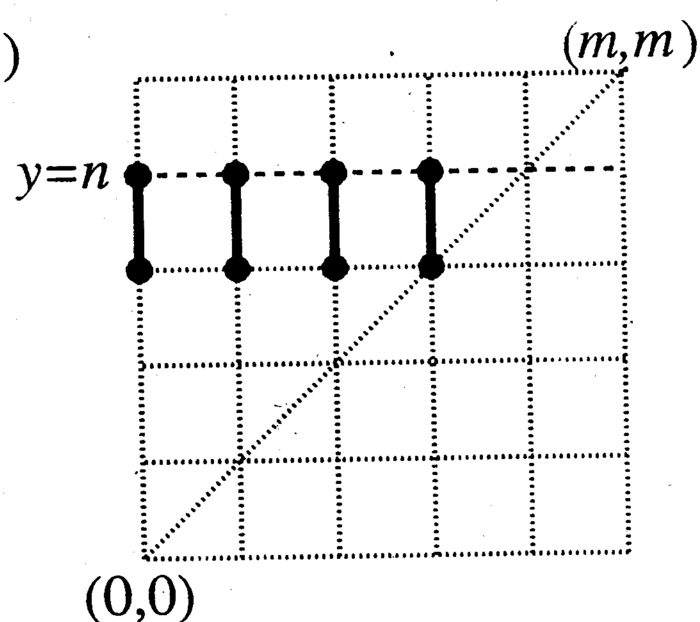
where

$$H_n(z) = \sum_{k=1}^{n} h_{n,k} z^k, H_n^*(z) = \sum_{k=2}^{n} h_{n,k} z^{k-1}$$

and $h_{n,k} = [x^{n-k}]\{R^{k-1}S\}$.



$x^n H_n^*(R) S$

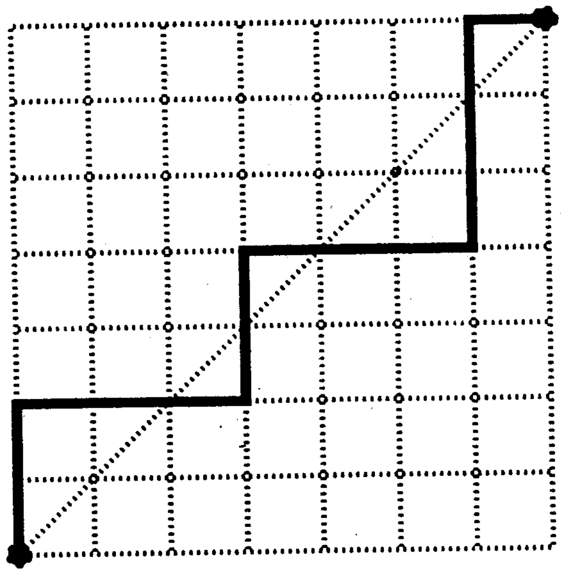


$x^n H_n(R) S$

Flaws

By *flaws* of a path π, we mean the steps of π that fall below the line $y = x$.

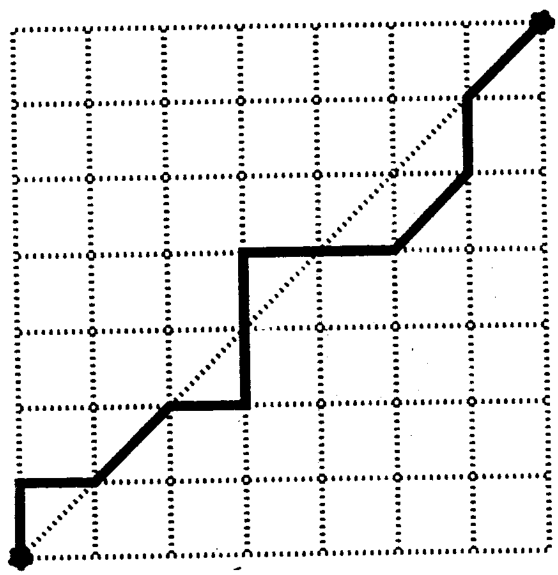
A path from $(0, 0)$ to (m, m) with the steps $\{(1, 0), (0, 1)\}$ is called an *m-Catalan path with n flaws* if the flaws of which, when concatenated and flipped over the line $y = x$, correspond to an n-Catalan path.



a 7-Catalan path with 3 flaws

Schröder paths with flaws

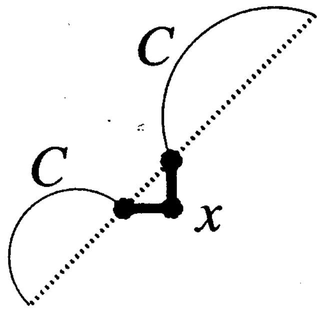
A path from $(0,0)$ to (m,m) with the steps $\{(1,0),(0,1),(1,1)\}$ is called an *m-Schröder path with n flaws* if the flaws of which, when concatenated and flipped over the line $y = x$, correspond to a small n-Schröder path.



a 7-Schröder path with 3 flaws

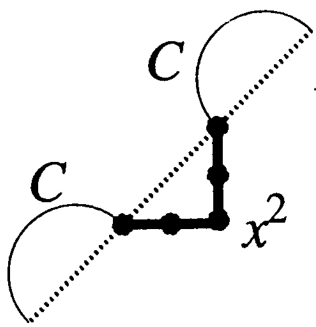
Enumeration of Catalan paths with flaws

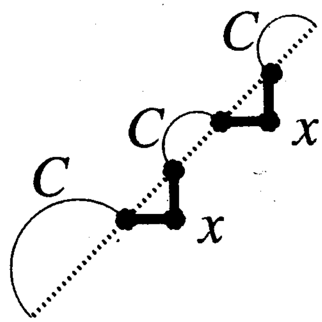
The m-Catalan paths with 1-flaw are in the form:



generating function: xC^2

The m-Catalan paths with 2 flaws are in one of the forms:





generating function: $x^2(C^2 + C^3)$

The generating functions for the number of m-Catalan paths with n flaws:

n	generating functions
1	xC^2
2	$x^2(C^2+C^3)$
3	$x^3(2C^2+2C^3+C^4)$
4	$x^4(5C^2+5C^3+3C^4+C^5)$

Theorem. *The generating function for the number of m-Catalan paths with n flaws coincides with the remainder of the n-th Taylor expansion of C.*

Classic Chung-Feller Theorem

Theorem. **(Chung-Feller)** *The number of Catalan paths of semilength m with n flaws is c_m $(0 \leq n \leq m)$, independent of n.*

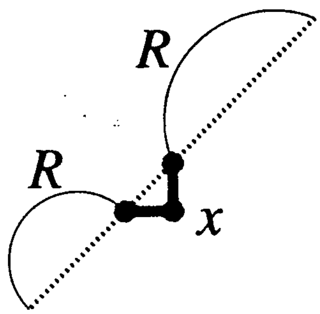
To see this, by the Taylor expansion of C

$$
\begin{aligned}
c_m = [x^m]C &= [x^m]\{xC^2\} \\
&= [x^m]\{x^2(C^2 + C^3)\} \\
&= [x^m]\{x^3(2C^2 + 2C^3 + C^4)\} \\
&= [x^m]\{x^4(5C^2 + 5C^3 + 3C^4 + C^5)\}
\end{aligned}
$$

Q: How to establish the bijection?

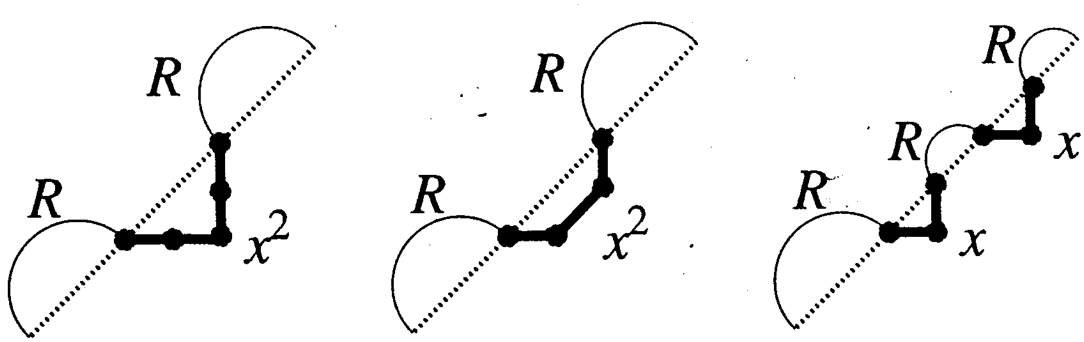
Enumeration of Schröder paths with flaws

An m-Schröder paths with 1 flaw is in the form:



generating function: xR^2

An m-Schröder paths with 2 flaws are in one of the forms:



generating function: $x^2(2R^2 + R^3)$

The generating functions for the number of m-Schröder paths with n flaws:

n	generating functions
1	xR^2
2	$x^2(2R^2+R^3)$
3	$x^3(6R^2+4R^3+R^4)$
4	$x^4(22R^2+16R^3+6R^4+R^5)$

Theorem. *The generating function for the number of m-Schröder paths with n flaws is*

$$x^n(G_n(R)R),$$

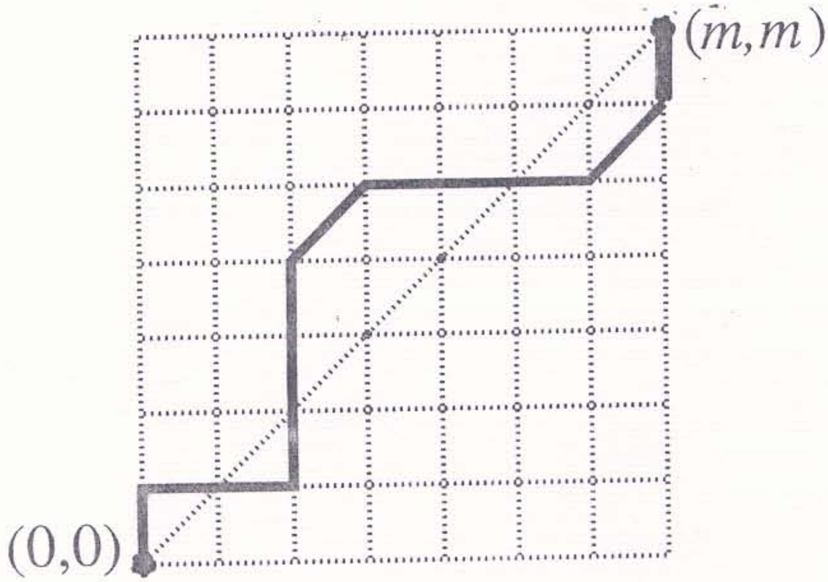
where

$$G_n(z) = \sum_{k=1}^{n} g_{n,k} z^k \text{ and } g_{n,k} = [x^{n-k}]R^k.$$

Remark: There is no Chung-Feller property for Schröder paths with flaws.

'Artificial' Chung-Feller Property

We create a variation of Chung-Feller theorem regarding flaws with coloring for large Schröder paths:



If we assign two possible colors to the north step from $(m, m-1)$ to (m, m) then the generating function for the number of large m-Schröder paths with n flaws will become

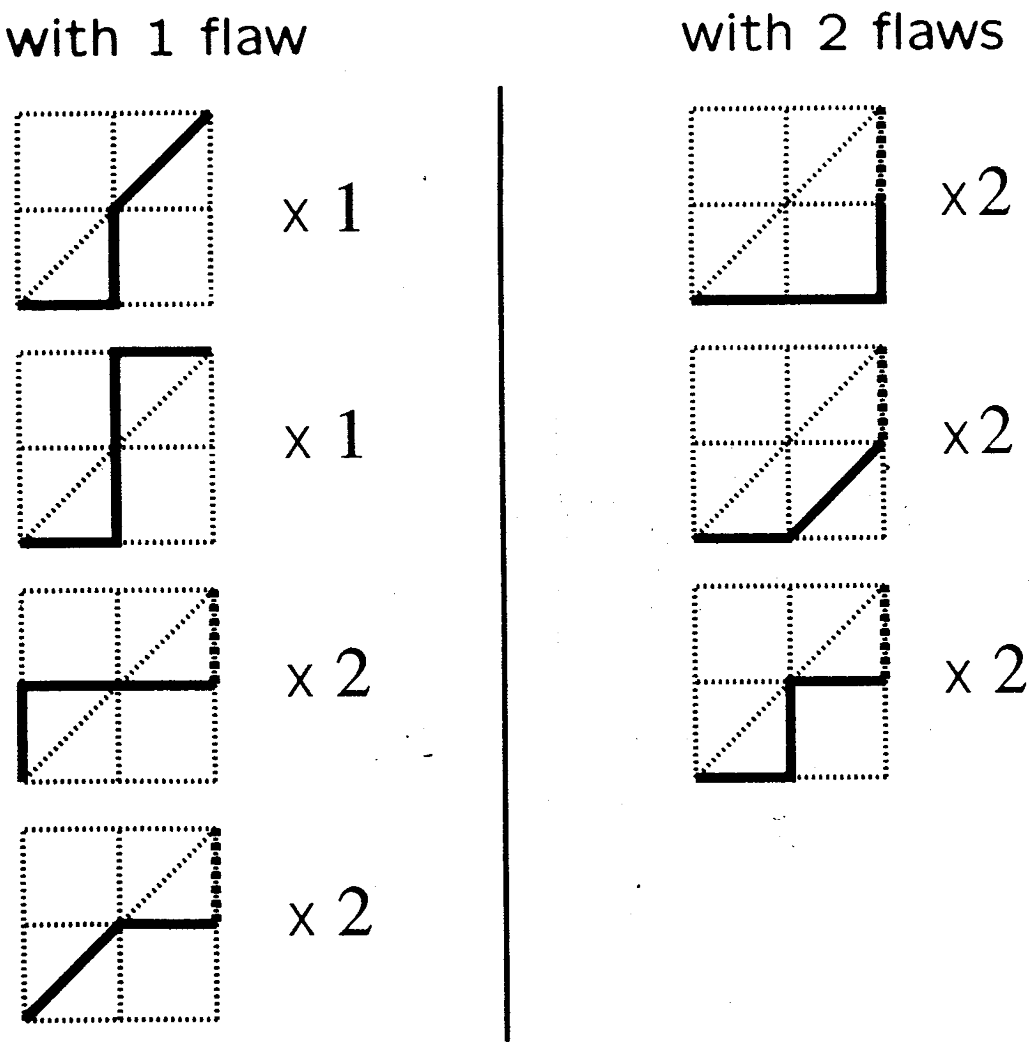
$$x^n(G_n(R)(1+R)),$$

which coincides with the remainder of the n-th expansion of R.

For example $r_2 = 6$

The number of 2-Schröder paths is 6.

The set of 2-Schröder paths with flaws are partitioned in the following two classes.



A fast enumeration of Schröder paths with flaws

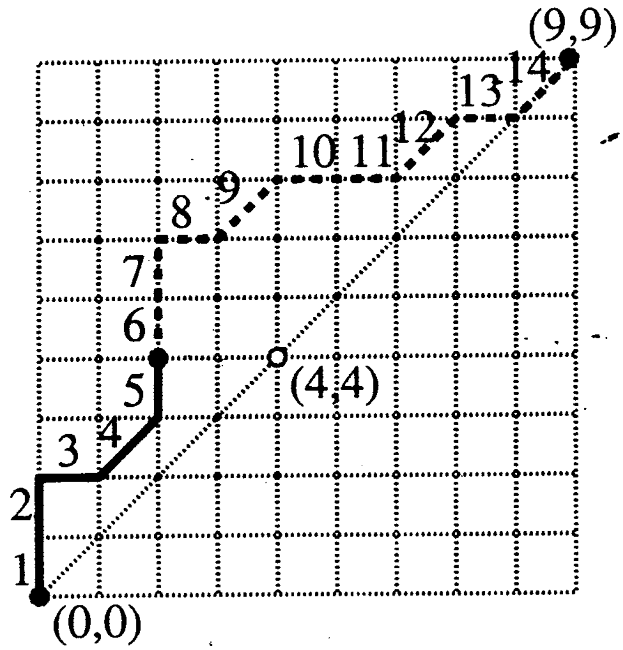
Theorem. *The number of m-Schröder paths with n flaws is*

$$r_m - \sum_{k=1}^{n} s_{k-1} r_{m-k}.$$

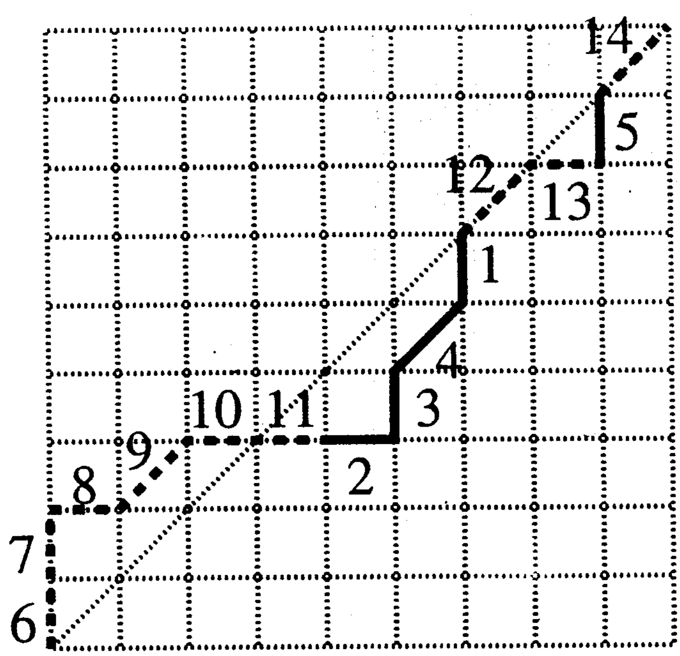
This formula is obtained by establishing a bijection between the following two sets.

Theorem. *The number of m-Schröder paths with an n-flaw is equal to the number of m-Schröder paths without diagonal steps in the segment from $(0,0)$ to (n,n) of the line $y=x$.*

Bijection



A 9-Schröder path without diagonal steps in the segment $(0,0)$ to $(4,4)$ of the line $y = x$



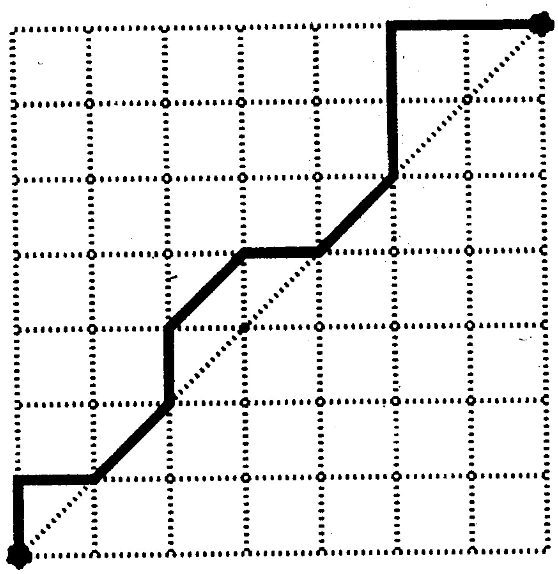
A 9-Schröder path with 4 flaws

Statistic returns of paths

A *return* (to the line $y = x$) of a path occurs whenever there is an east step that goes from line $y = x + 1$ to line $y = x$.

An *extended return* of a large Schröder path is either a diagonal step on the line $y = x$ or an east step that goes from line $y = x + 1$ to line $y = x$.

For example, the path shown below has 3 returns and 5 extended returns.



Enumeration results

paths	total number of returns
n-Catalan	$\frac{3n}{n+1}$ (by Deutsch & Shapiro)
small n-Schröder	$\frac{1}{2}(s_{n+1} - s_n)$
large n-Schröder	$r_{n+1} - 3r_n + r_{n-1}$ $r_{n+1} - 2r_n$ (extended)

The asymptotic growing rate of the large (or small) Schröder numbers is

$$\lim_{n\to\infty} \frac{r_{n+1}}{r_n} = \lim_{n\to\infty} \frac{s_{n+1}}{s_n} = 3 + 2\sqrt{2}.$$

Asymptotic average behaviors

paths	expected number of returns
Catalan	3
small Schröder	$1 + \sqrt{2}$
large Schröder	3 $1 + 2\sqrt{2}$ (extended)