SL

ENUMERATING PERMUTATIONS AVOIDING THREE BABSON-STEINGRÌMSSON PATTERNS

Antonio Bernini*

* Dipartimento di Sistemi e Informatica
via Lombroso 6/17, 50134 FIRENZE, ITALIA
e-mail: bernini@dsi.unifi.it

$$M = \{1\text{-}23,\ 12\text{-}3,\ 1\text{-}32,\ 13\text{-}2,\ 3\text{-}12,\ 31\text{-}2,\ 2\text{-}13,\ 21\text{-}3,\ 2\text{-}31,\ 23\text{-}1,\ 3\text{-}21,\ 32\text{-}1\}$$

$$|S_n(P)| = ? \qquad P \subseteq M$$

A. Claesson : the case $|P| = 1$

A. Claesson, T. Mansour: the case $|P| = 2$ and conjectures for $|P| \geq 3$

this work : the case $|P| = 3$ by means of ECO Method and a path-like representation of permutations.

$\binom{12}{3} = 220$ possibilities for $P \subseteq M$, $|P| = 3$ <u>SL 3</u>

$\pi \in S_n$ (reversal) $\pi^r_i = \pi_{n+1-i}$ (complement) $\pi^c_i = n+1-\pi_i$ $(i = 1, \dots, n)$

$P = \{p, q, r\}$ $P^r = \{p^r, q^r, r^r\}$ $P^c = \{p^c, q^c, r^c\}$ $P^{rc} = \{p^{rc}, q^{rc}, r^{rc}\}$

<u>ex</u> $q = 1\text{-}32$ $q^r = 23\text{-}1$ $q^c = 3\text{-}12$ $q^{rc} = 21\text{-}3$

$$|S_n(P)| = |S_n(P^r)| = |S_n(P^c)| = |S_n(P^{rc})|$$

conjectures

P	$\lvert S_n(P)\rvert$
{1-23,2-13,1-32}	F_n
{32-1,31-2,23-1}	
{3-21,2-31,3-12}	
{12-3,13-2,21-3}	
{1-23,13-2,3-12}	
{32-1,2-31,21-3}	
{3-21,31-2,1-32}	
{12-3,2-13,23-1}	
{1-23,2-13,13-2}	
{32-1,31-2,2-31}	
{3-21,2-31,31-2}	
{12-3,13-2,2-13}	
{1-23,1-32,3-12}	
{32-1,23-1,21-3}	
{3-21,3-12,1-32}	
{12-3,21-3,23-1}	
{1-23,1-32,31-2}	
{32-1,23-1,2-13}	
{3-21,3-12,13-2}	
{12-3,21-3,2-31}	
{1-23,13-2,31-2}	
{32-1,2-31,2-13}	
{3-21,31-2,13-2}	
{12-3,2-13,2-31}	
{1-23,21-3,13-2}	
{32-1,3-12,2-31}	
{3-21,23-1,31-2}	
{12-3,1-32,2-13}	
{12-3,21-3,2-13}	M_n
{3-21,3-12,31-2}	
{32-1,23-1,2-31}	
{1-23,1-32,13-2}	
{1-23,12-3,21-3}	
{32-1,3-21,3-12}	
{3-21,32-1,23-1}	
{12-3,1-23,1-32}	
{1-23,2-13,3-12}	n
{32-1,31-2,21-3}	
{3-21,2-31,1-32}	
{12-3,13-2,23-1}	
{1-23,2-13,2-31}	
{32-1,31-2,13-2}	
{3-21,2-31,2-13}	
{12-3,13-2,31-2}	

P	$\lvert S_n(P)\rvert$
{1-23,2-13,23-1}	n
{32-1,31-2,2-32}	
{3-21,2-31,21-3}	
{12-3,13-2,3-12}	
{1-23,23-1,31-2}	
{32-1,1-32,2-13}	
{3-21,21-3,13-2}	
{12-3,3-12,2-31}	
{1-23,1-32,3-21}	
{32-1,23-1,12-3}	
{3-21,3-12,1-23}	
{12-3,21-3,32-1}	
{1-23,2-13,31-2}	
{32-1,31-2,2-13}	
{3-21,2-31,13-2}	
{12-3,13-2,2-31}	
{1-23,21-3,2-31}	
{32-1,3-12,13-2}	
{3-21,23-1,2-13}	
{12-3,1-32,31-2}	
{1-23,21-3,23-1}	
{32-1,3-12,1-32}	
{3-21,23-1,21-3}	
{12-3,1-32,3-12}	
{1-23,21-3,3-12}	
{32-1,3-12,21-3}	
{3-21,23-1,1-32}	
{12-3,1-32,23-1}	
{1-23,21-3,31-2}	
{32-1,3-12,2-13}	
{3-21,23-1,13-2}	
{12-3,1-32,2-31}	
{2-13,2-31,1-32}	
{31-2,13-2,23-1}	
{2-31,2-13,3-12}	
{13-2,31-2,21-3}	
{2-13,2-31,13-2}	
{31-2,13-2,2-31}	
{2-31,2-13,31-2}	
{13-2,31-2,2-13}	
{2-13,23-1,1-32}	
{31-2,1-32,23-1}	
{2-31,21-3,3-12}	
{13-2,3-12,21-3}	

P	$\lvert S_n(P)\rvert$
{12-3,31-2,21-3}	n
{3-21,2-13,3-12}	
{32-1,13-2,23-1}	
{1-23,2-31,1-32}	
{12-3,31-2,2-13}	
{3-21,2-13,31-2}	
{32-1,13-2,2-31}	
{1-23,2-31,13-2}	
{1-23,2-31,3-12}	
{32-1,13-2,21-3}	
{3-21,2-13,1-32}	
{12-3,31-2,23-1}	
{1-23,2-31,31-2}	
{32-1,13-2,2-13}	
{3-21,2-13,13-2}	
{12-3,31-2,2-31}	
{12-3,3-12,21-3}	
{3-21,21-3,3-12}	
{32-1,1-32,23-1}	
{1-23,23-1,1-32}	
{12-3,3-12,2-13}	
{3-21,21-3,31-2}	
{32-1,1-32,2-31}	
{1-23,23-1,13-2}	
{2-13,23-1,13-2}	
{31-2,1-32,2-31}	
{2-31,21-3,31-2}	
{13-2,3-12,2-13}	
{2-31,21-3,1-32}	
{13-2,3-12,23-1}	
{2-13,23-1,3-12}	
{31-2,1-32,21-3}	
{2-31,21-3,13-2}	
{13-2,3-12,2-31}	
{2-13,23-1,31-2}	
{31-2,1-32,2-13}	
{13-2,21-3,23-1}	
{2-31,3-12,1-32}	
{31-2,23-1,21-3}	
{2-13,1-32,3-12}	
{23-1,21-3,3-12}	
{1-32,3-12,21-3}	
{21-3,23-1,1-32}	
{3-12,1-32,23-1}	

P	$\lvert S_n(P)\rvert$
{1-23,12-3,2-13}	2^{n-1}
{32-1,3-21,31-2}	
{3-21,32-1,2-31}	
{12-3,1-23,13-2}	
{1-23,12-3,23-1}	
{32-1,3-21,1-32}	
{3-21,32-1,21-3}	
{12-3,1-23,3-12}	
{1-23,2-13,21-3}	
{32-1,31-2,3-12}	
{3-21,2-31,23-1}	
{12-3,13-2,1-32}	
{1-23,3-12,31-2}	
{32-1,21-3,2-13}	
{3-21,1-32,13-2}	
{12-3,23-1,2-31}	
{2-13,21-3,2-31}	
{31-2,3-12,13-2}	
{2-31,23-1,2-13}	
{13-2,1-32,31-2}	
{2-13,21-3,23-1}	
{31-2,3-12,1-32}	
{2-31,23-1,21-3}	
{13-2,1-32,3-12}	
{2-13,21-3,1-32}	
{31-2,3-12,23-1}	
{2-31,23-1,3-12}	
{13-2,1-32,21-3}	
{2-13,21-3,13-2}	
{31-2,3-12,2-31}	
{2-31,23-1,31-2}	
{13-2,1-32,2-13}	
{2-13,21-3,3-12}	
{31-2,3-12,21-3}	
{2-31,23-1,1-32}	
{13-2,1-32,23-1}	
{2-13,21-3,31-2}	
{31-2,3-12,2-13}	
{2-31,23-1,13-2}	
{13-2,1-32,2-31}	
{1-23,21-3,1-32}	$\binom{n}{\lceil n/2\rceil}$
{32-1,3-12-23-1}	
{3-21,23-1,3-12}	
{12-3,1-32,21-3}	

P	$\lvert S_n(P)\rvert$
{1-23,2-13,3-21}	0
{32-1,31-2,12-3}	
{3-21,2-31,1-23}	
{12-3,13-2,32-1}	
{1-23,23-1,32-1}	
{32-1,13-2,1-23}	
{3-21,21-3,12-3}	
{12-3,3-12,3-21}	
{1-23,2-13,32-1}	
{32-1,31-2,1-23}	
{3-21,2-31,12-3}	
{12-3,13-2,3-21}	
{1-23,12-3,3-21}	
{32-1,3-21,12-3}	
{3-21,32-1,1-23}	
{12-3,1-23,32-1}	
{1-23,21-3,3-21}	
{32-1,3-12,12-3}	
{3-21,23-1,1-23}	
{12-3,1-32,32-1}	
{1-23,21-3,32-1}	
{32-1,3-12,1-23}	
{3-21,23-1,12-3}	
{12-3,1-32,3-21}	
{1-23,2-31,32-1}	
{32-1,13-2,1-23}	
{3-21,2-13,12-3}	
{12-3,31-2,3-21}	
{12-3,1-23,31-2}	$1+\binom{n}{2}$
{3-21,32-1,2-13}	
{32-1,3-21,13-2}	
{1-23,12-3,2-31}	
{1-23,2-31,23-1}	
{32-1,13-2,1-32}	
{3-21,2-13,21-3}	
{12-3,31-2,3-12}	
{12-3,2-13,32-1}	3
{3-21,31-2,1-23}	
{32-1,2-31,12-3}	
{1-23,13-2,3-21}	
{1-23,23-1,3-12}	$2^{n-2}+1$
{32-1,1-32,21-3}	
{3-21,21-3,1-32}	
{12-3,3-12,23-1}	

The ECO Method

Theorem:

Let S be a class of combinatorial objects;
let p be a parameter of S $(p: S \longrightarrow \mathbb{N}^+)$
and $S_n = \{x \in S : p(x) = n\}$;
let ϑ be an operator on S $(\vartheta : S_n \longrightarrow 2^{S_{n+1}}$,
where $2^{S_{n+1}}$ is the power set of S_{n+1}).
If ϑ satisfies the following conditions:

1) for each $y \in S_{n+1}$ there exists $x \in S_n$ such that $y \in \vartheta(x)$;

2) let $x_1, x_2 \in S_n$ and $x_1 \neq x_2$, then $\vartheta(x_1) \cap \vartheta(x_2) = \emptyset$;

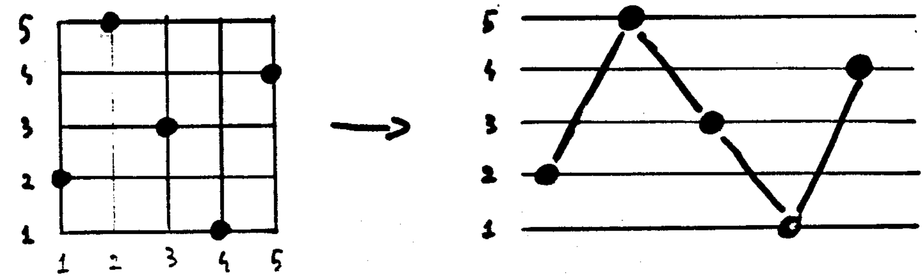
the following family of sets:

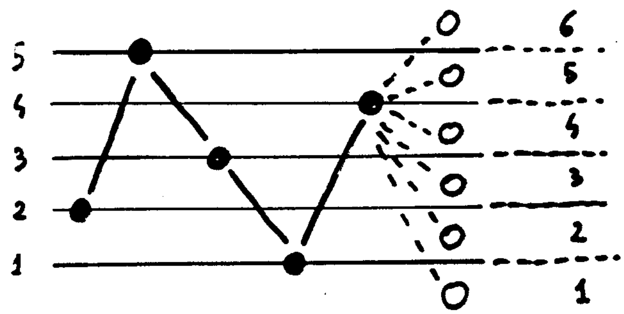
$$F_{n+1} = \{\vartheta(x) : \forall x \in S_n\}$$

is a partition of S_{n+1}.

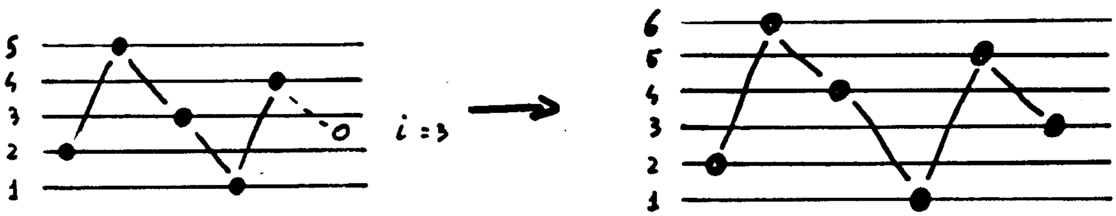
"STAFF REPRESENTATION" OF PERMUTATIONS

= 2 5 3 1 4





The plane is divided in $n+1$ regions



$\pi = 25314$

$\pi' = 264153$

Renaming rule:

1) $\pi'_{n+1} = i$

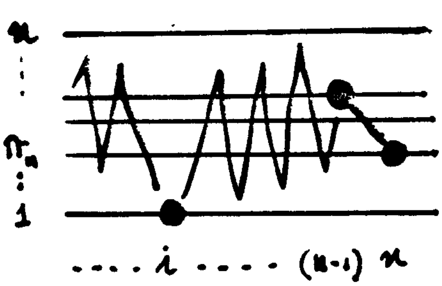
2) for $j = 1, \dots, n$

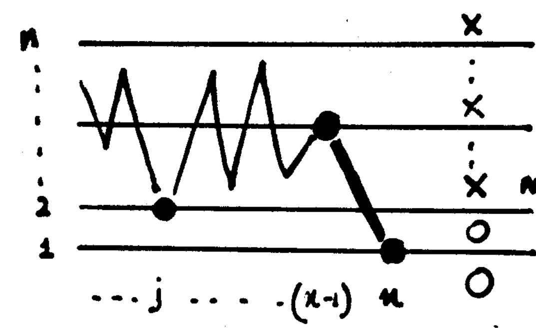
if $\pi_j < i$, then $\pi'_j = \pi_j$

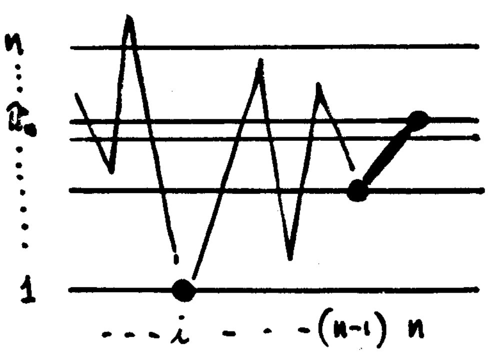
otherwise $\pi'_j = \pi_j + 1$

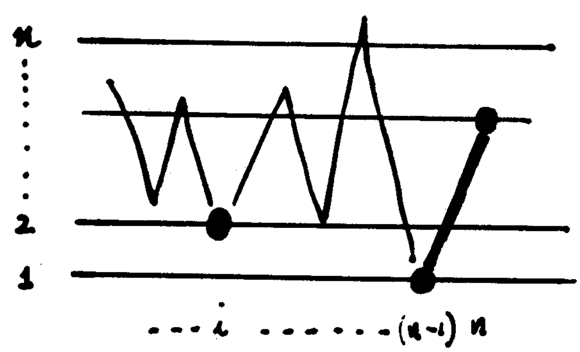
$S_n(\underbrace{1\text{-}23, 2\text{-}13, 1\text{-}32}_{P})$

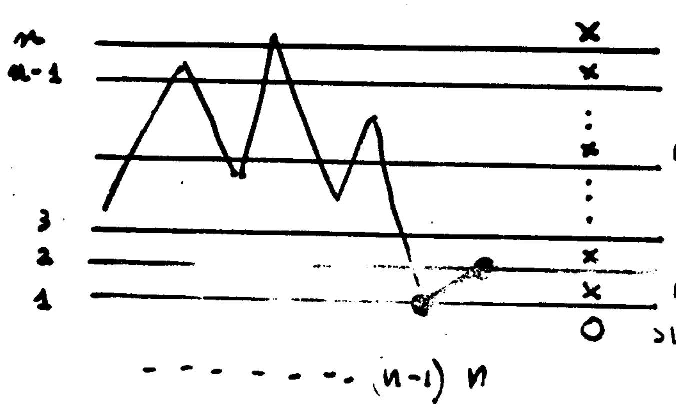
$|S_n(P)| = F_n$ pre LUC 3 dalla Tesi





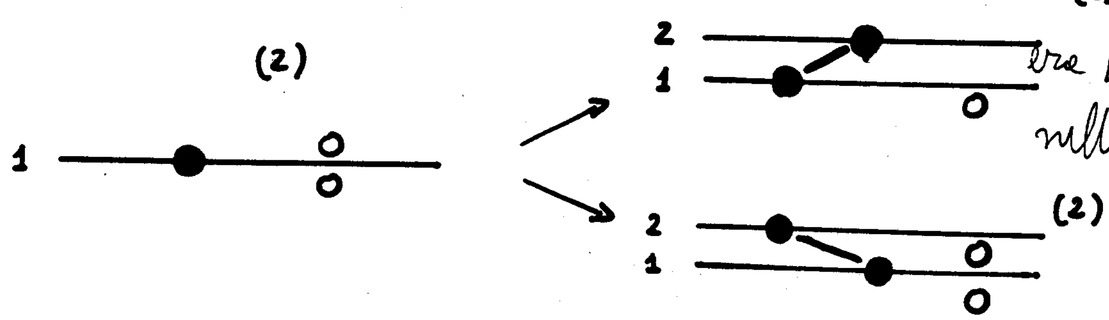




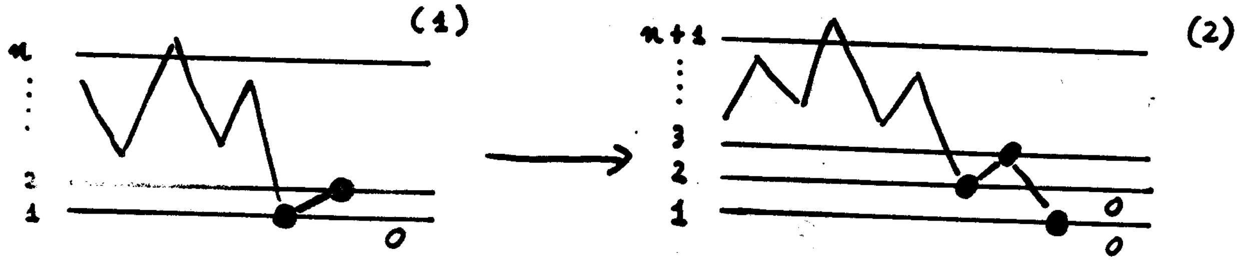


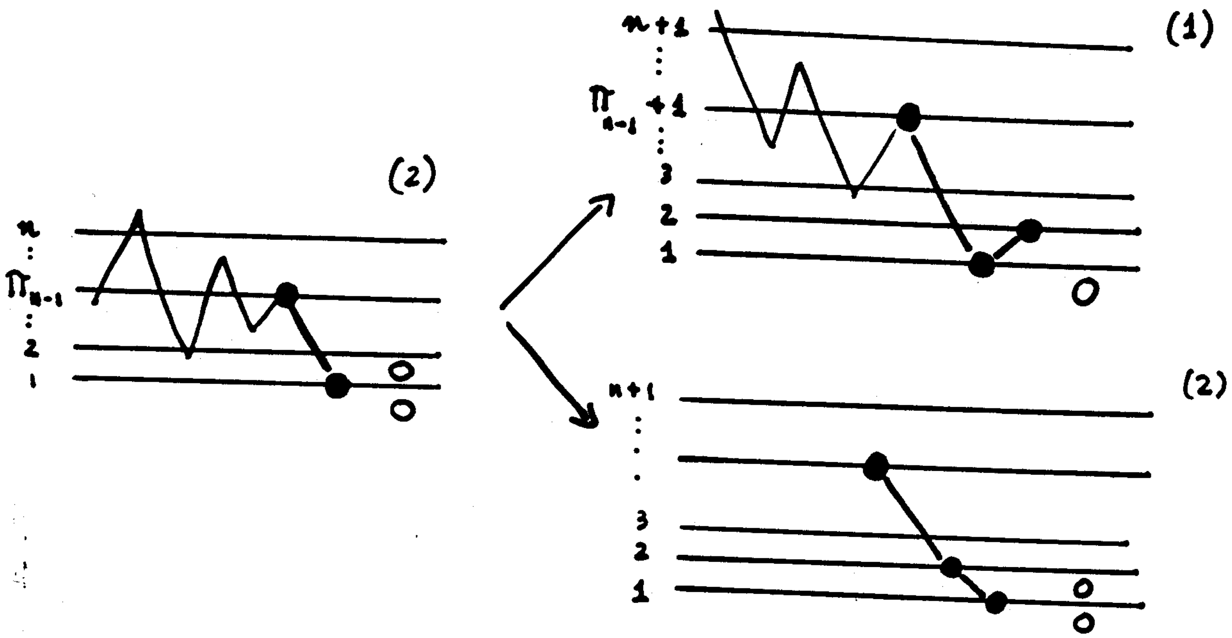
O ACTIVE SITES

D_n(1-23, …, 1-32) SL 8



vedi LUC 11 nella Tesi



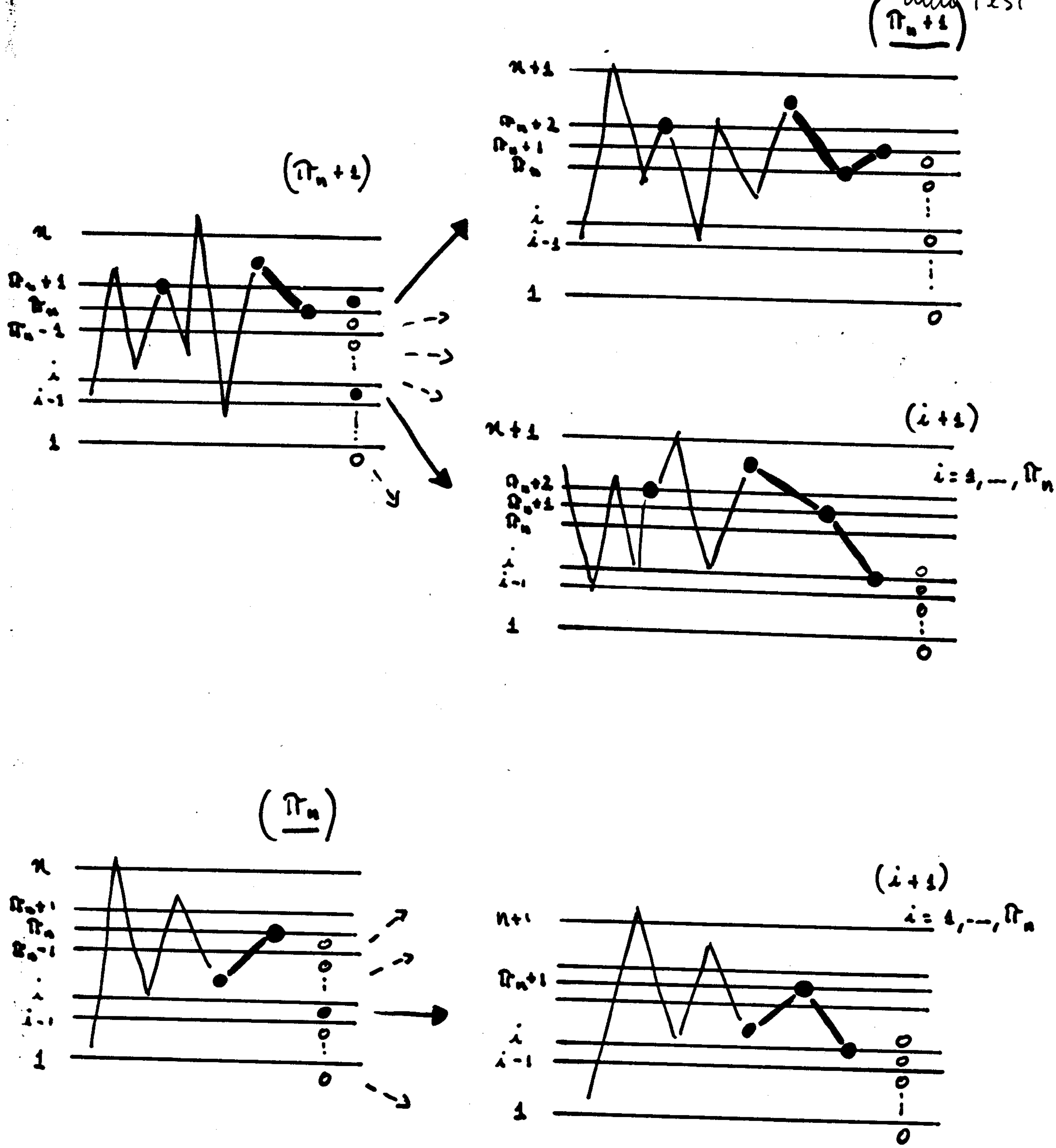


$$\begin{cases} (2) \\ (2) \rightsquigarrow (1)(2) \\ (1) \rightsquigarrow (2) \end{cases}$$

$$S_n(12\text{-}3,\, 21\text{-}3,\, 2\text{-}13) \equiv S_n(\underbrace{12\text{-}3,\, 2\text{-}13}_{\rho})$$

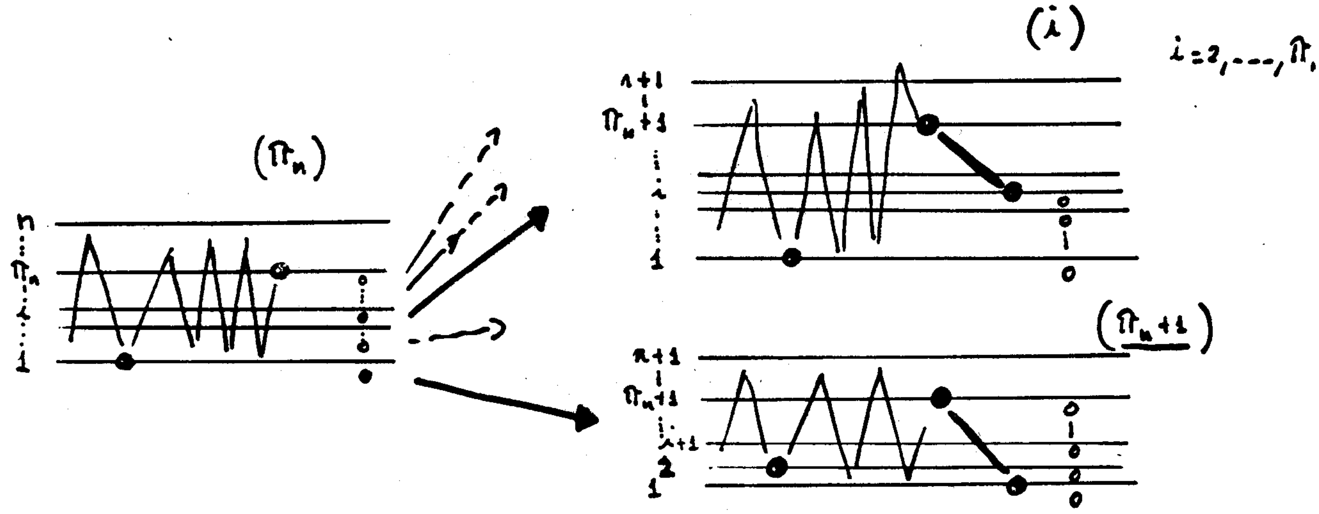
$$|S_n(\rho)| \overset{SL\ 9}{=} \pi_n$$

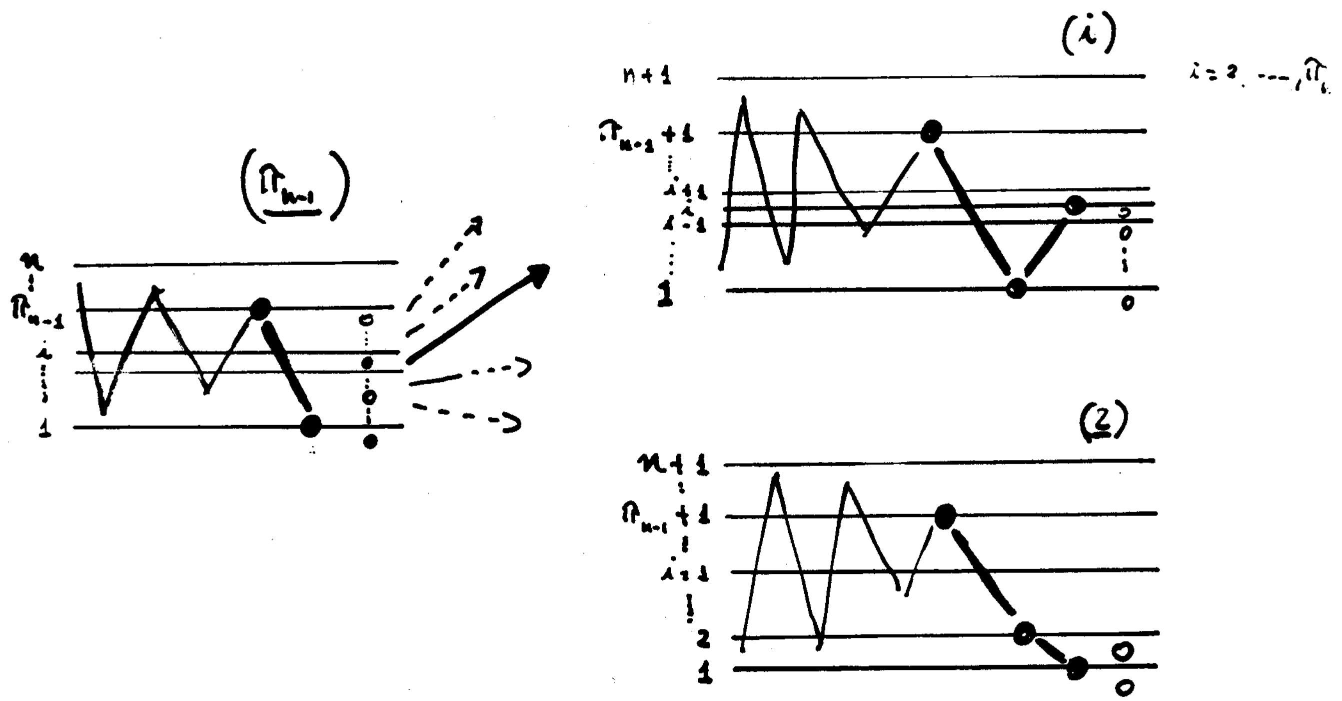
vra LUC 13 della Tesi



$$\begin{cases} (2) \\ (k) \rightsquigarrow (2)(3)\cdots(k)(\underline{k}) \\ (\underline{k}) \rightsquigarrow (2)(3)\cdots(k+1) \end{cases}$$

$$S_n(1\text{-}23,\ 12\text{-}3,\ 21\text{-}3) \equiv S_n(\underbrace{12\text{-}3,\ 21\text{-}3}_{\rho}) \qquad |S_n(\rho)| = {}^{S}\underline{M10_n}$$





$$\left\{\begin{array}{l} (\underline{2}) \\ (\underline{k}) \rightsquigarrow (\underline{2})(2)(3)\cdots(k) \\ (k) \rightsquigarrow (2)(3)\cdots(k)(\underline{k+1}) \end{array}\right.$$

The ECO Method permits to construct, in a recursive manner, all the objects of a class of combinatorial objects.
We can fix a parameter "p" of the objects and an ECO construction allows to make all the objects (and only one time (without repetition)) having the value of the parameter p equal to n+1, starting from the ones which have the value of the parameter equal to n.
Often this recursive construction can be used to find the generating function of the class as well as in all the cases of this work.
In particular we ~~fast~~ always find elegant and simple proofs of the conjectures of Mansour ~~do~~ and Claesson.