Recent Results on Paths, Permutations and Trees

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Joint work with

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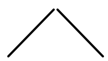
1. Dyck Paths and 3-Noncrossing Matchings

Dyck path : a lattice path on the plane consisting of

up steps  and down steps  which never go across the x-axis

peak : 

pair of noncrossing Dyck paths (P,Q) : two Dyck paths P and Q of the same length s.t. P never goes below Q

F_n denotes the set of all the pairs of noncrossing Dyck paths of length $2n$.

1. Dyck Paths and 3-Noncrossing Matchings

matching : a perfect matching which can be drawn a way that the elements $1, 2, \ldots, 2n$ are arranged on a line and the edges are drawn as arcs

– k-nonnesting matching : no k arcs covered one by another
– k-noncrossing matching : no k mutually crossing arcs

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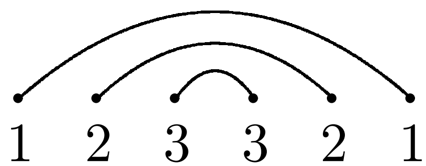
– k-nonnesting matching : no k arcs covered one by another

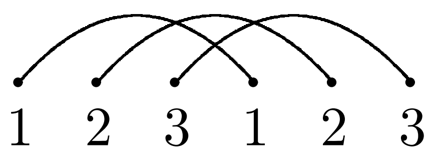
– k-noncrossing matching : no k mutually crossing arcs

– 3-nonnesting matching $\leftrightarrow$ *abccba*-free matching : no 3 arcs formed as



$M_n(abccba)$ denotes the set of all the 3-nonnesting matchings on $[2n]$.

– 3-noncrossing matching $\leftrightarrow$ *abcabc*-free matching : no 3 arcs formed as



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1. Dyck Paths and 3-Noncrossing Matchings

We will construct two bijections :

• 3-noncrossing matchings $\Longleftrightarrow$ pairs of noncrossing Dyck paths

M. Klazar first brought up the problem of enumerating 3-noncrossing partitions in

On abab-free and abba-free set partitions, Europ. J. Combin., 17 (1996), 53-68.

Later he pointed out that it is still unknown for the exact asymptotics or enumeration for k-noncrossing matchings, when $k \geq 3$, in

Bell numbers, their relatives, and algebraic differential equations, J. Combin. Theory, Series A, 102 (2003), 63-87.

1. Dyck Paths and 3-Noncrossing Matchings

We will construct two bijections :

- 3-noncrossing matchings $\Longleftrightarrow$ pairs of noncrossing Dyck paths
- 3-nonnesting matchings $\Longleftrightarrow$ pairs of noncrossing Dyck paths

The enumeration of 3-nonnesting matchings was first studied by D. Gouyou-Beauchamps in

D. Gouyou-Beauchamps, Standard Young tableaux of height 4 *and* 5*, European J. Combin., 10 (1989), no. 1, 69-82.*

He showed that there is a bijection between involutions with no decreasing sequence of length 5 and pairs of noncrossing Dyck left factors by an inductive construction. Hence he actually give the bijections between 3-nonnesting matchings and pairs of noncrossing Dyck paths, although he didn't mention the concept of nonnesting matchings.

$M_n(abcabc) \Longleftrightarrow F_n$

$M_n(abcabc) \Longrightarrow F_n$

Recursive construct :

For $M \in M_{n+1}(abcabc)$, $M' = M \setminus \{n+1\} \in M_n(abcabc)$, if
the first $(n+1)$ is after the i-th position of M',

the second $(n+1)$ is after the j-th position of M',

and suppose $(P', Q') \in F_n \leftrightarrow M' \in M_n(abcabc)$,

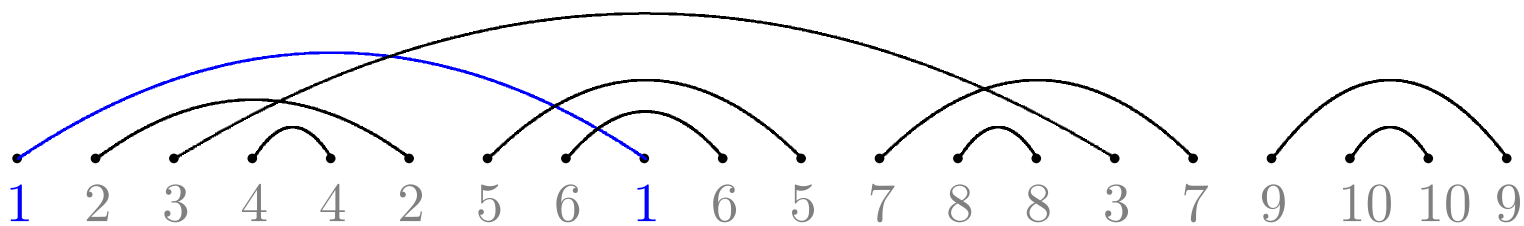
then
P : insert a peak after the i-th step of P',

Q : insert a peak after the j-th step of Q'.

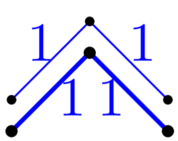
$(P, Q) \in F_{n+1}$ is the desired pair of Dyck paths.

$M_n(abcabc) \Longrightarrow F_n$

For a matching $M \in M_{10}(abcabc)$:



The corresponding pair of Dyck paths :



If $(n+1)_1$: after i-th position,

$(n+1)_2$: after j-th position,

then

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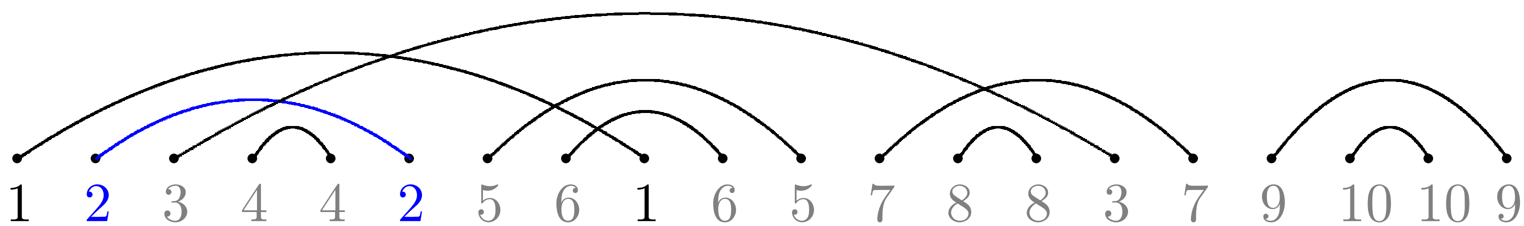

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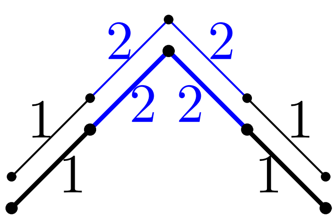

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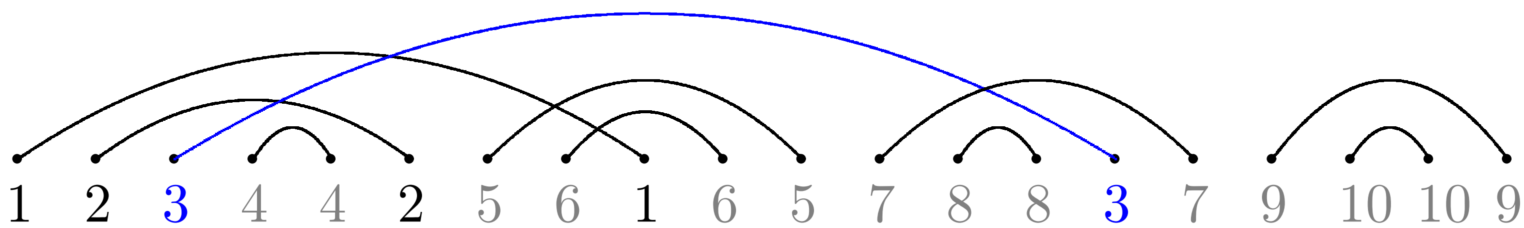

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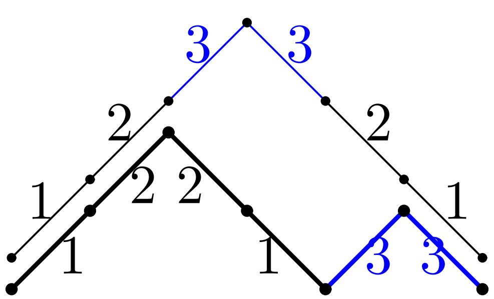

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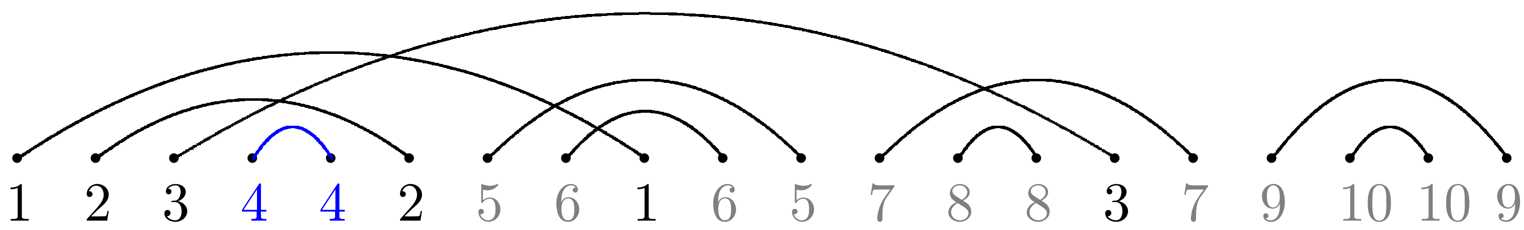

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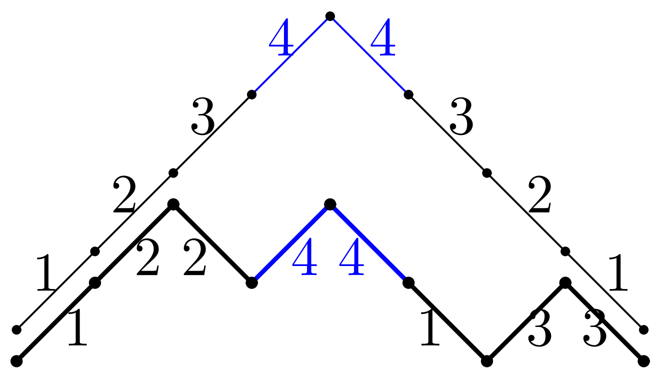

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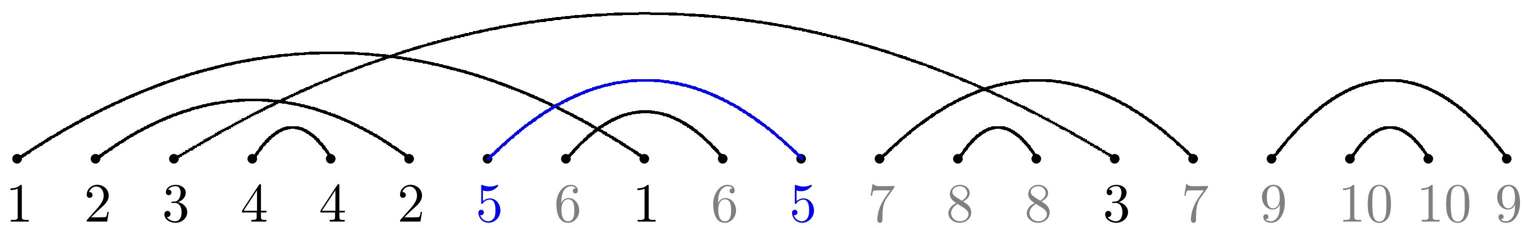

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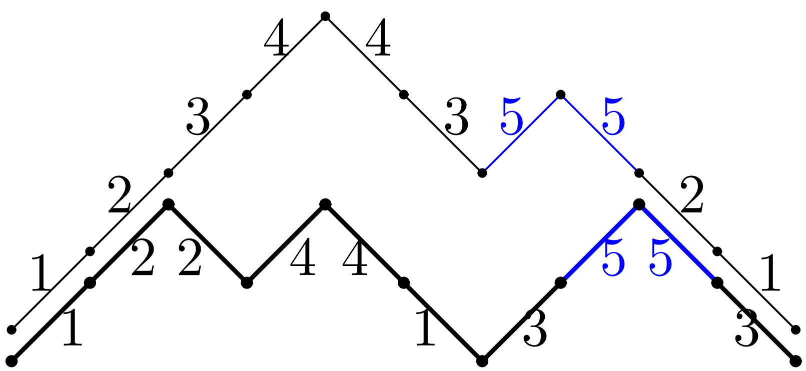

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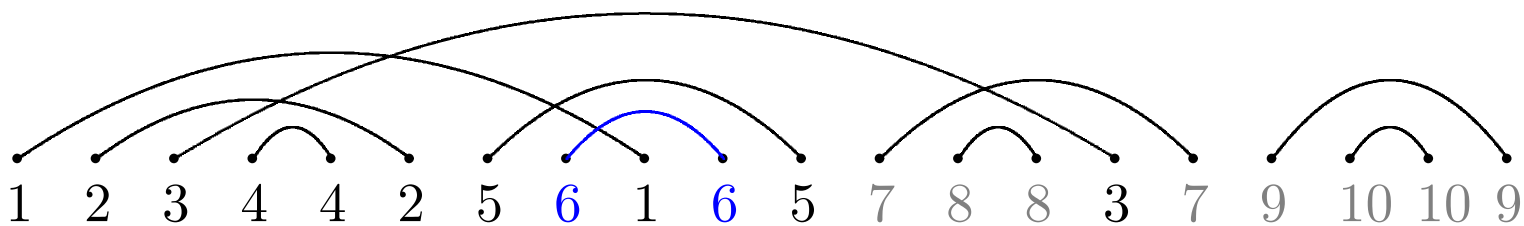

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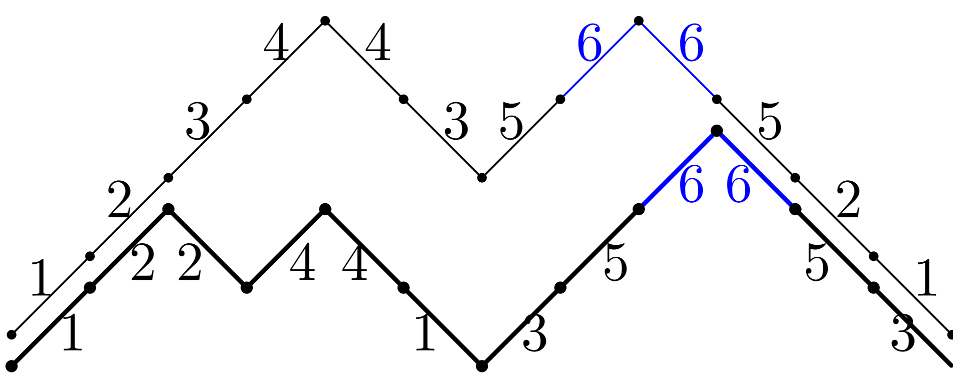

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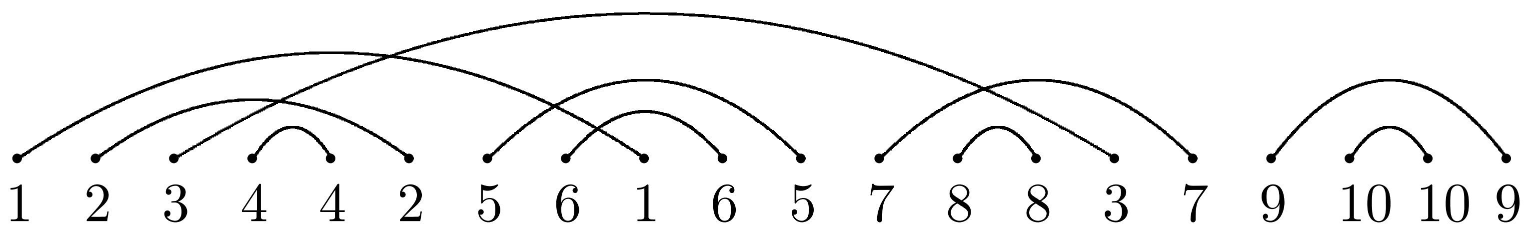

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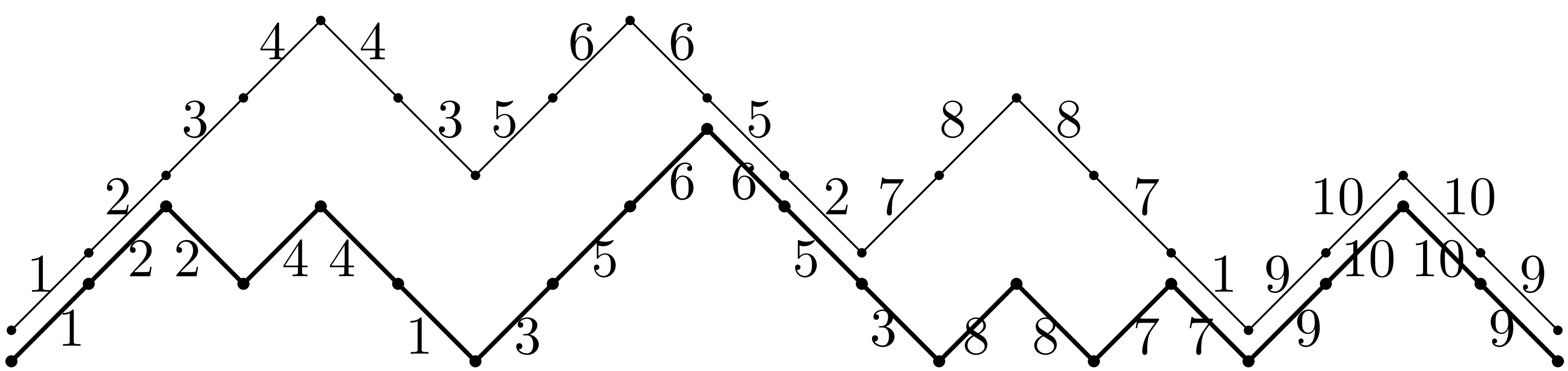


$M_n(abcabc) \Longleftarrow F_n$

Recursive construct :

Give $(P, Q) \in F_{n+1}$, suppose that

the i-th and $(i+1)$-th steps are the rightmost peak in P,

the j-th and $(j+1)$-th steps are the leftmost peak in Q s.t. $j \geq i$.

$(P', Q') \in F_n$ is obtained by deleting the above two peaks in P and Q respectively, and $M' \in M_n(abcabc)$ is the corresponding matching.

Now we

insert $(n+1)$ after the $(i-1)$-th elements of M',

insert $(n+1)$ after the $(j-1)$-th elements of M',

then we obtain a matching $M \in M_{n+1}(abcabc)$.

$M_n(abccba) \Longleftrightarrow F_n$

$M_n(abccba) \Longrightarrow F_n$

For $M \in M_n(abccba)$, suppose $M = X_1Y_1X_2Y_2 \cdots X_lY_l$,

X_i be the successive sequence of the beginnings of the arcs,

Y_i be the successive sequence of the ends of the arcs,

then

$P = u^{|X_1|}d^{|Y_1|}u^{|X_2|}d^{|Y_2|} \dots u^{|X_l|}d^{|Y_l|}$.

Q is determined by $\pi = Y_1Y_2 \cdots Y_l$ as C. Krattenthaler described :

Suppose $m_1, m_2, \cdots, m_s$ be the left-to-right maxima in π,

write $\pi = \omega_1 m_1 \omega_2 m_2 \cdots \omega_s m_s$,

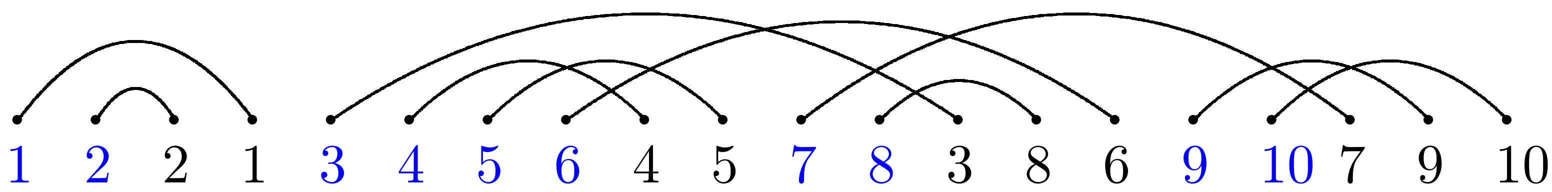
then m_i is translated into $m_i - m_{i-1}$ up steps ($m_0 = 0$),

ω_i is translated into $|\omega_i| + 1$ down steps,

we obtain the desired Dyck path Q.

$M_n(abccba) \Longrightarrow F_n$

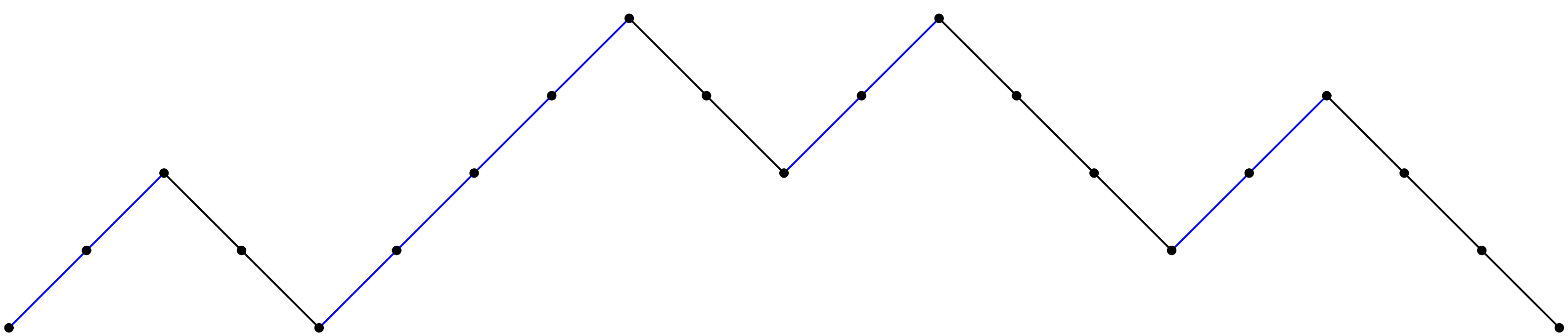
For a 3-nonnesting matching $M \in M_{10}(abccba)$:



The corresponding pair of Dyck paths :

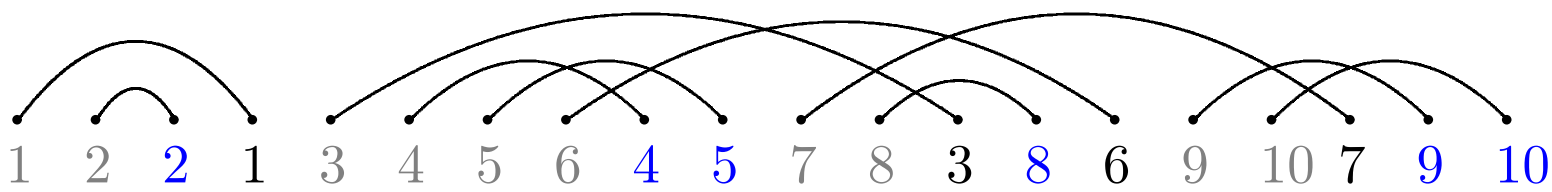
$$M = X_1 Y_1 X_2 Y_2 \cdots X_l Y_l$$

$$P = u^{|X_1|} d^{|Y_1|} u^{|X_2|} d^{|Y_2|} \cdots u^{|X_l|} d^{|Y_l|}$$



$M_n(abccba) \Longrightarrow F_n$

For example, for a 3-nonnesting matching $M \in M_{10}(abccba)$:

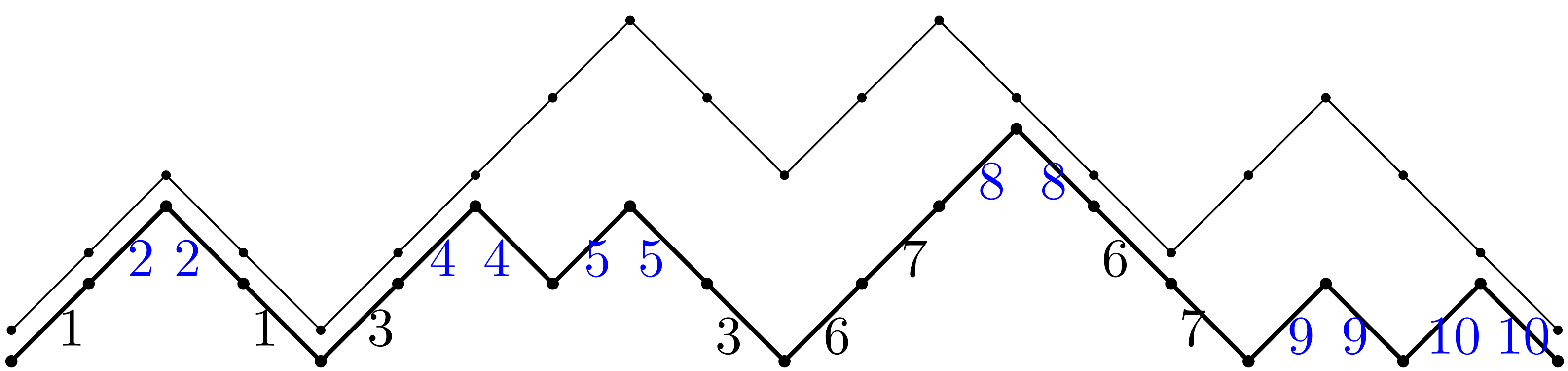


The corresponding pair of Dyck paths :

If $\pi = \omega_1 m_1 \omega_2 m_2 \cdots \omega_s m_s$,

then $m_i \rightarrow m_i - m_{i-1}$ up steps,

$\omega_i \rightarrow |\omega_i| + 1$ down steps



$M_n(abccba) \Longleftarrow F_n$

Give $(P, Q) \in F_n$.

Suppose $P = u^{x_1} d^{y_1} u^{x_2} d^{y_2} \cdots u^{x_l} d^{y_l}$,

then $\pi_1 = X_1 X_2 \cdots X_l$ such that $|X_i| = x_i$, $\pi_1 = 1\ 2\ \cdots\ n$.

In Q, the up steps are labelled with $\{1, 2, \cdots, n\}$ in order ; for the down steps d, if d follows an up step u, then label d with the same label as the label of u, otherwise we label d with $\min\{l_u \setminus l_d\}$. Then we read the labels of all the down steps of the path from left to right

$\pi_2 = Y_1 Y_2 \cdots Y_l$ such that $|Y_i| = y_i$ for $i = 1, 2, \ldots l$.

Now we merge π_1 and π_2 together and get
$M = X_1 Y_1 X_2 Y_2 \cdots X_l Y_l \in M_n(abccba)$.

Lemma (D. Gouyou-Beauchamps) The number of pairs of noncrossing Dyck paths of length $2n$ is : $|F_n| = c_n c_{n+2} - c_{n+1}^2$, where c_n is the nth Catalan number.

Theorem The numbers of 3-noncrossing and 3-nonnesting partitions on $[2n]$ are

$$|M_n(abcabc)| = |M_n(abccba)| = c_n c_{n+2} - c_{n+1}^2.$$

For the poor partitions, that is, unmatched singletons are allowed, we easily obtain :

Corollary The numbers of 3-nonnesting and 3-noncrossing poor partitions on $[n]$ are equal to

$$\sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} |F_i| = \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{3!n!(2i+2)!}{(n-2i)!i!(i+1)!(i+2)!(i+3)!}.$$

We are also working on the bijection :

k-noncrossing matchings $\Longleftrightarrow$ k-nonnesting matchings.

2. Restricted Partitions and Labelled 2-Motzkin paths

A **partition** P of $[n] = \{1, 2, ..., n\}$ is a collection $(B_1, B_2, ..., B_k)$ of nonempty disjoint subsets of [n], called *blocks*, whose union is $[n]$.

We use a directed graph on the vertex set $[n]$ to represent the partition P. For each block B_i, we associate it with a directed path starting with the minimum element in B_i, and going through elements in B_i in the increasing order. Note that when a block B_i has only one element, the corresponding path is an isolated vertex. And if the element i is in the block B_j, we label the point of the graph with the number i.

2. Restricted Partitions and Labelled 2-Motzkin paths

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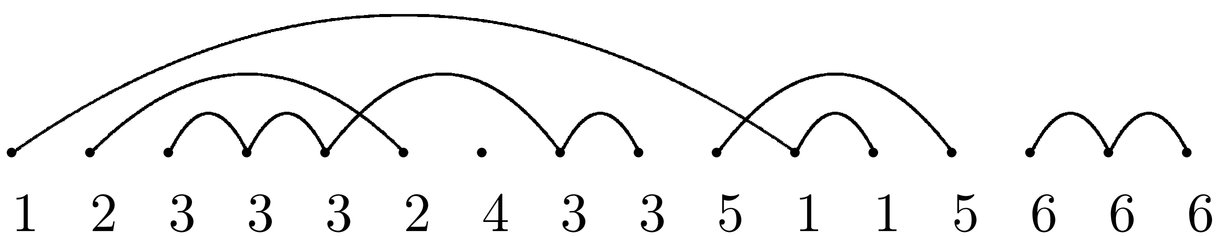
For example, the partition

$P = 1, 11, 12 \setminus 2, 6 \setminus 3, 4, 5, 8, 9 \setminus 7 \setminus 10, 13 \setminus 14, 15, 16$:



2. Restricted Partitions and Labelled 2-Motzkin paths

partition

– *abab*-free partition : partition does not contain a subsequence of the form $\cdots a \cdots b \cdots a \cdots b \cdots$, denote $\mathcal{P}_n(abab)$ for the set of *abab*-free partitions of length n.
– *abba*-free partition : partition does not contain a subsequence of the form $\cdots a \cdots b \cdots b \cdots a \cdots$, denote $\mathcal{P}_n(abba)$ for the set of *abba*-free partitions of length n.
– $abba\bar{b}$-free partition : partition s.t. every subsequence with the form $\cdots a \cdots b \cdots b \cdots a \cdots$ can be extend to a subsequence of the form $\cdots a \cdots b \cdots b \cdots a \cdots b \cdots$, denote $\mathcal{P}_n(abba\bar{b})$ for the set of $abba\bar{b}$-free partitions of length n.

2. Restricted Partitions and Labelled 2-Motzkin paths

2-Motzkin path : Motzkin path whose horizontal steps are colored with two distinct colors.

– $\mathcal{M}_n(\infty)$: the set of 2-Motzkin paths without any wave step on the x-axis,

– $\mathcal{M}_n(j)$: the set of 2-Motzkin paths with the wave steps only on the levels $1, 2, ..., j$,

– $\mathcal{L}_n(j)$: the set of 2-Motzkin paths with the wave steps only on the levels $0, 1, ..., j-1$,

where $j = 1, \cdots, \infty$, the level of a step is the larger y-coordinate of its two ends.

Mapping φ : partitions $\longleftrightarrow$ 2-Motzkin paths

– the start point on a directed path $\leftrightarrow$ up step,

– the end point on a directed path $\leftrightarrow$ down step,

– the other point on a directed path $\leftrightarrow$ wave step,

– the isolated point $\leftrightarrow$ horizontal step.

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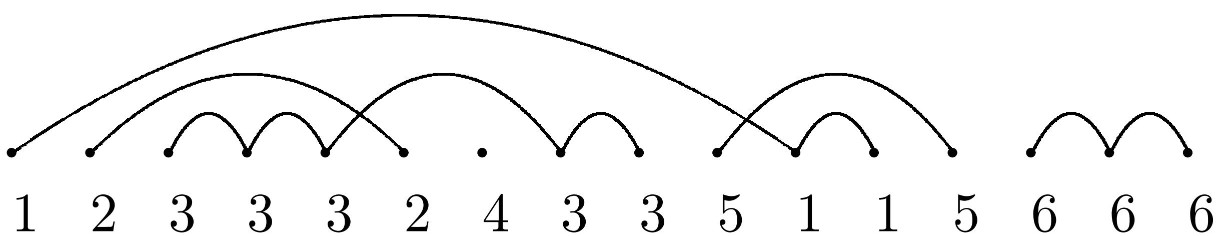
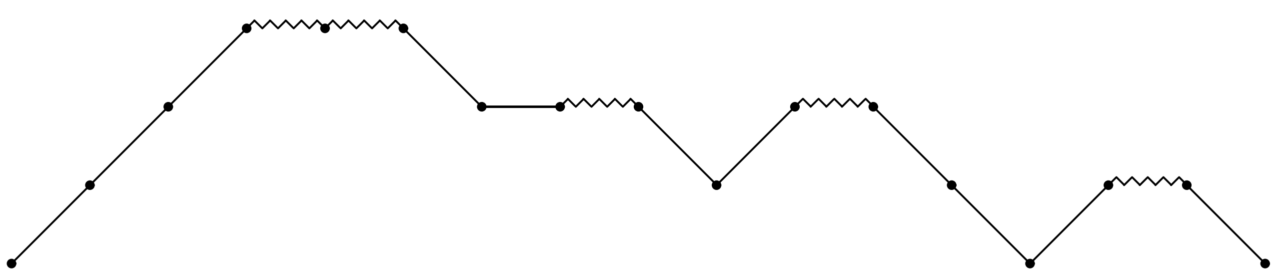
For example, the partition :



the corresponding 2-Motzkin path :



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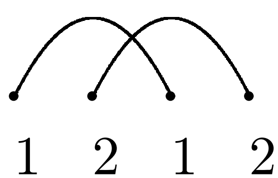
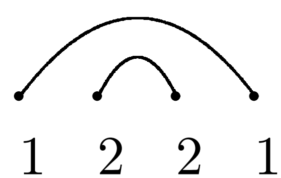
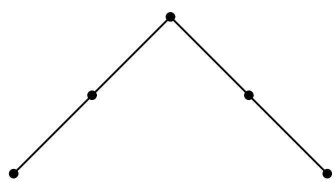
The mapping is neither surjective nor injective, since

both 

and 

are correspondence to the same 2-Motzkin path :



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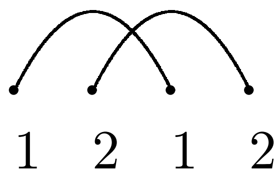
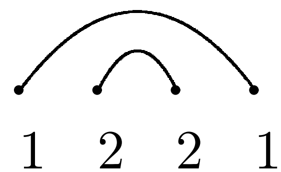
both 

and 

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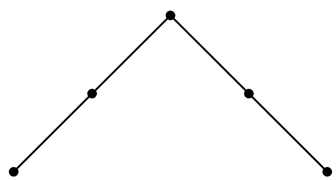
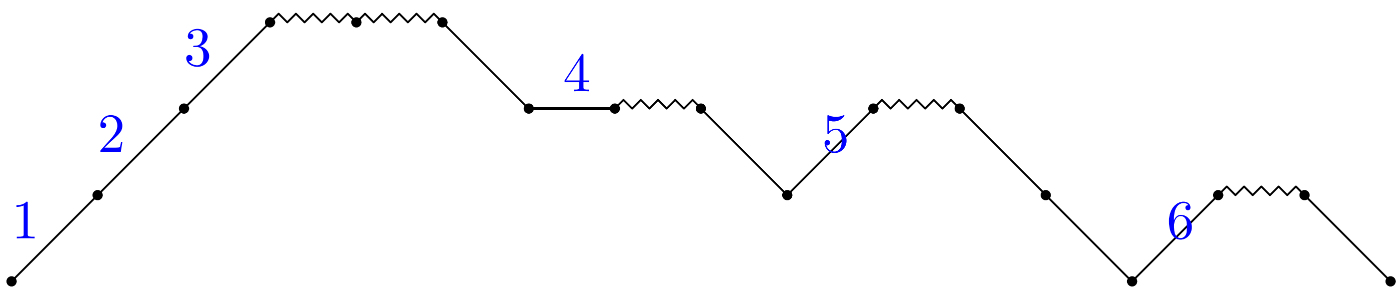


But if we give certain label rules for each steps of 2-Motzkin path, we will get some beautiful bijections between certain pattern free partitions and restricted 2-Motzkin paths.

Nearest Rule For a 2-Motzkin path, we label the up steps and horizontal steps with the alphabet $\{1, 2, 3, \cdots\}$ in order from left to right.

For example, the nearest labelled 2-Motzkin path :



Nearest Rule For a 2-Motzkin path, we label the up steps and horizontal steps with the alphabet $\{1, 2, 3, \cdots\}$ in order from left to right. For a wave or down step, label it with the same alphabet as the nearest up step before it and on the same level, for a wave step on the x-axis, we label it with 0.

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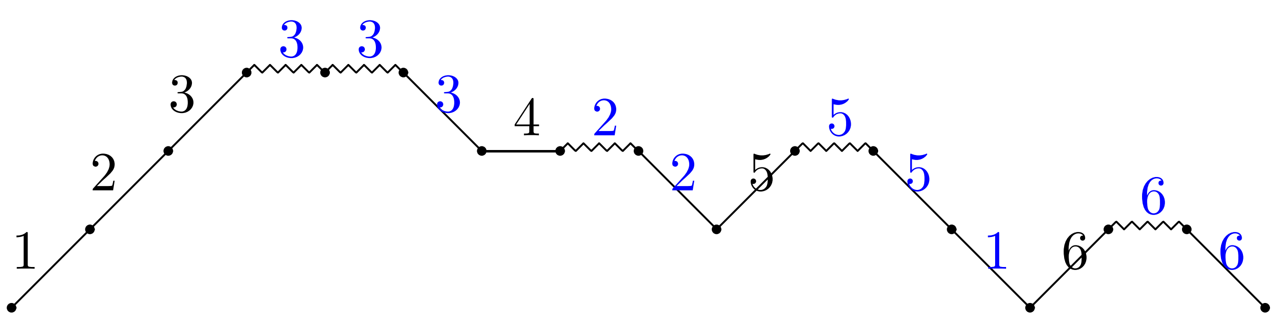

Nearest Rule

Now the mapping φ is a bijection between $\mathcal{P}_n(abab)$ and the paths in $\mathcal{M}_n(\infty)$ labelled by the nearest rule.

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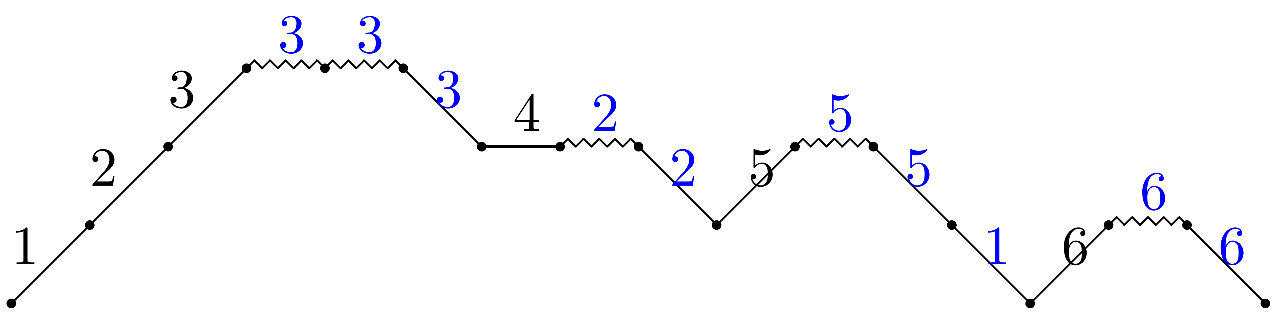
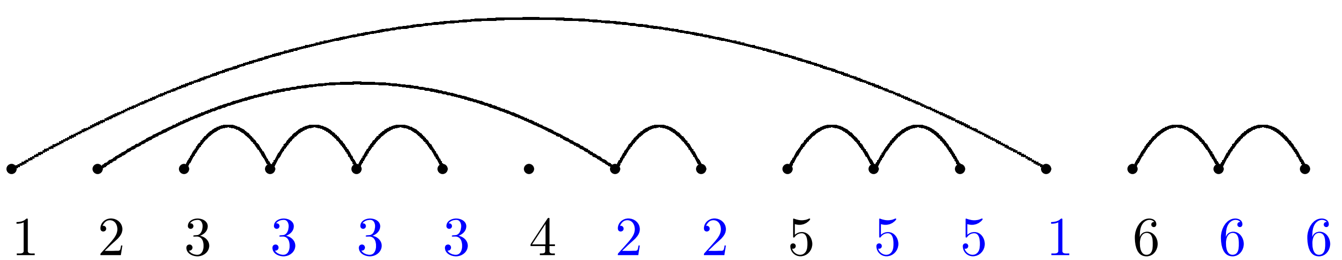


The corresponding *abab*-free partition :



Nearest Rule

We have the following theorem :

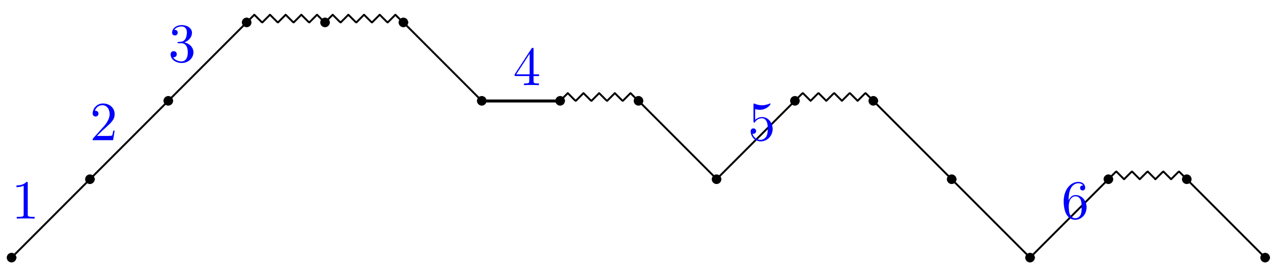
Theorem φ is a bijection between $\mathcal{P}_n(abab)$ and the paths in $\mathcal{M}_n(\infty)$ labelled by the nearest rule.

Corollary

1. φ is a bijection between $\mathcal{P}_n(abab, a_1 \cdots a_j ccca_j \cdots a_1)$ and paths in $\mathcal{M}_n(j)$ labelled by the nearest rule, where $j = 1, 2, \cdots, \infty$.
2. φ is a bijection between $\mathcal{CP}_{n+2}(abab)$ and all the 2-Motzkin paths in $\mathcal{M}_n$ labelled by the nearest rule.
3. φ is a bijection between $\mathcal{CP}_{n+2}(abab, a_1 \cdots a_j ccca_j \cdots a_1)$ and 2-Motzkin paths in $\mathcal{L}_n(j)$ labelled by the nearest rule.

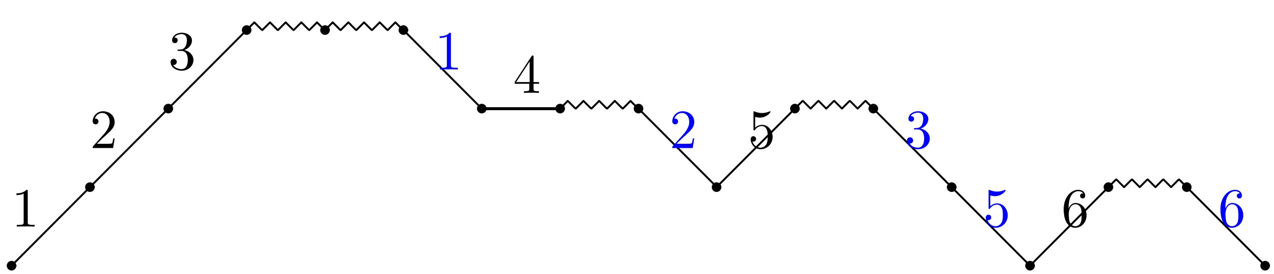
Furthest Rule : For a 2-Motzkin path, we label the up steps and horizontal steps with the alphabet $\{1, 2, 3, \cdots\}$ in order from left to right.

For example, we have the following furthest labelled 2-Motzkin path :



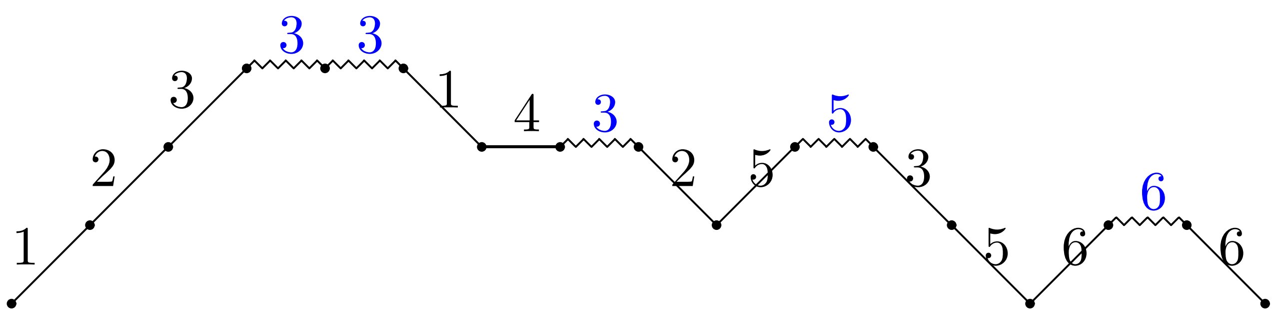
Furthest Rule : For a 2-Motzkin path, we label the up steps and horizontal steps with the alphabet $\{1, 2, 3, \cdots\}$ in order from left to right ; for the down steps, we label them in order from left to right with the label sets of the up steps.

For example, the furthest labelled 2-Motzkin path :



Furthest Rule : For a 2-Motzkin path, we label the up steps and horizontal steps with the alphabet $\{1, 2, 3, \cdots\}$ in order from left to right ; for the down steps, we label them in order from left to right with the label sets of the up steps ; and a wave step on the x-axis, we label it with 0, otherwise, we label it with the label of the right most up step on the left of it.

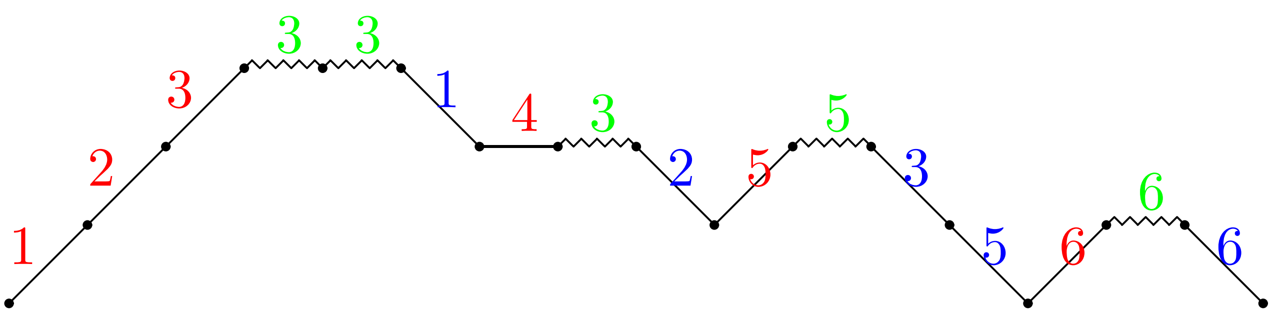
For example, the furthest labelled 2-Motzkin path :



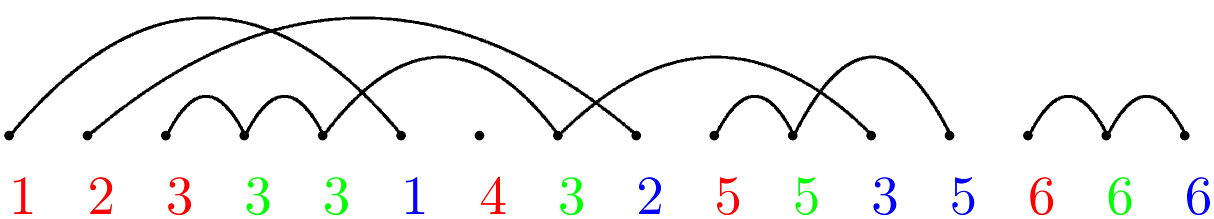
Furthest Rule

Now the mapping φ is a bijection between $\mathcal{P}_n(abba\bar{b})$ and the paths in $\mathcal{M}_n(\infty)$ labelled by the furthest rule.

For example, the furthest labelled 2-Motzkin path :



the corresponding $abba\bar{b}$-free partition :



Further Rule

We have the following theorem :

Theorem

1. φ is a bijection between $\mathcal{P}_n(abba\bar{b})$ and 2-Motzkin paths in $\mathcal{M}_n(\infty)$ labelled by the furthest rule,
2. φ is a bijection between $\mathcal{P}_n(abba\bar{b}, a_1 \cdots a_j cca_1 \cdots a_j c)$ and 2-Motzkin paths in $\mathcal{M}_n(j)$ labelled by the furthest rule, where $j = 1, 2, \cdots, \infty$.
3. φ is a bijection between $\mathcal{CP}_{n+2}(abba\bar{b}, a_1 \cdots a_j cca_1 \cdots a_j c)$ and 2-Motzkin paths in $\mathcal{L}_n(j)$ labelled by the furthest rule, where $j = 1, 2, \cdots, \infty$.

Enumeration

We denote the cardinality of $\mathcal{M}_n(j)$ by $m_n^{(j)}$, and the cardinality of $\mathcal{L}_n(j)$ by $l_n^{(j)}$, where $j = 0, 1, \cdots, \infty$. Then we have the following relations :

$$m_n^{(0)} = l_n^{(0)} = m_n; \; m_n^{(\infty)} = l_{n-1}^{(\infty)} = c_n;$$

$$l_n^{(j)} = 2l_{n-1}^{(j)} + \sum_{i=2}^{n} l_{i-2}^{(j-1)} l_{n-i}^{(j)}, j = 1, 2, \cdots, \infty;$$

$$l_n^{(j)} = m_n^{(j-1)} + \sum_{i=1}^{n} m_{i-1}^{(j-1)} l_{n-i}^{(j)}, j = 1, 2, \cdots, \infty;$$

$$m_n^{(j)} = m_{n-1}^{(j)} + \sum_{i=2}^{n} l_{i-2}^{(j)} m_{n-i}^{(j)}, j = 0, 1, \cdots, \infty.$$

Now we set $r_n^{(k)}$ as follows

$$r_0^{(2j)} = 1;$$

$$r_n^{(2j)} = l_n^{(j)}, j, n \geq 1;$$

$$r_n^{(2j+1)} = m_n^{(j)}, j, n \geq 0.$$

Denote $R^{(k)}(x) = \sum_{n=0}^{\infty} r_n^{(k)} x^n$ as the generating function of $r_n^{(k)}$, then we can get

$$R^{(k)}(x) = 1 + xR^{(k-1)}(x)R^{(k)}(x), \qquad j \geq 2.$$

Using the above recursive relation for the generating function, we found that the sequences $r_n^{(k)}$ are the same as the sequences by E. Barcucci, A. D. Lungo, E. Pergola and R. Pinzani in

From Motzkin to Catalan permutations, Discrete Mathemathcis, 217 (2000), 33-49..

These number sequences lie between the Motzkin and the Catalan numbers, and provide a "discrete continuity" between these two sequences.

The first few numbers of $r_n^{(k)}$

n	0	1	2	3	4	5	6	7	8	$\cdots$
$r_n^{(1)} = m_n$	1	1	2	4	9	21	51	127	323	$\cdots$
$r_n^{(2)} = l_{n-1}^{(1)}$	1	1	2	5	13	35	96	267	750	$\cdots$
$r_n^{(3)} = m_n^{(1)}$	1	1	2	5	14	41	123	374	1147	$\cdots$
$r_n^{(4)} = l_{n-1}^{(2)}$	1	1	2	5	14	42	131	418	1352	$\cdots$
$r_n^{(5)} = m_n^{(2)}$	1	1	2	5	14	42	132	428	1417	$\cdots$
$r_n^{(6)} = l_{n-1}^{(3)}$	1	1	2	5	14	42	132	429	1429	$\cdots$
$\cdots$										$\cdots$
$r_n^{(2j)} = l_{n-1}^{(j)}$										$\cdots$
$r_n^{(2j+1)} = m_n^{(j)}$										$\cdots$
$\cdots$										$\cdots$
$r_n^{(\infty)} = c_n$	1	1	2	5	14	42	132	429	1430	$\frac{1}{n+1}\binom{2n}{n}$

Use the labeling method above, we have

Theorem There is a bijection between 2-regular *abba*-free partitions and connected *abba*-free partitions of the same length.

2-regular partition : partition without two connected elements in one block

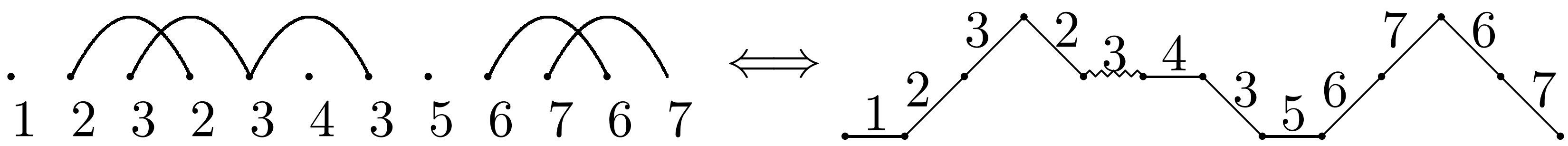
$\mathcal{P}_n(abba, 2)$ denotes the set of 2-regular *abba*-free partitions of length n.

connected partition : partition s.t. every element except for the beginning and ending is covered by an arc

$\mathcal{CP}_n(abba)$ denotes the set of the connected *abba*-free partitions of length n

$\mathcal{P}_n(abba, 2) \Longleftrightarrow \mathcal{CP}_n(abba)$

We use an example to show the proof :

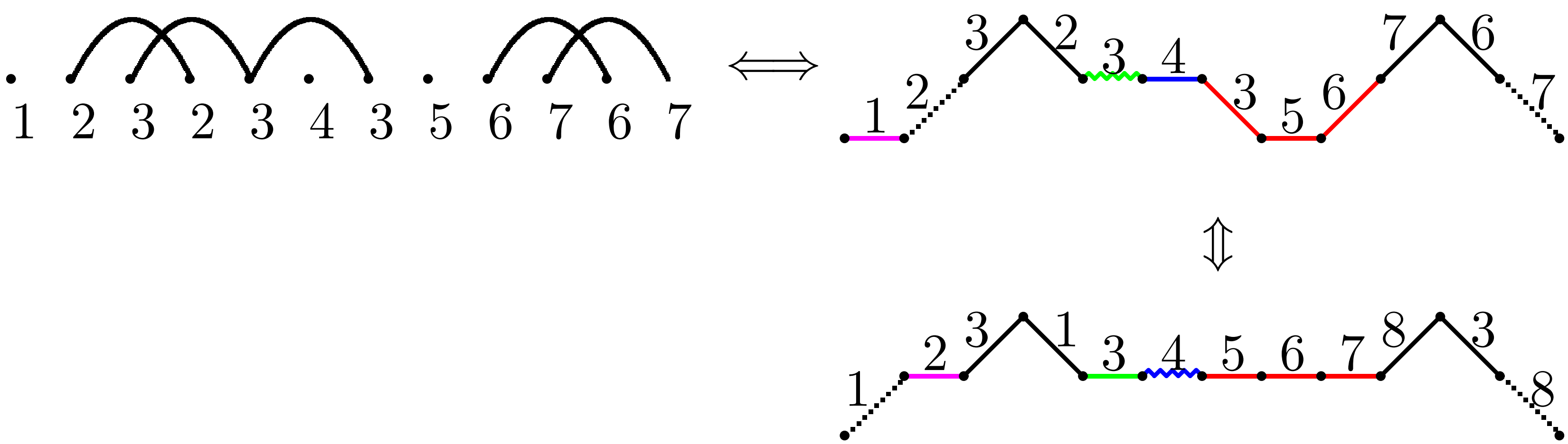


Use the Furthest Rule, there is a bijection :

$\varphi(\mathcal{P}_n(abba, 2)) \Longleftrightarrow$ those paths in $\mathcal{M}_n(1)$ which contains no pairs of successive steps uw, ud, ww or wd.

$\mathcal{P}_n(abba, 2) \Longleftrightarrow \mathcal{CP}_n(abba)$

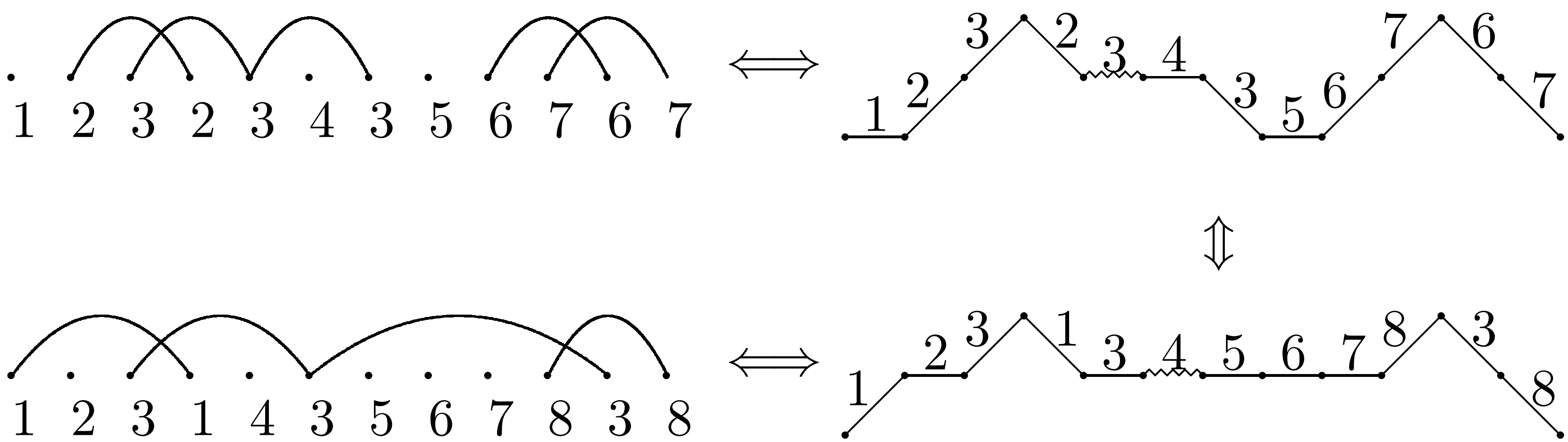
We use an example to show the proof :



$\mathcal{M}(1)$ containing of neither uw, ww, wd nor ud on the level 1 $\Longleftrightarrow$
$\mathcal{M}(1)$ containing of no step on the level 0

$\mathcal{P}_n(abba, 2) \Longleftrightarrow \mathcal{CP}_n(abba)$

We use an example to show the proof :



Use the Furthest Rule, there is a bijection :

$\mathcal{CP}_n(abba) \Longleftrightarrow$ 2-Motzkin paths with w only on level 1 and no step on level 0

Now we get the bijection : $\mathcal{P}_n(abba, 2) \Longleftrightarrow \mathcal{CP}_n(abba)$

Němeček and Klazar gave a bijection between the 2-regular *abba*-free partition and nonnegative words in

A bijection between nonegative words and sparse abba-free partitions, Discrete Mathematics, 265(1-3) (2003), 411-416.

In the above example, if we disregard the first u and the last d in the path in $\mathcal{CP}_n(abba)$, and define the following map $\psi : \{U, D, H, W\} \longmapsto \{1, 0, -1\}$ as follows :

$$\psi(U) = \psi(H) = 1, \psi(D) = -1, \psi(W) = 0.$$

then we get a nonnegative word $1, 1, -1, 1, 0, 1, 1, 1, 1, -1$ of length 10 with the alphabet $1, 0, -1$, which corresponds to $P_{12} = 123234356767 \in \mathcal{P}_{12}(abba, 2)$. If we use the bijection by Němeček and Klazar, we get the same nonnegative word.

3. A bijection between 3214-avoiding involutions and Motzkin paths

$\mathcal{S}_n$: the set of permutations of $[n] = \{1, \ldots, n\}$

For $\pi \in \mathcal{S}_n$ and $\sigma \in \mathcal{S}_k$,

involution : $\pi^{-1} = \pi$.

π *avoids* σ : whenever π contains no subsequence with all of the same pairwise comparisons as σ.

$\mathcal{I}_n(\sigma)$: the set of involutions in $\mathcal{S}_n$ which avoid the pattern σ.

3. A bijection between 3214-avoiding involutions and Motzkin paths

Guibert proved that

$|\mathcal{I}_n(1234)| = |\mathcal{I}_n(1243)| = |\mathcal{I}_n(3412)| = |\mathcal{I}_n(4321)| = m_n,$

where m_n is the n-th Motzkin number, in

Combinatoire des permutations à motifs exclus en liaison avec mots, cartes planaires et tableaux de Young, Ph. D. Thesis, University Bordeaux I, France, 1995.

And he further conjectured that $|\mathcal{I}_n(2143)| = |\mathcal{I}_n(1432)| = m_n$.

Later, Guibert, Pergola and Pinzani proved that $|\mathcal{I}_n(2143)| = m_n$ by generating trees in

Vexillary involutions are enumerated by Motzkin numbers, Ann. Combin. 5 (2001) 153–174.

3. A bijection between 3214-avoiding involutions and Motzkin paths

A.D. Jaggard gave an affirmative answer to this conjecture by introducing the equivalence of $\mathcal{I}_n(1234)$ and $\mathcal{I}_n(3214)$ in

Prefix exchanging and pattern avoidance by involutions, Elect. J. Combin. 9 (2003) #R16.

But it is still elusive to find a bijection between $\mathcal{I}_n(3214)$ and the set of Motzkin paths of length n.

Note that $1432 \xrightarrow{reverse} 2341 \xrightarrow{complement} 3214$.

generating tree : a rooted, labelled tree in which the size and labels of the set of children of each vertex x are determined solely by the label of x.
succession system : a set of succession rules consisting of a basis which is the label of the root, and an inductive step which is a set of succession rules that yields a multiset of labelled children upon which is depends.
Thus, any particular generating tree can be recursively defined by a succession system.

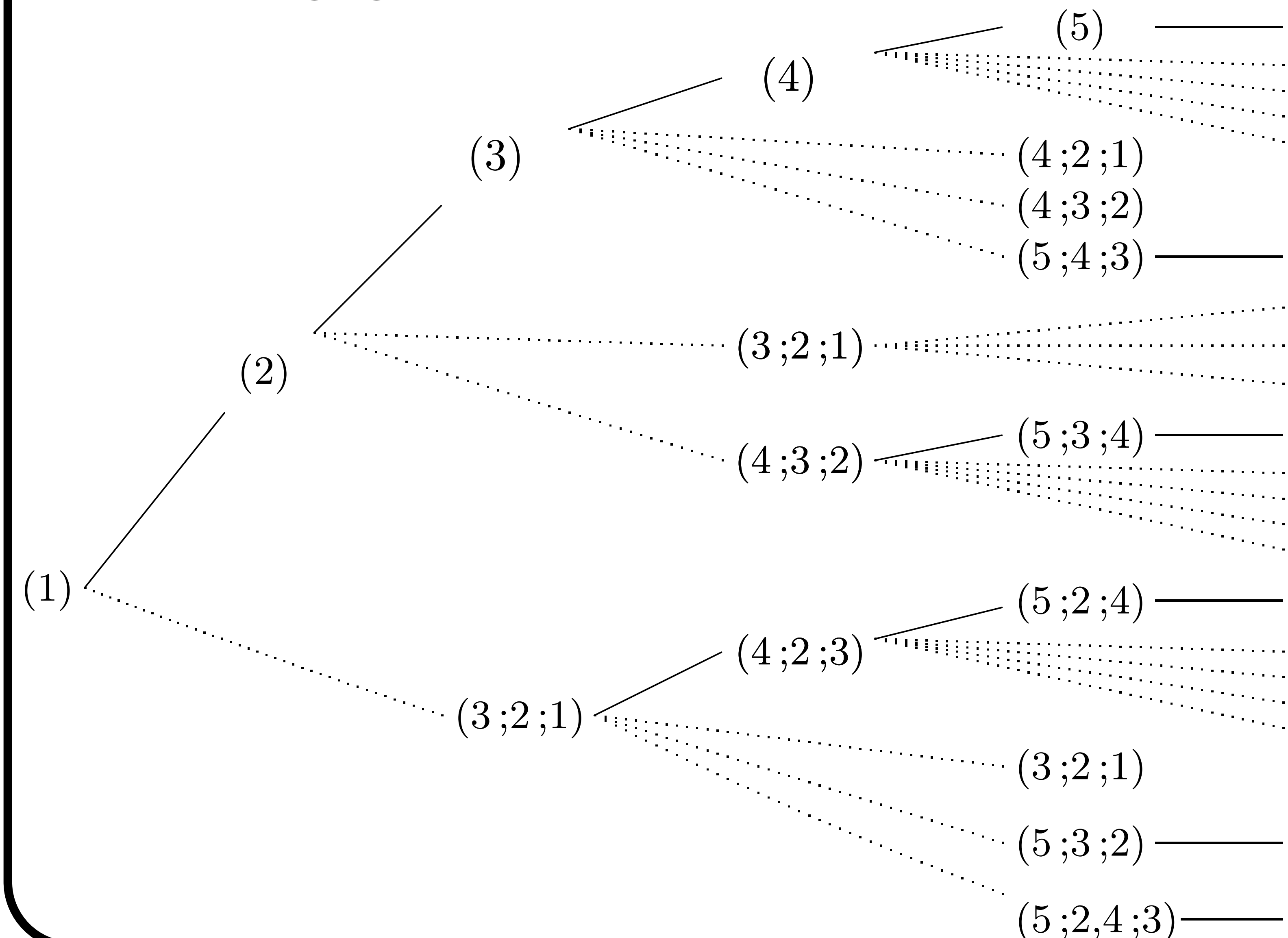
we construct a special generating tree T in which nodes on the nth level have label $(n+1)$ or $(t; d_1, d_2, \ldots, d_h; p)$. Moreover, T is characterized by succession system whose label is on the nth level of this generating tree.

The following is the generating tree for both $\mathcal{I}_n(3214)$ and Motzkin paths of length n :

Succession System :

$$
\left\{
\begin{array}{lll}
(1) & & \\
(n+1) & \xrightarrow{1} & (n+2) \\
 & \xrightarrow{2} & (n+2;2;1),(n+2;3;2),\ldots,(n+2;n+1;n), \\
 & & (n+3;n+2;n+1) \\
\text{when} & & t \leq n \\
(t;d_1,d_2, & \xrightarrow{2} & (d_1+1;2;1),(d_1+1;3;2),\ldots,(d_1+1;d_1;d_1-1), \\
\ldots,d_h;p) & & (d_2+1;d_1+1;d_1),\ldots,(d_2+1;d_1,d_2;d_2-1), \\
 & & \ldots, \\
 & & (d_h+1;d_1,\ldots,d_{h-2},d_{h-1}+1;d_{h-1})\ldots,(d_h+1;d_1,\ldots,d_{h-1},d_h;d_h-1) \\
 & & (t+1;d_1,\ldots,d_{h-1},d_h+1;d_h),\ldots,(t+1;d_1,\ldots,d_h,t+1;t) \\
\text{when} & & p < n \\
(n+1;d_1, & \xrightarrow{1} & (n+2;d_1,\ldots,d_h;n+1) \\
\ldots,d_h;p) & \xrightarrow{2} & (d_1+1;2;1),(d_1+1;3;2),\ldots,(d_1+1;d_1;d_1-1), \\
 & & (d_2+1;d_1+1;d_1),\ldots,(d_2+1;d_1,d_2;d_2-1), \\
 & & \ldots, \\
 & & (d_h+1;d_1,\ldots,d_{h-2},d_{h-1}+1;d_{h-1}),(d_h+1;d_1,\ldots,d_{h-1},d_h;d_h-1), \\
 & & (n+3;d_1,\ldots,d_{h-1},d_h+1;d_h),\ldots,(n+3;d_1,\ldots,d_h,n+2;n+1) \\
(n+1;d_1, & \xrightarrow{1} & (n+2;d_1,\ldots,d_h;n+1) \\
\ldots,d_h;n) & \xrightarrow{2} & (d_1+1;2;1),(d_1+1;3;2),\ldots,(d_1+1;d_1;d_1-1), \\
 & & (d_2+1;d_1+1;d_1),\ldots,(d_2+1;d_1,d_2;d_2-1), \\
 & & \ldots, \\
 & & (d_h+1;d_1,\ldots,d_{h-2},d_{h-1}+1;d_{h-1})\ldots,(d_h+1;d_1,\ldots,d_{h-1},d_h;d_h-1) \\
 & & (n+2;d_1,\ldots,d_h+1;d_h),\ldots,(n+2,d_1,\ldots,d_h,n+1;n), \\
 & & (n+3;d_1,\ldots,d_h,n+2;n+1).
\end{array}
\right.
$$

The following figure describes the first five levels of T :

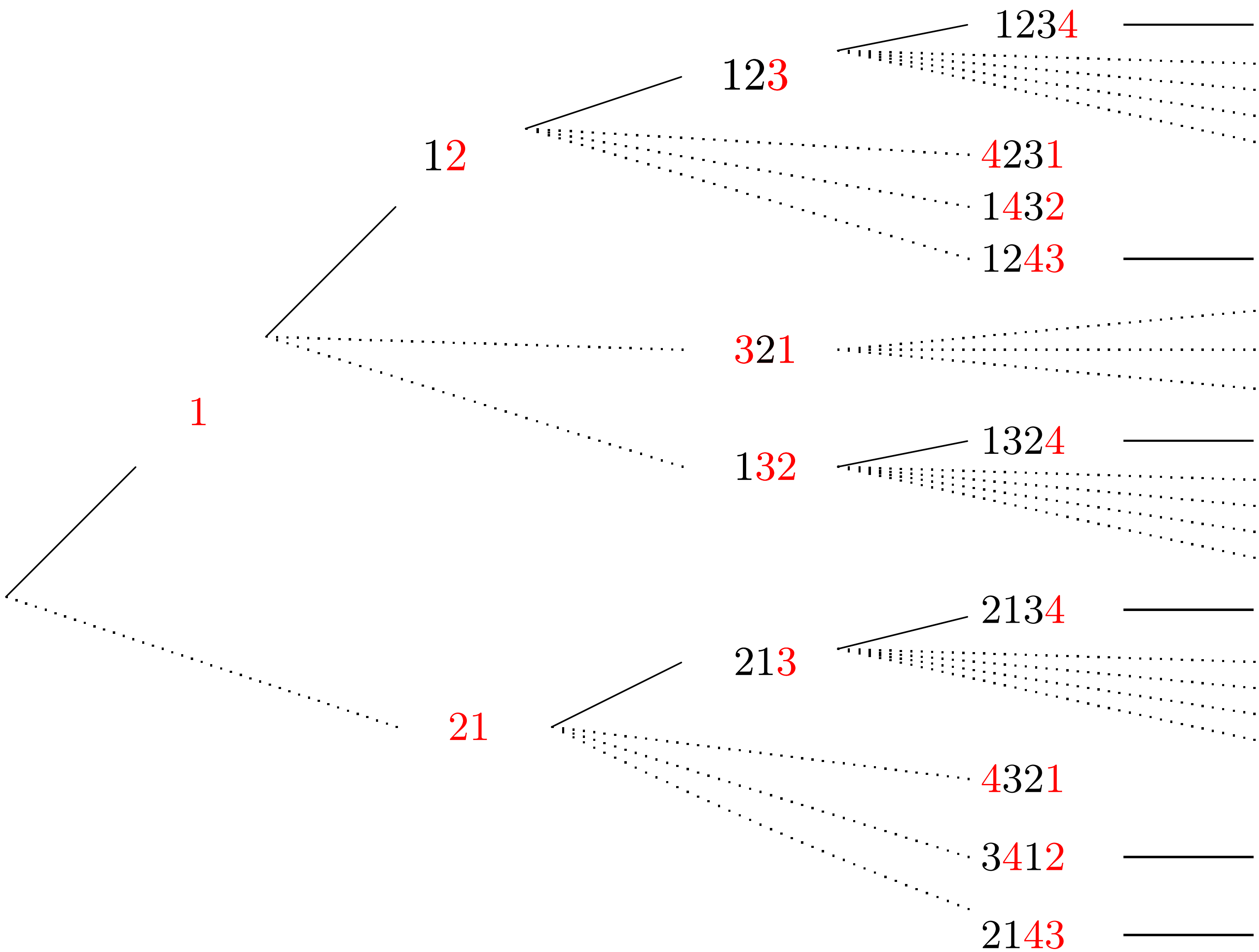


3214-avoiding involutions and generating tree

Definition : its labels (t) or $(t; d_1, d_2, \ldots, d_h; p)$ corresponding to a 3214-avoiding involution π of length n are defined by :

- t : the number of active sites of π. That is , the number of 3214-avoiding involutions of length $n+2$ obtained by inserting a cycle of length 2 according to the generating tree for involutions.
- p is the position of n in the involution π. That is, $n = \pi_p$.
- d_i is the ith least element in the set $\{k : \pi_{k-1} > \pi_k\}$ satisfying $2 \le d_i \le p+1$, for each $1 \le i \le h$ where $\pi \ne 12 \ldots n$.

The figure describes the first five levels for 3214-avoiding involutions :



Motzkin paths and generating tree :

Definition : In a Motzkin path M of length n,

- t : $t = n + 1$, if there exists no nonzero level H,
  $t = \min\{i + k : M(i + k) = D, \text{for } \ k \geq 1\}$, otherwise, where M_i be its leftmost nonzero level H.
- p : If $M(n) = H$,
  - $p = n$, if there exists no nonzero H,
  - $p = t - j - k$, otherwise, where j is the least positive integer s.t. $M(t - j) = H$, k is the least positive integer s.t. $M(t - j - k) = U$.

  If $M(n) = D$,
  - $p = t - j$, if there exists a least positive integer j s.t. $M(t - j) = U$ and $M(1) \cdots M(t - j - 1)M(t - j + 1) \cdots M(n - 1)$ is still a Motzkin path,
  - $p = t - j - k$, otherwise, where j is the least positive integer s.t. $M(t - j) = H$, k is the least positive integer s.t. $M(t - j - k) = U$.
- Des : $Des = \{d_1, d_2, \ldots, d_h\}$, where
  - $1 < d_1 < d_2 < \ldots < d_h \leq p + 1$,
  - $M(d_i - 1) = U$ and $M(d_i) \neq U$, for $1 \leq i \leq h$.

growing rule of Motzkin paths :

Case A. For the special Motzkin path made up of n H's with label $(n+1)$, we consider the two following cases in order to make it growing :

A.1 Adding a H to the right, we obtain a Motzkin path labelled $(n+2)$;

$$\underbrace{HH\ldots H}_{n} \longrightarrow \underbrace{HH\ldots H}_{n}\,H$$

A.2 Inserting U to the pth site and adding D to the end for $1 \le p \le n+1$, we obtain a Motzkin path $M^+ = H^{p-1}UH^{n-p+1}D$ labelled $(n+2; p+1; p)$ when $p \le n$ and $(n+3; p+1; p)$ when $p = n+1$.

$$H^{p-1}H^{n-p+1} \longrightarrow H^{p-1}UH^{n-p+1}D$$

growing rule of Motzkin paths :

Case B. A Motzkin path having at least one nonzero level H with label $(t; d_1, d_2, \ldots, d_h; p)$. So $t \leq n$. Note that $d_0 = 1$. In order to make it growing, we insert a U to the kth site for $1 \leq k \leq t$ and consider the subpath $M' = M_k M_{k+1} \ldots M_t$:

B.1 If M_{k+i} is the first U in the subpath M', then there exists M_{k+i+j} satisfying that M_{k+i+j} is the first letter which is not U for $k \leq k+i < k+i+j \leq t$:

B.1.1 If $M_{k+i+j} = D$, M_{k+i} is replaced by H and an H is added to the end.

$$M_1 \cdots M_{k-1} \underbrace{M_k \cdots M_{k+i-1}}_{\text{no ``U"}} U U^{j-1} D M_{k+i+j+1} \cdots M_n \longrightarrow$$

$$M_1 \cdots M_{k-1} U \underbrace{M_k \cdots M_{k+i-1}}_{\text{no ``U"}} H U^{j-1} D M_{k+i+j+1} \cdots M_n H$$

growing rule of Motzkin paths :

B.1.2 If $M_{k+i+j} = H$, let M_r be the first D such that the number of U is equal to that of D in M for $r > k+i+j$. Then M_{k+i}, M_{k+i+j} and M_r are changed to H, D and U, respectively. Moreover, we add a D to the end;

$$\underbrace{M_1 \cdots M_{k-1} \overbrace{M_k \cdots M_{k+i-1}}^{\text{no "U"}} U U^{j-1} H M_{k+i+j+1} \cdots D}_{|U|=|D|} M_{r+1} \cdots M_n \longrightarrow$$

$$M_1 \cdots M_{k-1} U \underbrace{M_k \cdots M_{k+i-1}}_{\text{no "U"}} H U^{j-1} D M_{k+i+j+1} \cdots U M_{r+1} \cdots M_n D$$

B.2 If there exists no U in the subpath M', then we add a D in the end. $M_1 \cdots M_{k-1} \underbrace{M_k \cdots M_t}_{\text{no "U"}} M_{t+1} \cdots M_n \longrightarrow$

$$M_1 \cdots M_{k-1} U \underbrace{M_k \cdots M_t}_{\text{no "U"}} M_{t+1} \cdots M_n D$$

growing rule of Motzkin paths :

Case C. We now consider a Motzkin path without nonzero level H with label $(t; d_1, d_2, \ldots, d_h; p)$. According to definition we have $t = n + 1$. It is convenient to assume that $d_0 = 1$. We consider the two following cases in order to make it growing :

C.1 Adding a H to the end, we obtain a Motzkin path.

$$\underbrace{M_1 M_2 \cdots M_n}_{\text{no nonzero level "H"}} \longrightarrow M_1 M_2 \cdots M_n H$$

growing rule of Motzkin paths :

C.2.1 Inserting a U to the kth site of the original path for $1 \leq k < d_h$, since the original path has no nonzero H,we can obtain a new Motzkin path similar to the rule of B.1.1 :

$$M_1 \cdots M_{k-1} \underbrace{M_k \cdots M_{k+i-1}}_{\text{no ``U"}} U U^{j-1} D M_{k+i+j+1} \cdots M_n \longrightarrow$$

$$M_1 \cdots M_{k-1} U \underbrace{M_k \cdots M_{k+i-1}}_{\text{no ``U"}} H U^{j-1} D M_{k+i+j+1} \cdots M_n H$$

growing rule of Motzkin paths :

C.2.2 If $p < n$, then $p = d_h - 1$, and $M_{p+j} = D$ for any $1 \le j \le n-p$. So we insert a U to the kth site of the original path for $d_h \le k \le n+1$, we obtain a new Motzkin path by adding a D to the end :

$$\underbrace{M_1 M_2 \cdots M_{p-1}}_{\text{no nonzero level ``H''}} D^{n-p+1} \longrightarrow \underbrace{M_1 M_2 \cdots M_{p-1}}_{\text{no nonzero level ``H''}} D^{k-p} U D^{n-k+1} D$$

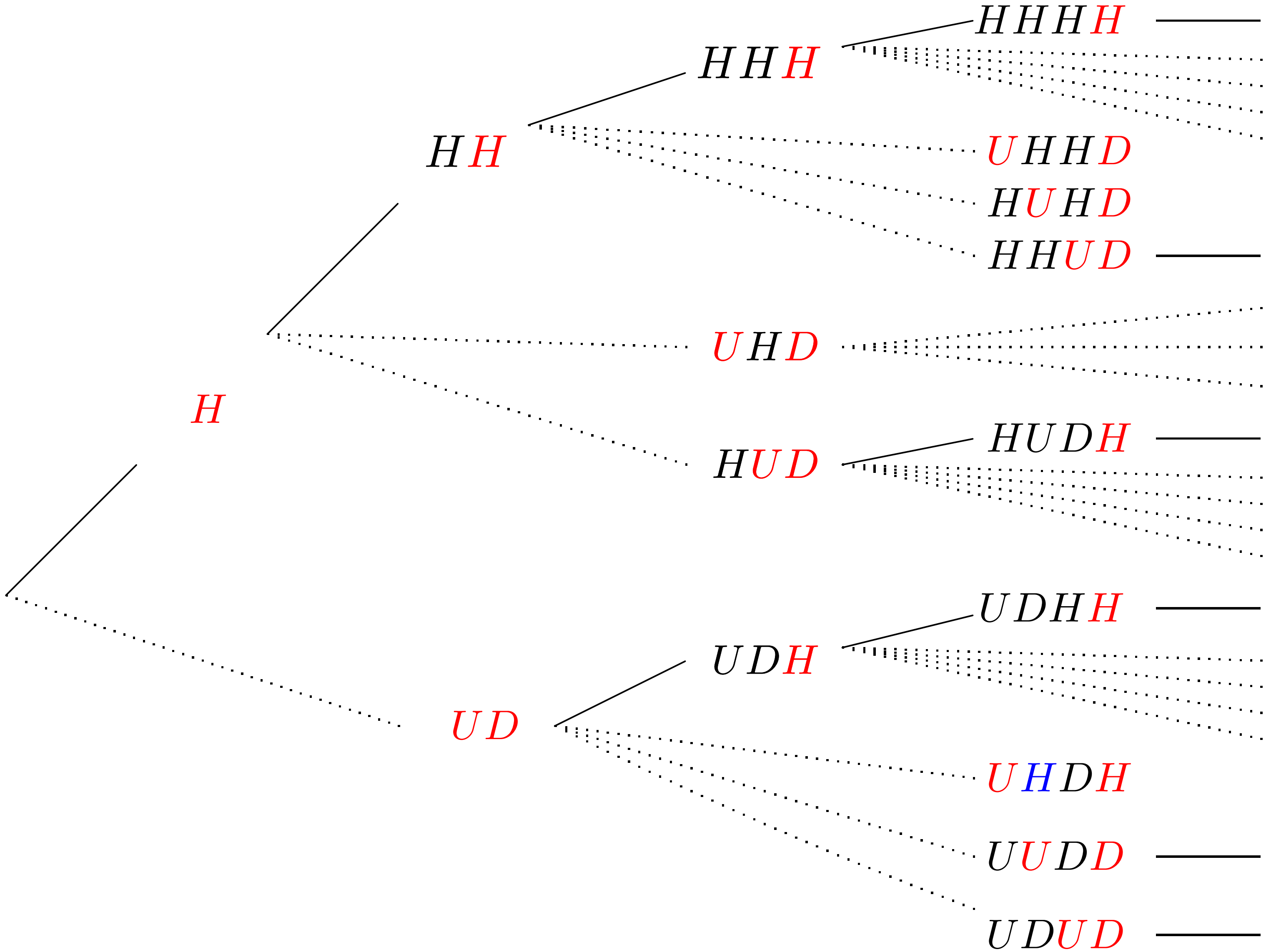
growing rule of Motzkin paths :

C.2.3 If $p = n$, then $M_n = H$ and $M(d_h + j) \neq U$ for any nonnegative integer j. So we insert a U to the kth site of the original word for $d_h \leq k \leq n$, we obtain a new Motzkin word by adding a D to the end.

$$M_1 M_2 \cdots M_{d_h} \underbrace{M_{d_h+1} \cdots M_{k-1} M_k \cdots M_{n-1} H}_{\text{no ``U"}} \longrightarrow$$

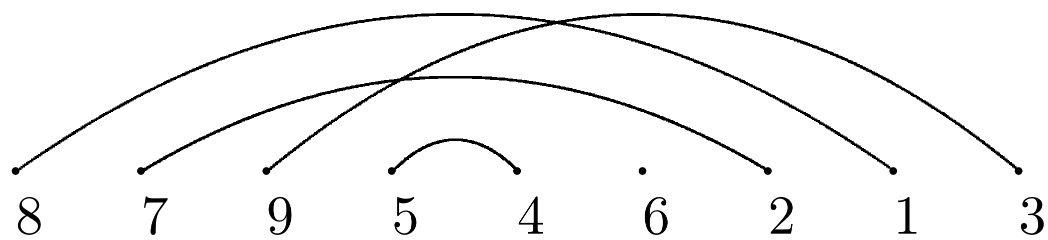
$$M_1 M_2 \cdots M_{d_h} M_{d_h+1} \cdots M_{k-1} U M_k \cdots M_{n-1} H D$$

The figure describes the first five levels for Motzkin paths :

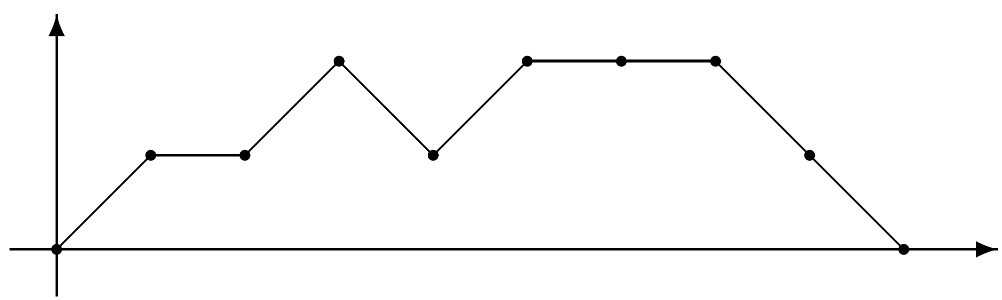


Theorem There is a bijection between $\mathcal{I}_n(3214)$ and the set of Motzkin paths of length n.

For example, the involution avoiding 3214 :



the corresponding Motzkin path :



Corollary(*R. Simion, F.W. Schmidt*) The number of 321-avoiding involutions of length n is counted by $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Corollary(*E. Deutsch, A. Robertson, D. Saracino*) The number $I_n^k(321)$ of 321-avoiding involutions of length n with k fixed points is counted by

$$I_n^k(321) = \begin{cases} \frac{k+1}{n+1}\binom{n+1}{\frac{n-k}{2}} & \text{for n-k even} \\ 0 & \text{for n-k odd} \end{cases}$$

Corollary For $n > 1$ and $1 \leq k < 2n$, $\frac{k}{2n-k}\binom{2n-k}{n}$ counts the number of 321-avoiding involutions of length $2n$ without fixed points satisfying k is the least descent.

Corollary For $n > 1$ and $1 \leq k \leq n$, $\frac{1}{n}\binom{n}{k}\binom{n}{k-1}$ counts the number of 321-avoiding involutions of length $2n$ without fixed points having exactly k descents.

4. Colored combinatorial objects

(r, s, t)-colored Dyck path : every peak point is colored by one letter of alphabet $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, every double-rise point is colored by one letter of the alphabet $\{\beta_1, \beta_2, \ldots, \beta_s\}$, and every double-fall point is colored by one letter of alphabet $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$.

$\mathcal{D}_{r,s,t}(n)$ denotes the set of all (r, s, t)-colored Dyck paths of semilength n

Theorem 1 : The number of (r, s, t)-colored Dyck paths of semilength n is counted by

$$\frac{1}{n}\binom{n}{k}\binom{n}{k-1}r^k(st)^{n-k}.$$

(r, s, t)**-colored plane tree** : the leave is labelled by (x) and the non-root internal node is labelled by (y, z), where $x \in \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, $y \in \{\beta_1, \beta_2, \ldots, \beta_s\}$ and $z \in \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$.

$\mathcal{T}_{r,s,t}(n)$ denotes the set of all (r, s, t)-colored plane trees with n edges.

hilly poor noncrossing partition : a poor noncrossing partition satisfying that each isolated vertex is covered by some arc in its linear presentation.

(r, s, t)**-colored hilly poor noncrossing partition** : in the linear representation, each non-void block is colored by one letter of alphabet $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$ and each void block's shape belongs to the alphabet $\{\beta_1, \beta_2, \ldots, \beta_s\}$ and its color belongs to the alphabet $\{\gamma_1, \gamma_2, \ldots, \gamma_t\}$.

$\mathcal{H}_{r,s,t}(n)$ denotes the set of all (r, s, t)-colored hilly poor noncrossing partitions with n blocks

Theorem 2 : The number of all (r, s, t)-colored hilly poor noncrossing partitions is given by

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} r c_k \binom{n-1}{2k} (rst)^k (st+r)^{n-1-2k}.$$

$\Phi : \mathcal{D}_{r,s,t}(n) \Longleftrightarrow \mathcal{T}_{r,s,t}(n)$

Sketch of proof

Traverse a (r, s, t)-colored plane tree in preorder :

- When an edge is traversed from a vertex to its son vertex, if the son vertex is an internal vertex with label (y, z), we draw a rise step and color its right lattice point with y ; if the son vertex is a leaf with label (x), we draw a rise step and color its right lattice point with (x).
- When an edge is traversed from a vertex to its non-root father vertex with label (y, z), if the vertex is not the rightmost son vertex, we draw a fall step without any color ; if the vertex is the rightmost son vertex, we draw a fall step and color its right lattice point with z.

Therefore, we obtain a (r, s, t)-colored Dyck path.

$\Psi : \mathcal{T}_{r,s,t}(n) \Longleftrightarrow \mathcal{P}_{r,s,t}(n)$

Sketch of proof

Induction on n.

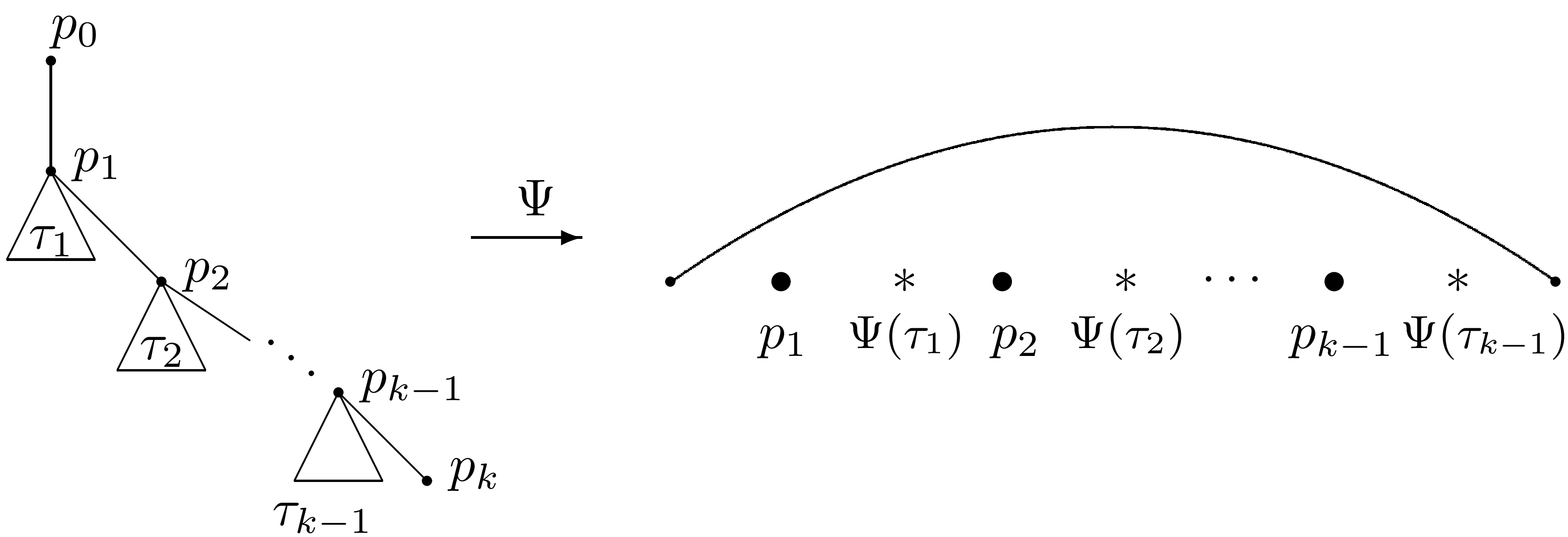
Given $T \in \mathcal{T}_{r,s,t}(n)$ with root degree k, it can be decomposed into k subtrees $\{T_i\}_{i=1}^{k}$ listed from left to right with root degree 1 from its root, denote by $\Psi(T) = \Psi(T_1)\Psi(T_2)\ldots\Psi(T_k)$.

It suffices to construct the bijection Ψ between the set of (r, s, t)-colored plane trees of n edges with root degree 1 and the set of (r, s, t)-colored hilly poor noncrossing partitions of n blocks with only one big arc.

Let τ be a (r, s, t)-colored plane tree of n edges with root degree 1 and $p = p_0p_1p_2\ldots p_{k-1}p_k$ be its rightmost path and p_k be its rightmost leaf. we consider the decomposition of τ.

Ψ **:** $\mathcal{T}_{r,s,t}(n) \Longleftrightarrow \mathcal{P}_{r,s,t}(n)$

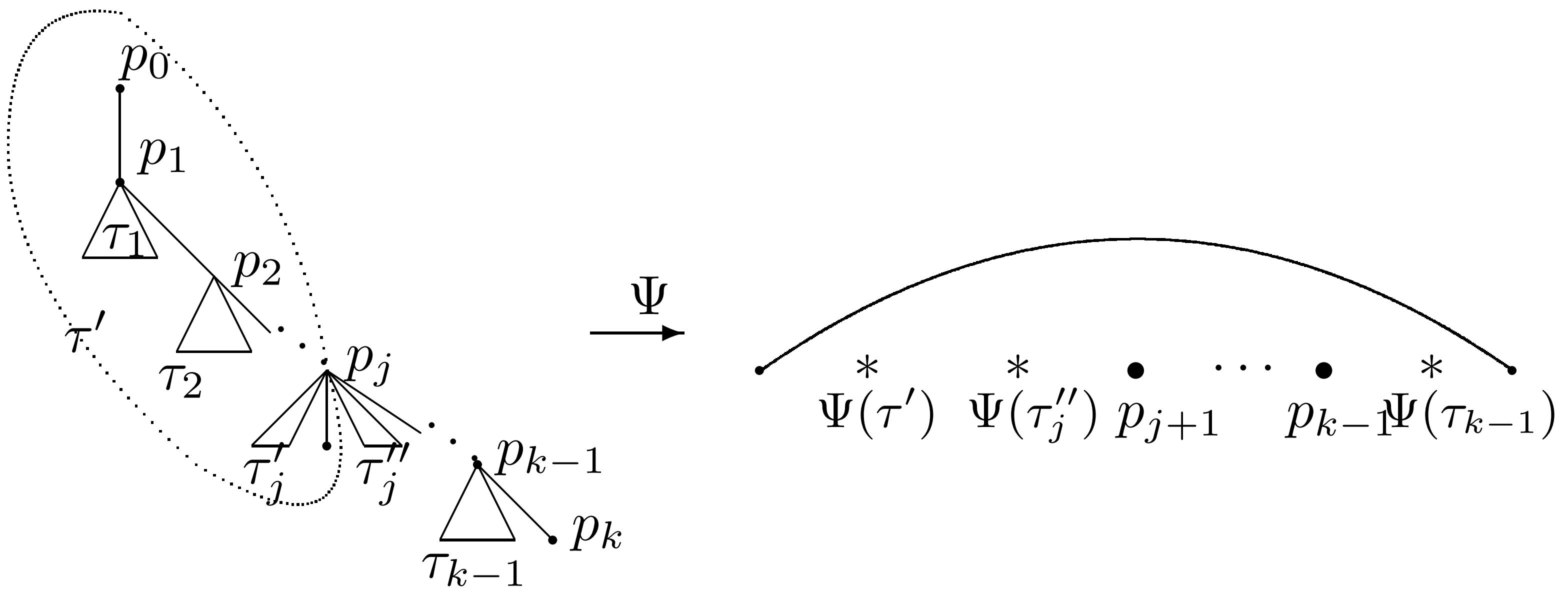
Case I : If there exists no near-leaf node in the rightmost path, the mapping Ψ is described as :



In this case, p_k corresponds to a big arc which is colored by the label of p_k. And for any $1 \leq i \leq k-1$, the node p_i with label (y) corresponds to a void block of s-color.

$\Psi : \mathcal{T}_{r,s,t}(n) \Longleftrightarrow \mathcal{P}_{r,s,t}(n)$

Case II : If there is at least one near-leaf node : Let j be the largest index such that p_j is near-leaf, then the mapping is described as :



where τ''_j contains no leaf child of the vertex p_j.

In this case, the big arc of $\Psi(\tau)$ is colored by the label of p_k. And for any $j+1 \leq i \leq k-1$, the node p_i labelled by (y) corresponds to a void block is of s-color.

By $\mathcal{D}_{r,s,t}(n) \Longleftrightarrow \mathcal{T}_{r,s,t}(n) \Longleftrightarrow \mathcal{P}_{r,s,t}(n)$ and Theorem 1 and 2, we have

$$\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} r^k (st)^{n-k} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} rc_k \binom{n-1}{2k} (rst)^k (st+r)^{n-1-2k}.$$

Setting $r = 1, s = t = 2$ in the above formula implies an answer to a question mentioned by C. Coker

Enumerating a class of lattice paths, Discrete Math. 271 (2003) 13-28.

(i,j)-Motzkin paths

(i, j)**-Motzkin path** : Motzkin path that each up step is colored by one of the i colors, and each horizontal step is colored by one of the j colors.

$\mathcal{M}_{i,j}(n)$ denotes the set of (i, j)-Motzkin paths of length n.

π : $\mathcal{P}_{r,s,t}(n) \Longleftrightarrow \mathcal{M}_{rst,st+r}(n)$ starting with a horizontal step colored by one letter of $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$

Sketch of proof :

The out arc colored by x of a basic pair and its corresponding rightmost block of shape y and color z correspond to an up step colored by ternary letters (x, y, z) and a down step, respectively ; for the remaining blocks, a non-void block colored by x corresponds to a horizontal step colored by x ; a void block of shape y and color z corresponds to a horizontal step colored by a pair (y, z), where $x \in \{\alpha_1, \alpha_2, \ldots, \alpha_r\}$, $y \in \{\beta_1, \beta_2, \ldots, \beta_s\}$ and $z \in \{\gamma_1, \gamma_2, \ldots, \gamma_t\}$.

It is clear that

$\pi \cdot \Psi \cdot \Phi^{-1}$: $\mathcal{D}_{r,s,t}(n) \Longleftrightarrow \mathcal{M}_{rst,st+r}(n)$ starting with a horizontal step colored by one letter of $\{\alpha_1, \alpha_2, \ldots, \alpha_r\}$

$\mathcal{H}'_{u,v;r,s;t}(n)$ denotes the set of hilly poor noncrossing partitions with n blocks s.t. the first arc has u choices for type and v choices for color, other arcs have r choices for type and s choices for color, and isolated vertices have t choices for color.

By another paper of ours (submit), we have

Theorem 3 : The number of partitions in $\mathcal{H}'_{u,v;r,s;t}(n+1)$ satisfying that all rightmost blocks are arcs is counted by

$$uv \sum_{k=0}^{n} c_{k+1} \binom{n}{k} (rs)^k t^{n-k}.$$

Restrict π to the set $\mathcal{H}'_{u,v;r,s;t}(n)$ whose rightmost blocks are arcs and $\mathcal{M}_{r^2s^2,2rs+t}(n)$ starting with a horizontal step colored by one of uv colors, we obtain the bijection π', we obtain a combinatorial interpretation for

$$\sum_{k=0}^{\lfloor (n-1)/2 \rfloor} c_k \binom{n-1}{2k} (r^2s^2)^k (2rs+t)^{n-2k-1} = \sum_{k=0}^{n-1} c_{k+1} \binom{n-1}{k} (rs)^k t^{n-k-1}.$$

Specially, when $u = v = x$, $r = s - 1 = x$ and $t = 1$, we have

$$x^2 \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} c_k \binom{n-1}{2k} x^{2k} (x+1)^{2k} (2x^2 + 2x + 1)^{n-2k-1}$$

$$= x^2 \sum_{k=0}^{n-1} c_{k+1} \binom{n-1}{k} x^k (x+1)^k$$

Restricting the bijection $\pi \cdot \Psi \cdot \Phi^{-1}$ to the set $\mathcal{D}_{x^2,x+1,x+1}(n)$ and $(x^2(x+1)^2, (x+1)^2+x^2)$-colored Motzkin paths of length n starting with a horizontal step colored by one letter of x^2 colors, we obtain a combinatorial explanation of

$$\sum_{k=1}^{n} \frac{1}{n}\binom{n}{k}\binom{n}{k-1} x^{2k}(1+x)^{2(n-k)} = x^2 \sum_{k=0}^{n-1} c_{k+1}\binom{n-1}{k} x^k (1+x)^k,$$

which implies an answer to a question mentioned by C. Coker in

Enumerating a class of lattice paths, Discrete Math. 271 (2003) 13-28.

Thanks !