Paths, Permutations and Trees

Center for Combinatorics
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Enumerating Permutations Avoiding Three Babson - Steingrímsson Patterns

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Abstract.
Generalized patterns were introduced by Babson and Steingrímsson to study Mahonian statistics([1]). The difference with "classical patterns" is that the generalized ones require that two adjacent integers in a pattern must be adjacent in the permutation. For instance, the permutation contains the generalized pattern $1-32$ if the elements corresponding to the 3 and 2 are adjacent, as they are in the pattern. It is evident that there exist twelve different generalized patterns of length three, namely:

$$\begin{aligned}\mathcal{P} \quad &= \{1-23, 12-3, 1-32, 13-2, 3-12, 31-2, 2-13, 21-3,\\ &\quad 2-31, 23-1, 3-21, 32-1\}\end{aligned}$$

These patterns are of type $(1,2)$ or $(2,1)$, referring to the number of integers preceding or following the dash. We consider the problem of pattern avoidance for such patterns. Claesson presented a complete solution for the number of permutations avoiding any single pattern in $\mathcal{P}$ (see [3] and table 1. Subsequently Claesson and Mansour ([4]) gave the solution for the number of permutations avoiding a pair of generalized patterns of type (2,1) or (1,2) (see tables 2 and 3). They also formulated conjectures for the number of permutations avoiding the patterns in any subset of three or more patterns of $\mathcal{P}$. In our work the case $|P| = 3$ has been completely solved by means of the ECO Methodology ([2]) and a particular graphical representation of permutations.

In the following we recall briefly the basics of ECO and the above mentioned graphical method for permutations, also giving the detailed

proof of a specific case. At the end of the paper in tables 4 and 5 the complete results of our work can be found. In tables 1, 2, 3, 4 and 5 we use the following notation:

- P is a subset of $\mathcal{P}$;
- $|S_n(P)|$ is the number of the permutations of S_n avoiding the patterns specified in the set P;
- B_n is the n-th Bell number;
- C_n is the n-th Catalan number;
- b_n satisfies $b_0 = 1$ and, for $n \geq -2$, $b_{n+2} = b_{n+1} + \sum_{k=0}^{n} \binom{n}{k}$;
- I_n is the number of involutions of S_n;
- M_n is the n-th Motzkin number;
- a_n is the number of strongly monotone partitions of $[n]$;
- B_n^* is the n-th Bessel number;
- F_n is the n-th Fibonacci number.

The ECO method is based on the following

Proposition 1 *Let S be a class of combinatorial objects; let p be a parameter of S $(p : S \longrightarrow \mathbb{N}^+)$ and $S_n = \{x \in S : p(x) = n\}$; let ϑ be an operator on S $(\vartheta : S_n \longrightarrow 2^{S_{n+1}}$, where $2^{S_{n+1}}$ is the power set of $S_{n+1})$. If ϑ satisfies the following conditions:*

(1) *for each $Y \in S_{n+1}$ there exists $X \in S_n$ such that $Y \in \vartheta(X)$;*

(2) *let $X_1, X_2 \in S_n$ and $X_1 \neq X_2$, then $\vartheta(X_1) \bigcap \vartheta(X_2) = \emptyset$;*

then the following family of sets: $\mathcal{F}_{n+1} = \{\vartheta(X) : \forall X \in S_n\}$ is a partition of S_{n+1}.

Therefore, such an operator on S yields a recursive description of the class S. We can describe the recursive construction by means of a *succession rule*:

Definition 1 *Let ϑ be an operator satisfying the previous proposition. Let $|\vartheta(X)| = k$ and $\vartheta(X) = \{X_1, X_2, \ldots, X_k\}$. A class S of combinatorial objects satisfies the* succession rule

$$\begin{cases} (b) \\ (k) \rightsquigarrow (r_1)\ldots(r_k) \end{cases}$$

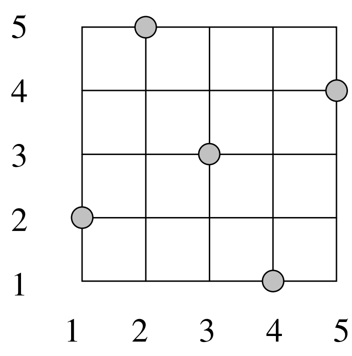
if:

(1) *the minimal object (referring to the parameter p) of S generates b objects by means of ϑ;*

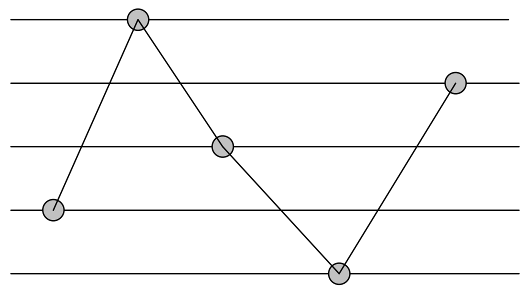
(2) *$\forall X \in S$ with $|\vartheta(X)| = k$, then $|\vartheta(X_i)| = r_i$ and $X_i \in \vartheta(X), 1 \leq i \leq k$.*

For our proofs we also use a particular and simple graphical representation of permutations which we explain with an example. Let $\pi = 25314 \in S_5$, we can represent it as a matrix as follows:

$\pi = 2\ 5\ 3\ 1\ 4$



A new graphical representation (which we would like to call "staff representation") is easily obtained by the preceding one by deleting the vertical lines and joining each vertex to the following one by a segment:



We keep the horizontal line and enumerate them from the bottom; every integer of π becomes a "node" on the line corresponding to its value; every pair of adjacent nodes $\pi_i\pi_{i+1} \quad (i = 1,\ldots,n-1)$ is linked by an ascending or descending segment according to $\pi_i < \pi_{i+1}$ (the pair is an "ascent") or $\pi_i > \pi_{i+1}$ (the pair is a "descent").

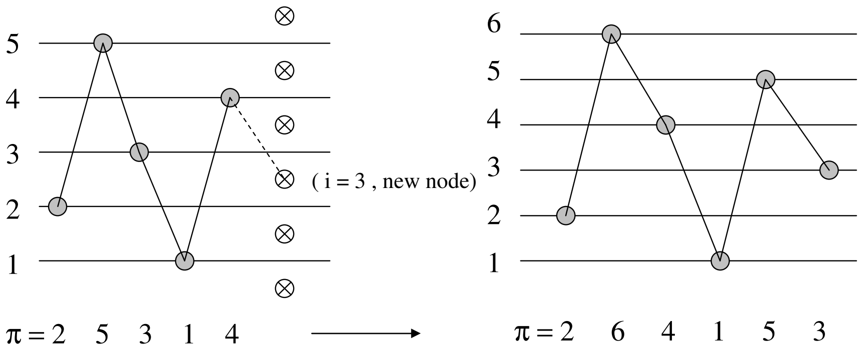
If $\pi \in S_n$, the n horizontal lines of the permutation in the above form divide the plane in $n+1$ regions. Therefore we can obtain $n+1$ permutations belonging to S_{n+1} starting from π, by inserting a new node in each of these regions and renaming the integers of the new permutation $\pi' \in S_{n+1}$ according to the following rule: if we insert the node into the $i-$th region, then

(1) $\pi'_{n+1} = i$

(2) for $j = 1,\ldots,n$

   (a) if $\pi_j < i$ then $\pi'_j = \pi_j$;

   (b) otherwise $\pi'_j = \pi_j + 1$.



Now let's prove that $|S_n(1-23,2-13,1-32)| = F_n$, as an example of the strategy used in all our proofs.

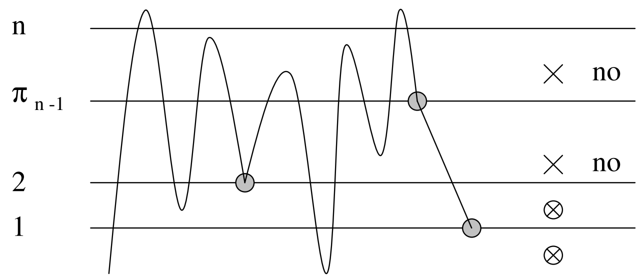
Let $\pi \in S_n(P)$, being $P = \{1-23, 2-13, 1-32\}$. We observe that:

- if $\pi_{n-1} > \pi_n$ $(n > 2)$ then $\pi_n = 1$, otherwise pattern $1-32$ would appear in π. Then, the new node can be inserted only in one of the following regions:

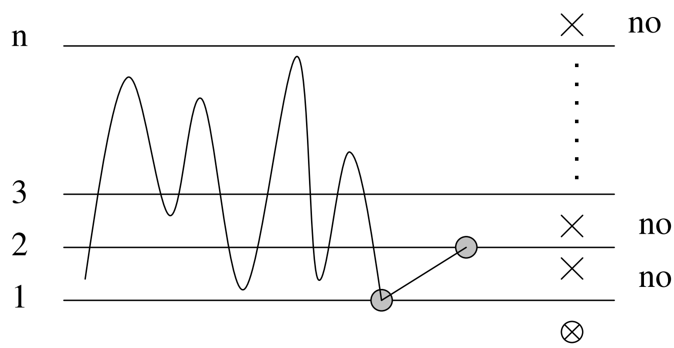
  (1) under the line 1 (only patterns $32-1, 23-1, 3-21$ can appear, which are not forbidden);

  (2) between line 1 and line 2 (only patterns $32-1, 23-1, 3-12$ can appear, which are allowed).

  Any other insertion would create a pattern $2-13$ (which must be avoided):



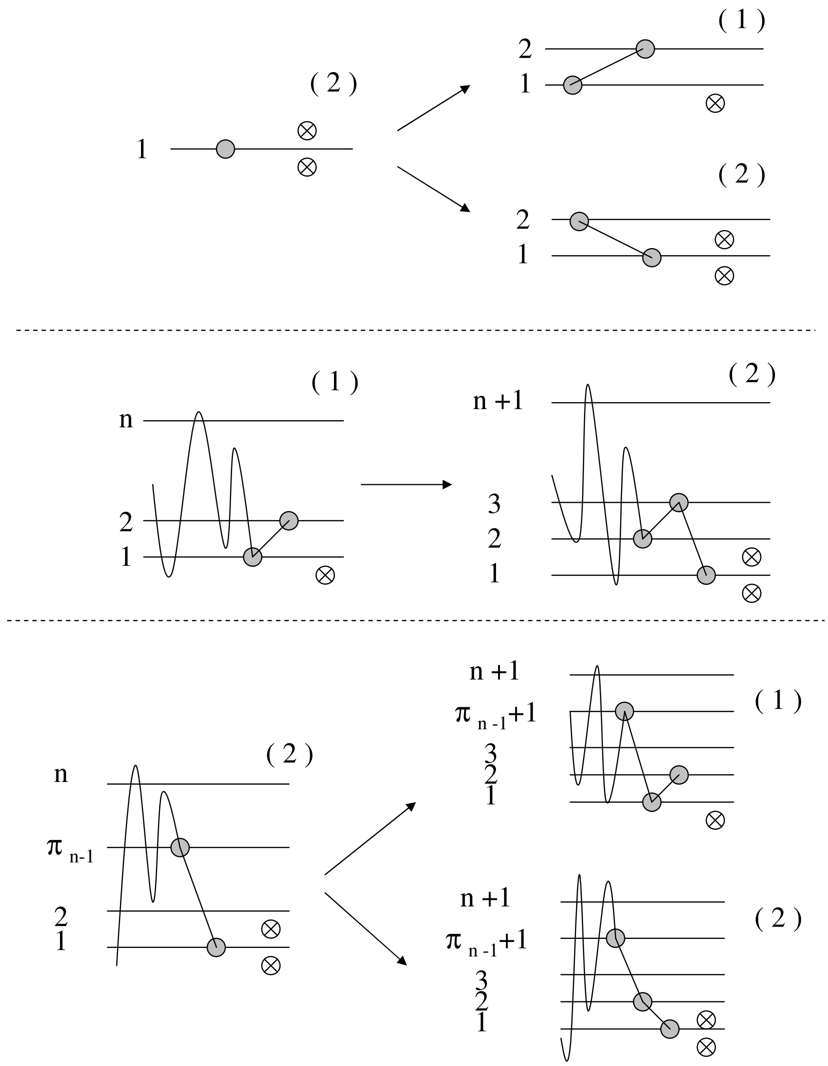
- if $\pi_{n-1} < \pi_n$ $(n > 2)$ then $\pi_{n-1}\pi_n = 12$, otherwise patterns $2-13$ or $1-23$ would appear in π. Then the new node can be inserted only under the line 1 while any other insertion is not allowed because patterns $1-23$ or $1-32$ would certainly appear if we consider the nodes π_{n-1}, π_n and the new one.



In this way, an operator ϑ has been defined which allows us to construct permutations belonging to $S_{n+1}(P)$ starting from $\pi \in S_n(P)$. It is easy to prove that such a ϑ satisfies the two conditions of Proposition 1. We also observe that the class $S(P)$ satisfies the following succession rule:

$$\begin{cases} (2) \\ (2) \rightsquigarrow (1)(2) \\ (1) \rightsquigarrow (2) \end{cases}$$

as shown in the next figure:



It is easy to prove that the previous succession rule leads to the generating function for Fibonacci numbers $\{F_n\}_{n\geq 1}$.

References

1. E. Babson, E Steingrímsson *Generalized permutation patterns and a classification of the Mahonian statistics*, Seminaire Lotharingien de Combinatoire, Article B44b, 18 pp, 2000.

2. E.Barcucci, A. Del Lungo, E. Pergola, R. Pinzani *ECO: A Methodology for the Enumeration of Combinatorial Objects*, Journal of Difference Equations and Applications, 5, pp 435-490, (1999).

3. A. Claesson *Generalized pattern avoidance*, European Journal of Combinatorics, 22, pp 961-971, (2001).

4. A.Claesson, T. Mansour *Enumerating permutations avoiding a pair of Babson-Steingrímsson patterns*, Ars Combinatoria, to appear, preprint math. C0/0303138

P	$\|S_n(P)\|$
$\{1-23\}$	B_n
$\{3-21\}$	
$\{12-3\}$	
$\{32-1\}$	
$\{1-32\}$	
$\{3-12\}$	
$\{21-3\}$	
$\{23-1\}$	
$\{2-13\}$	C_n
$\{2-31\}$	
$\{13-2\}$	
$\{31-2\}$	

Table 1: the case $|P| = 1$

P	$\|S_n(P)\|$
$\{1-23, 2-13\}$	0
$\{3-21, 2-31\}$	
$\{12-3, 13-2\}$	
$\{32-1, 31-2\}$	
$\{1-23, 23-1\}$	
$\{3-21, 21-3\}$	
$\{12-3, 3-12\}$	
$\{32-1, 1-32\}$	
$\{1-23, 31-2\}$	
$\{3-21, 13-2\}$	
$\{12-3, 2-31\}$	
$\{32-1, 2-13\}$	
$\{1-32, 2-13\}$	
$\{3-12, 2-31\}$	
$\{13-2, 21-3\}$	
$\{23-1, 31-2\}$	
$\{1-32, 2-31\}$	

P	$\|S_n(P)\|$
$\{3-12, 2-13\}$	0
$\{31-2, 21-3\}$	
$\{23-1, 13-2\}$	
$\{1-32, 3-12\}$	
$\{23-1, 21-3\}$	
$\{1-32, 23-1\}$	
$\{3-12, 21-3\}$	
$\{1-32, 31-2\}$	
$\{3-12, 13-2\}$	
$\{21-3, 2-13\}$	
$\{23-1, 2-13\}$	
$\{2-13, 2-31\}$	
$\{31-2, 13-2\}$	
$\{2-13, 13-2\}$	
$\{2-13, 31-2\}$	
$\{2-13, 31-2\}$	
$\{2-31, 13-2\}$	

Table 2: the case $|P| = 2$

P	$\|S_n(P)\|$
$\{1-23,2-31\}$	$\binom{n}{2}+1$
$\{3-21,2-13\}$	
$\{12-3,31-2\}$	
$\{32-1,13-2\}$	
$\{1-23,3-12\}$	b_n
$\{3-21,1-32\}$	
$\{23-1,12-3\}$	
$\{32-1,21-3\}$	
$\{1-23,1-32\}$	I_n
$\{3-21,3-12\}$	
$\{21-3,12-3\}$	
$\{32-1,23-1\}$	
$\{1-32,13-2\}$	C_n
$\{3-12,31-2\}$	
$\{21-3,2-13\}$	
$\{23-1,2-31\}$	

P	$\|S_n(P)\|$
$\{1-23,32-1\}$	0
$\{3-21,12-3\}$	
$\{1-23,3-21\}$	$2(n-1)$
$\{32-1,12-3\}$	
$\{1-23,13-2\}$	M_n
$\{3-21,31-2\}$	
$\{12-3,2-13\}$	
$\{32-1,2-31\}$	
$\{1-23,21-3\}$	
$\{3-21,23-1\}$	
$\{12-3,1-32\}$	
$\{32-1,3-12\}$	
$\{1-32,21-3\}$	a_n
$\{3-12,23-1\}$	
$\{1-23,12-3\}$	B_n^*
$\{3-21,32-1\}$	

Table 3: the case $|P|=2$

P	$\lvert S_n(P)\rvert$
{1-23,2-13,1-32}	F_n
{32-1,31-2,23-1}	
{3-21,2-31,3-12}	
{12-3,13-2,21-3}	
{1-23,13-2,3-12}	
{32-1,2-31,21-3}	
{3-21,31-2,1-32}	
{12-3,2-13,23-1}	
{1-23,2-13,13-2}	
{32-1,31-2,2-31}	
{3-21,2-31,31-2}	
{12-3,13-2,2-13}	
{1-23,1-32,3-12}	
{32-1,23-1,21-3}	
{3-21,3-12,1-32}	
{12-3,21-3,23-1}	
{1-23,1-32,31-2}	
{32-1,23-1,2-13}	
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{12-3,21-3,2-31}	
{1-23,13-2,31-2}	
{32-1,2-31,2-13}	
{3-21,31-2,13-2}	
{12-3,2-13,2-31}	
{1-23,21-3,13-2}	
{32-1,3-12,2-31}	
{3-21,23-1,31-2}	
{12-3,1-32,2-13}	
{12-3,21-3,2-13}	M_n
{3-21,3-12,31-2}	
{32-1,23-1,2-31}	
{1-23,1-32,13-2}	
{1-23,12-3,21-3}	
{32-1,3-21,3-12}	
{3-21,32-1,23-1}	
{12-3,1-23,1-32}	
{1-23,2-13,3-12}	n
{32-1,31-2,21-3}	
{3-21,2-31,1-32}	
{12-3,13-2,23-1}	
{1-23,2-13,2-31}	
{32-1,31-2,13-2}	
{3-21,2-31,2-13}	
{12-3,13-2,31-2}	

P	$\lvert S_n(P)\rvert$
{1-23,2-13,23-1}	n
{32-1,31-2,3-32}	
{3-21,2-31,21-3}	
{12-3,13-2,3-12}	
{1-23,23-1,31-2}	
{32-1,1-32,2-13}	
{3-21,21-3,13-2}	
{12-3,3-12,2-31}	
{1-23,1-32,3-21}	
{32-1,23-1,12-3}	
{3-21,3-12,1-23}	
{12-3,21-3,32-1}	
{1-23,2-13,31-2}	
{32-1,31-2,2-13}	
{3-21,2-31,13-2}	
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{2-31,2-13,31-2}	
{13-2,31-2,2-13}	
{2-13,23-1,1-32}	
{31-2,1-32,23-1}	
{2-31,21-3,3-12}	
{13-2,3-12,21-3}	

P	$\lvert S_n(P)\rvert$
{12-3,31-2,21-3}	n
{3-21,2-13,3-12}	
{32-1,13-2,23-1}	
{1-23,2-31,1-32}	
{12-3,31-2,2-13}	
{3-21,2-13,31-2}	
{32-1,13-2,2-31}	
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{23-1,21-3,3-12}	
{1-32,3-12,21-3}	
{21-3,23-1,1-32}	
{3-12,1-32,23-1}	

Table 4: the case $|P| = 3$

<table>
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<tr><td>{1-23,12-3,2-13}
{32-1,3-21,31-2}
{3-21,32-1,2-31}
{12-3,1-23,13-2}</td><td rowspan="10">2^{n-1}</td></tr>
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{12-3,1-23,3-12}</td></tr>
<tr><td>{1-23,2-13,21-3}
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{13-2,1-32,2-31}</td></tr>
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{3-21,23-1,3-12}
{12-3,1-32,21-3}</td><td>$\binom{n}{\lceil n/2\rceil}$</td></tr>
</table>

<table>
<tr><th>P</th><th>$|S_n(P)|$</th></tr>
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{3-21,2-31,1-23}
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{12-3,3-12,3-21}</td></tr>
<tr><td>{1-23,2-13,32-1}
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{1-23,12-3,2-31}</td><td rowspan="2">$1+\binom{n}{2}$</td></tr>
<tr><td>{1-23,2-31,23-1}
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{12-3,31-2,3-12}</td></tr>
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{3-21,21-3,1-32}
{12-3,3-12,23-1}</td><td>$2^{n-2}+1$</td></tr>
</table>

Table 5: the case $|P| = 3$