Euler-Mahonian Statistics on Ordered Partitions and q-Meixner Polynomials

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Ordered partitions

Set $[n] = \{1, \ldots, n\}$.

$$\pi = \{2,9\}/\{3\}/\{1,4,8\}/\{5,6\}/\{7\}$$

is an ordered partition of $[9]$ with 5 blocks.

$$\#\text{of partitions of } [n] \text{ into } k \text{ blocks} = S(n,k)$$

Stirling numbers of the 2nd kind. So

$$\#\text{ of ordered partitions of } [n] \text{ with } k \text{ blocks} = k!S(n,k)$$

The total number of $\mathcal{OP}$ of $[n]$ is equal to

$$\tilde{B}_n = \sum_k k!S(n,k) \Longrightarrow \sum_{n\geq 0} \tilde{B}_n \frac{z^n}{n!} = \frac{1}{2-e^z}.$$

Preferential arrangements

A $\mathcal{PA}$ of $[n] := \{1, 2, \ldots, n\}$ is a *permutation* σ of $[n]$ with some ascents ($\sigma_i < \sigma_{i+1}$) underlined.

e.g. $\sigma = 2\,9\,3\,1\,4\,8\,5\,\underline{6}\,7$

is a preferential arrangement of $[9]$.

Each preferential arrangement can be identified with an **ordered partition** of $[n]$ by creating a new block after each descent and underlined ascent. For the above example,

$$\sigma \mapsto \pi_\sigma = \{2,9\}/\{3\}/\{1,4,8\}/\{5,\underline{6}\}/\{7\}$$

is an ordered partition of $[9]$ with 5 blocks.

$\mathcal{PA} \longleftrightarrow \mathcal{OP}$ for $n = 3$

$$1\,2\,3 \longrightarrow \{1,2,3\}, \qquad \underline{1}\,2\,3 \longrightarrow \{1\}/\{2,3\},$$

$$1\,\underline{2}\,3 \longrightarrow \{1,2\}/\{3\}, \qquad \underline{1}\,\underline{2}\,3 \longrightarrow \{1\}/\{2\}/\{3\},$$

$$2\,1\,3 \longrightarrow \{2\}/\{1,3\}, \qquad 2\,\underline{1}\,3 \longrightarrow \{2\}/\{1\}/\{3\},$$

$$2\,3\,1 \longrightarrow \{2,3\}/\{1\}, \qquad \underline{2}\,3\,1 \longrightarrow \{2\}/\{3\}/\{1\},$$

$$1\,3\,2 \longrightarrow \{1,3\}/\{2\}, \qquad \underline{1}\,3\,2 \longrightarrow \{1\}/\{3\}/\{2\},$$

$$3\,1\,2 \longrightarrow \{3\}/\{1,2\}, \qquad 3\,\underline{1}\,2 \longrightarrow \{3\}/\{1\}/\{2\},$$

$$3\,2\,1 \longrightarrow \{3\}/\{2\}/\{1\}.$$

The Eulerian polynomials are defined by

$$A_n(x) = \sum_{\sigma \in S_n} x^{1+|\{i:\sigma(i)>\sigma(i+1)\}|}$$

or

$$\sum_{n\geq 0} A_n(x)\frac{z^n}{n!} = \frac{1-x}{1-xe^{(1-x)z}}.$$

A mapping $\varphi : S_n \longrightarrow \mathbb{N}$ is **Eulerian** if

$$\sum_{\sigma \in S_n} x^{\varphi(\sigma)} = A_n(x),$$

is **Mahonian** if

$$\sum_{\sigma \in S_n} x^{\varphi(\sigma)} \quad = \quad 1 \cdot (1+q) \cdots (1+q+\cdots+q^{n-1}) := [n]_q!.$$

Foata, Zeilberger, Simion, Stanton, ...

Let $\mathcal{OP}_n$ be the set of $\mathcal{OP}$s of $[n]$ identified with $\mathcal{PA}$. Then a mapping

$$\mathrm{mah} : \mathcal{OP}_n \longrightarrow \mathbb{N}$$

is called *Mahonian* if

$$\sum_{\pi \in \mathcal{OP}_n} q^{\mathrm{mah}\,\pi} u^{\mathrm{Nun}(\pi)} = \sum_{k \geq 1} [k]! S_q(n,k) u^{n-k}, \qquad (1)$$

where $\mathrm{Nun}(\pi)$ is the number of non-underlined ascents in π.

E. Steingrímsson made some conjectures.

Remark: When $u \to 0$

$$\sum_{k=1}^{n} [k]_q! S_q(n,k) u^{n-k} \longrightarrow [n]_q!$$

The q-Meixner polynomials

Let

$$[b;n]_q = b + q + q^2 + \cdots + q^{n-1}, \quad [1;n]_q = [n]_q,$$

and

$$[b;n]_q! = [b;n]_q[b;n-1]_q \cdots [b;1]_q, \quad [1;n]_q! = [n]_q!.$$

The polynomial

$$M_n(x;b,u;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-u)^{n-k} q^{\binom{n-k}{2}} \prod_{i=1}^{n-k} (b-1+[k+i]_q) \prod_{i=0}^{k-1} (x-[i]_q).$$

is a rescaled version of classical q-Meixner polynomials :

$${}_2\phi_1 \left(\begin{matrix} q^{-n}, & x \\ bq & \end{matrix} \middle| q; -\frac{q^{n+1}}{c} \right).$$

When $b \to \infty$ the polynomial $M_n(x; b, u/b; q)$ reduces to *q*-Charlier polynomial:

$$C_n(x; u) = \sum_{k=0)}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (-u)^{n-k} q^{\binom{n-k}{2}} \prod_{i=0}^{k-1} (x - [i]_q).$$

When $u \to \infty$ the polynomial $u^{-n} M_n(ux; b, u; q)$ reduces to *q*-Laguerre polynomial:

$$L_n(x; b) = \sum_{k=0)}^{n} (-1)^{n-k} \begin{bmatrix} n \\ k \end{bmatrix}_q q^{\binom{n-k}{2}} \prod_{i=1}^{n-k} (b - 1 + [k+i]_q).$$

The combinatorics of these polynomials have been studied by Simion, Stanton, Wachs , White, etc.

If $\{p_n(x)\}$ is a sequence of monic orthogonal polynomials with respect to a linear functionnal $\mathcal{L} : \mathbb{C}[x] \to \mathbb{C}$:

$$\mathcal{L}(p_n p_m) = 0 \quad \text{if} \quad n \neq m,$$

then there exist b_n and λ_n such that

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x), \quad n \geq 0$$

with $p_0(x) = 1$ and $p_{-1}(x) = 0$.

The moments $\mu_n = \mathcal{L}(x^n)$ satisfy

$$\sum_{n\geq 0} \mu_n t^n = \cfrac{1}{1 - b_0\, t - \cfrac{\lambda_1\, t^2}{1 - b_1\, t - \cfrac{\lambda_2\, t^2}{\ddots}}}$$

The q-Meixner polynomials $M_n(x) := M_n(x; b, u; q)$ satisfy

$$M_{n+1}(x) = (x - b_n)M_n(x) - \lambda_n M_{n-1}(x)$$

where $M_{-1}(x) = 0$, $M_0(x) = 1$ and

$$\begin{cases} b_n = uq^n(b + q[n]_q) + [n]_q(1 + uq^n) \\ \lambda_n = uq^{n-1}[n]_q(b + q[n-1]_q)(1 + uq^n). \end{cases} \tag{2}$$

So the moments $\mu_n(q)$ of q-Meixner polynomials satisfy

$$\sum_{n\geq 0} \mu_n(q)t^n = \cfrac{1}{1 - b_0\,t - \cfrac{\lambda_1\,t^2}{1 - b_1\,t - \cfrac{\lambda_2\,t^2}{\ddots}}}$$

where b_n and λ_n are as in (2).

There are two classical q-Stirling numbers of second kind $S_q(n,k)$ and $\tilde{S}_q(n,k)$:

$$S_q(n,\, k) = S_q(n-1,\, k-1) + [k]_q S_q(n-1,\, k);$$

with $S_q(n,k) = \delta_{n\,k}$ if $n = 0$ or $k = 0$, and

$$\tilde{S}_q(n,k) = q^{\binom{n}{2}} S_q(n,k).$$

Theorem 1 *The n-th moment of q-Meixner polynomials is equal to*

$$\mu_n(q) = \sum_{k=1}^{n} [b;k]_q!\, S_q(n,k) u^k.$$

Sketch of Proof.

$$\sum_{n \geq k} S_q(n,k)\, z^n = \frac{z^k}{(1-z)(1-[2]_q\, z)\cdots(1-[k]_q\, z)}.$$

So

$$\begin{aligned} f(b,z) &:= \sum_{n,k} [b;k]_q! S_q(n,k)\, u^k\, z^n \\ &= \sum_{k \geq 0} \frac{[b;k]_q!\, u^k\, z^k}{(1-z)(1-[2]_q\, z)\cdots(1-[k]_q\, z)}. \end{aligned}$$

It follows that

$$f(b,z) = 1 + \frac{ubz}{1-z} f\left(\frac{b}{q}+1,\ \frac{qz}{1-z}\right)$$

Suppose that

$$f(b,z) = \cfrac{1}{1 - \cfrac{c_1(b)\,z}{1 - \cfrac{c_2(b)\,z}{\ddots}}}$$

We derive that

$$c_{2i+1}(b) = uq^i(b + q + \cdots q^i), \qquad c_{2i}(b) = [i]_q(1 + uq^i) \qquad i \geq 0.$$

Therefore, by contraction we obtain

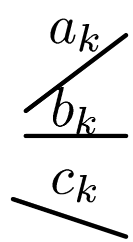
$$f(b,z) = \cfrac{1}{1 - b_0\,z - \cfrac{\lambda_1\,z^2}{1 - b_1\,z - \cfrac{\lambda_2\,z^2}{\ddots}}},$$

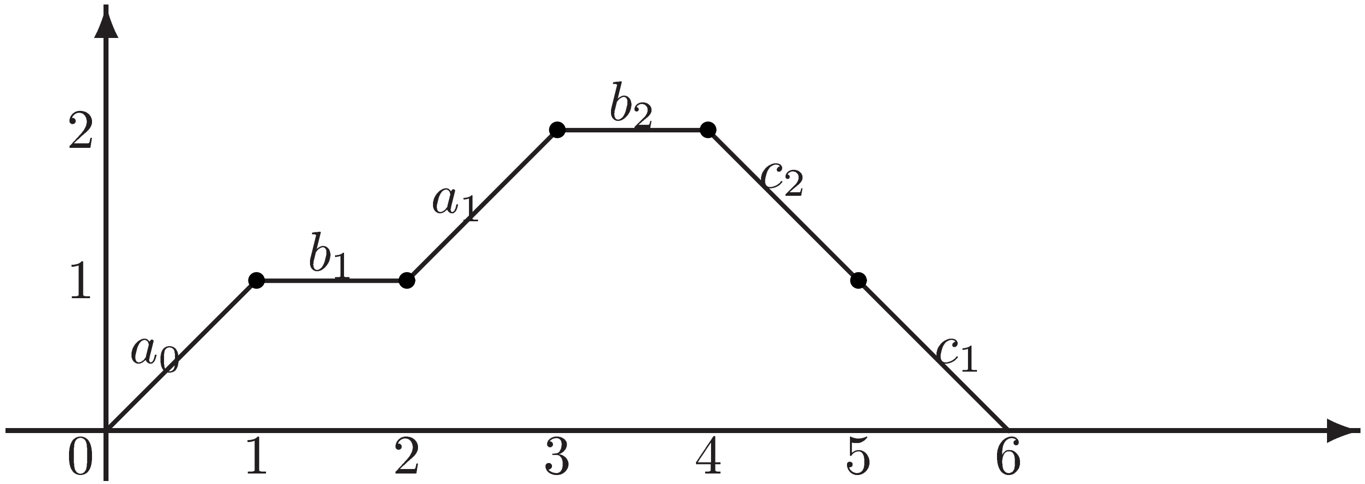
where b_i and λ_i are as in (2).

Motzkin Path

Weight:

level $k \cdots\cdots$





$$w(\gamma) = a_0 a_1 b_1 b_2 c_1 c_2.$$

Let $\Gamma(n)$ be the set of Motzkin paths from $(0,0)$ to $(n,0)$.

Then it's well-known that

$$1 + \sum_{n\geq 1} \left(\sum_{\gamma\in\Gamma(n)} w(\gamma) \right) t^n = \cfrac{1}{1 - b_0\, t - \cfrac{a_0 c_1\, t^2}{1 - b_1\, t - \cfrac{a_1 c_2\, t^2}{\ddots}}},$$

From weighted Motzkin paths to $\mathcal{OP}$

There is a classical bijection due to Franccon and Viennot between the permutations groupe S_n and the set Γ_n of weighted Motzkin paths of length n, that we can adapt to give a bijection between $\mathcal{OP}_n$ and Γ_n.

Definitions

$\sigma = \sigma_1\sigma_2\ldots\sigma_n \in S_n$, by convention $\sigma_0 = +\infty$, $\sigma_{n+1} = 0$. For $1 \le i \le n$, σ_i is a

- *valley* if $\sigma_{i-1} > \sigma_{i-1} < \sigma_{i+1}$
- *peak* if $\sigma_{i-1} < \sigma_i > \sigma_{i+1}$
- *double ascent* if $\sigma_{i-1} < \sigma_i < \sigma_{i+1}$
- *double descent* if $\sigma_{i-1} > \sigma_i > \sigma_{i+1}$

e.g. $\pi = 4/1\,6\,9/7\,8/3\,5/2$

$$V(\pi) = \{1,3,7\}, P(\pi) = \{5,8,9\}, DA(\pi) = \{6\}, DD(\pi) = \{2\}$$

For $\sigma = \sigma_1\sigma_2\ldots\sigma_n \in \underline{S}_n$, $\sigma_0 = +\infty$, $\sigma_{n+1} = 0$. the elements σ_i fall into six classes:

- double ascents — da
- underlined double ascents — $\underline{\text{da}}$
- valleys — va
- underlined valleys — $\underline{\text{va}}$
- double descents — dd
- peaks — pe

e.g. $\sigma = 2\,9\,3\,1\,4\,8\,5\,\underline{6}\,7$,

$$\text{da}(\sigma) = \{4\}, \qquad \underline{\text{da}}(\sigma) = \{6\}$$
$$\text{va}(\sigma) = \{1, 2, 5\}, \qquad \underline{\text{va}}(\sigma) = \emptyset$$
$$\text{dd}(\sigma) = \{3\}, \quad \text{pe}(\sigma) = \{7, 8, 9\}$$

For $\sigma \in \underline{S}_n$, a run is a maximum increasing string.

e.g. $\sigma = 2\,9/3/1\,4\,8/5\,\underline{6}/7$ has 5 runs.

$$\begin{aligned} \mathrm{lsg}(i) &= \text{\#of runs of } \sigma \text{ strictly to the left of } i \text{ which} \\ &\quad\ \text{contain elements smaller and greater than } i \\ \mathrm{rsg}(i) &= \ldots \text{right of } i \ldots \end{aligned}$$

Define

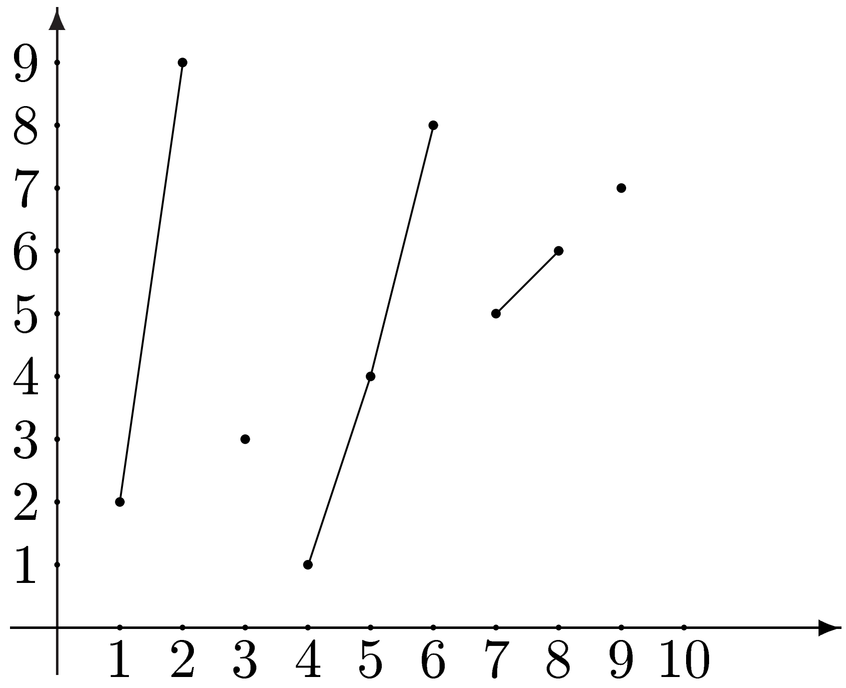
$$\mathrm{lsg}(\sigma) = \sum_{i=1}^{n} \mathrm{lsg}(i), \quad \mathrm{rsg}(\sigma) = \sum_{i=1}^{n} \mathrm{rsg}(i)$$

We also define the lsg and rsg and on the sets da, $\underline{\text{da}}$, $\ldots$, pe of σ:

$$\mathrm{lsg}(\mathrm{da})(\sigma) = \sum_{i \in \mathrm{da}(\sigma)} \mathrm{lsg}(i), \quad \mathrm{rsg}(\mathrm{da})(\sigma) = \sum_{i \in \mathrm{da}(\sigma)} \mathrm{rsg}(i)$$

The statistics lsg and rsg have analogous definitions on each of the remaining five classes of elements.

e.g. if $\sigma = 2\,9/3/1\,4\,8/5\,\underline{6}/7$ then



$\mathrm{lsg}(3) = 1, \quad \mathrm{rsg}(3) = 1, \quad \mathrm{lsg}(4) = 1, \quad \mathrm{rsg}(4) = 0.$

$$\mathrm{lsg}(\mathrm{da})(\sigma) = 1, \qquad \mathrm{rsg}(\mathrm{da})(\sigma) = 0$$

Generalized q-Meixner polynomials

Introduce the following notation :

$$[n]_{x,y} = \frac{x^n - y^n}{x - y}.$$

It is easy to check that

$$[n]_{q^2,q} = q^{n-1}[n]_q, \qquad [n]_{qx,qy} = q^{n-1}[n]_{x,y}.$$

Define $P_n(x)$ by

$$P_{n+1}(x) = (x - b_n)P_n(x) - \lambda_{n+1}(x)$$

where

$$\begin{cases} b_n & = & a[n+1]_{r,s} + b[n]_{t,z} + c[n]_{x,y}, \\ \lambda_{n+1} & = & (\alpha[n+1]_{p,q} + \beta[n+1]_{\xi,\eta})\, \gamma[n+1]_{v,w}. \end{cases}$$

Theorem 2 *The n-th moment μ_n for the generalized q-Meixner polynomials is*

$$\begin{aligned}\mu_n = \sum_{\pi \in \underline{S_n}} & a^{\mathrm{dd}(\sigma)} b^{\mathrm{da}(\sigma)} c^{\underline{\mathrm{da}}(\sigma)} \alpha^{\mathrm{va}(\sigma)} \beta^{\underline{\mathrm{va}}(\sigma)} \gamma^{\mathrm{pe}(\sigma)} \\ & \times r^{\mathrm{lsg}(\mathrm{dd})(\sigma)} s^{\mathrm{rsg}(\mathrm{dd})(\sigma)} t^{\mathrm{lsg}(\mathrm{da})(\sigma)} z^{\mathrm{rsg}(\mathrm{da})(\sigma)} \\ & \times x^{\mathrm{lsg}(\underline{\mathrm{da}})(\sigma)} y^{\mathrm{rsg}(\mathrm{da})(\sigma)} p^{\mathrm{lsg}(\mathrm{va})(\sigma)} q^{\mathrm{rsg}(\mathrm{va})(\sigma)} \\ & \times \xi^{\mathrm{lsg}(\underline{\mathrm{va}})(\sigma)} \eta^{\mathrm{rsg}(\underline{\mathrm{va}})(\sigma)} v^{\mathrm{lsg}(\mathrm{pe})(\sigma)} w^{\mathrm{rsg}(\mathrm{pe})(\sigma)}\end{aligned}$$

There are several possibilities to specialize the parameters in order to obtain a combinatorial interpretation of mah.

Remark: When $c = \beta = 0$, $\alpha = a$ and $\gamma = b$ we recover the case studied by Simion and Stanton.

Set $a = \gamma = \eta = 1$, $\beta = c = u$, $r = p = t = x = v = q^2$ and $b = \alpha = s = z = y = \xi = w = q$, then b_n and λ_{n+1} reduce to

$$\begin{cases} b_n &= q^n[n+1] + (u+q^n)[n], \\ \lambda_{n+1} &= (q^{n+1}+u) \cdot q^n[n+1]^2. \end{cases} \tag{3}$$

Theorem 3 *For $\sigma \in \underline{S}_n$, let*

$$s(\sigma) = \#\,\mathrm{da}(\sigma) + \#\,\mathrm{va}(\sigma) + (2\,\mathrm{lsg} + \mathrm{rsg} - \mathrm{lsg}(\underline{\mathrm{da}}) - \mathrm{rsg}(\underline{\mathrm{da}}))(\sigma).$$

Then

$$\sum_{\pi \in \underline{S}n} u^{\#\underline{\mathrm{da}}(\sigma)+\#\underline{\mathrm{va}}(\sigma)} q^{s(\sigma)} = \sum_{k=1}^{n} [k]! S_q(n,k) u^{n-k}.$$

Remark Setting $u = 0$, since $\#va + \#da = n - \text{run}$, we recover a result of Simion and Stanton.

Set $b = \alpha = u$, $a = \gamma = z = \eta = 1$ and $\beta = c = s = t = w = q$ and $x = r = v = \xi = q^2$, then b_n and λ_{n+1} reduce to (3).

Theorem 4 *For $\sigma \in \underline{S}_n$, let*

$$s'(\sigma) = (\#\,\mathrm{va} + \#\underline{\mathrm{da}} + 2\,\mathrm{lsg} + \mathrm{rsg} - \mathrm{lsg}(\mathrm{da}) - \mathrm{lsg}(\mathrm{va}) - \mathrm{rsg}(\mathrm{va}))(\sigma).$$

Then

$$\sum_{\pi \in \underline{S}_n} u^{\mathrm{da}(\sigma)+\mathrm{va}(\sigma)} q^{s'(\sigma)} = \sum_{k=1}^{n} [k]! S_q(n,k) u^{n-k}.$$

Steingrimsson's conjectures

Let $\mathcal{OP}_n^k$ be the set of ordered partitions of the set $[n]$ into k blocks.

The problem is to find some *Euler-mahonian* statistics on $\mathcal{OP}_n^k$ whose generating fonctions are equal to $[k]_q!S_q(n,k)$.

Definition 1 *Let $\pi = B_1/\cdots/B_k \in \mathcal{OP}_n^k$. We define a partial order on blocks as follows : $B_i > B_j$ if all the letters of B_i are greater than those of B_j. We say that i is a block descent in π if $B_i > B_{i+1}$. The block major index of π, denoted* $\mathrm{bmaj}(\pi)$, *is the sum of the block descents in π. A block inversion in π is a pair (i,j) such that $i < j$ and $B_i > B_j$.*

$$\mathrm{binv}\,\pi = \#\ \textit{of block inversions in}\quad \pi.$$

Let

$$w_i = \text{index of the block (counting from the left) containing } i.$$

Example 1 *If* $\pi = 4\,7 \;-\; 3 \;-\; 1\,5\,9 \;-\; 6\,8 \;-\; 2$, *then*

$$w(\pi) = w_1 w_2 \ldots w_9 = 3\,5\,2\,1\,3\,4\,1\,4\,3.$$

and $\mathcal{O}(\pi) = \{4, 3, 1, 6, 2\}$, $\mathcal{F}(\pi) = \{7, 3, 9, 8, 2\}$.

Define eight coordinate statistics on $\mathcal{OP}_n^k$:

$$
\begin{aligned}
\mathrm{ros}_i(\pi) &= \#\{j \in \mathcal{O}(\pi) \,|\, i > j,\, w_j > w_i\}, \\
\mathrm{rob}_i(\pi) &= \#\{j \in \mathcal{O}(\pi) \,|\, i < j,\, w_j > w_i\}, \\
\mathrm{rcs}_i(\pi) &= \#\{j \in \mathcal{F}(\pi) \,|\, i > j,\, w_j > w_i\}, \\
\mathrm{rcb}_i(\pi) &= \#\{j \in \mathcal{F}(\pi) \,|\, i < j, w_j > w_i\}, \\
\mathrm{los}_i(\pi) &= \#\{j \in \mathcal{O}(\pi) \,|\, i > j, w_j < w_i\}, \\
\mathrm{lob}_i(\pi) &= \#\{j \in \mathcal{O}(\pi) \,|\, i < j, w_j < w_i\}, \\
\mathrm{lcs}_i(\pi) &= \#\{j \in \mathcal{F}(\pi) \,|\, i > j, w_j < w_i\}, \\
\mathrm{lcb}_i(\pi) &= \#\{j \in \mathcal{F}(\pi) \,|\, i < j, w_j < w_i\}.
\end{aligned}
$$

We then define ros, rob, rcs, rcb, lob, los, lcs and lcb as the sum of their coordinate statistics, e.g. $\mathrm{ros}(\pi) = \sum_i \mathrm{ros}_i(\pi)$.

Inspired by a statistic mak due to Foata & Zeilberger on the permutations, Steingrímsson introduced its analogue on $\mathcal{OP}$.

Definition 2 *For all* $\pi \in \mathcal{OP}_n^k$, *let*

$$\begin{aligned}
\mathrm{mak}(\pi) &= \mathrm{ros}(\pi) + \mathrm{lcs}(\pi), \\
\mathrm{lmak}'(\pi) &= n(k-1) - [\mathrm{lcb}(\pi) + \mathrm{rob}(\pi)], \\
\mathrm{mak}'(\pi) &= \mathrm{lob}(\pi) + \mathrm{rcb}(\pi), \\
\mathrm{lmak}\,(\pi) &= n(k-1) - [\mathrm{los}(\pi) + \mathrm{rcs}(\pi)].
\end{aligned}$$

Conjecture 1 (Steingrímsson) *The following statistiques are Euler-mahoniennes :*

$$\begin{array}{ll} \mathrm{mak} + \mathrm{bmaj}, & \mathrm{lmak}' + \mathrm{bmaj}, \\ \mathrm{mak}' + \mathrm{bmaj}, & \mathrm{lmak} + \mathrm{bmaj}, \\ \mathrm{mak} + \mathrm{binv}, & \mathrm{lmak}' + \mathrm{binv}, \\ \mathrm{mak}' + \mathrm{binv}, & \mathrm{lmak} + \mathrm{binv}. \end{array}$$

In other words, the GF of the above statistics over $\mathcal{OP}_n^k$ are equal to

$$q^{\binom{k}{2}}[k]_q!S_q(n,k).$$

Proposition 1 (Ksavrelof-Zeng) *The following equations hold :*

$$\begin{array}{lll} \mathrm{mak} = \mathrm{lmak}', & & \mathrm{mak}' = \mathrm{lmak}, \\ \mathrm{mak} = \mathrm{lmak}', & & \mathrm{mak}' = \mathrm{lmak}. \end{array}$$