Lecture Hall Theorems, q-series and Truncated Objects

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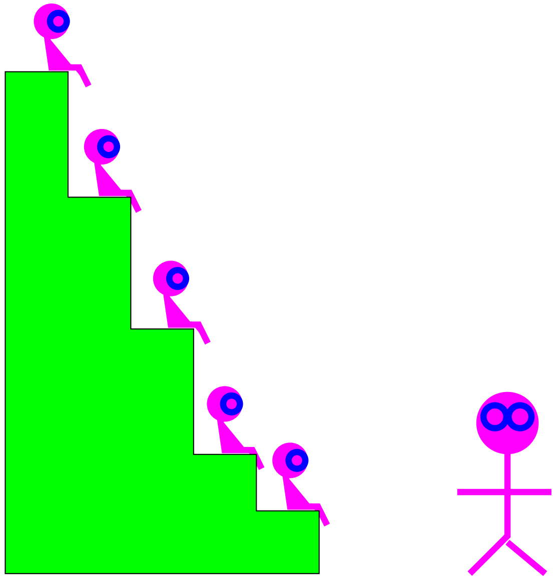
http://www.prism.uvsq.fr/users/syl



Lecture Hall partitions L_n

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_{n-1}}{2} \geq \frac{\lambda_n}{1} \geq 0,$$

Ex: $\lambda = (8,6,4,2,1) \in L_5$

$|\lambda| = \lambda_1 + \lambda_2 + \ldots + \lambda_n$

$|\lambda| = 21$

$$L_n(q) \triangleq \sum_{\lambda \in L_n} q^{|\lambda|} = \frac{1}{(q;q^2)_n}; \qquad (a;q)_n = \prod_{i=0}^{n-1}(1 - aq^i)$$

Algebra - Coxeter groups Bousquet-Mélou, Eriksson

Combinatorial interpretation

Euler's Theorem

Partitions into distinct parts

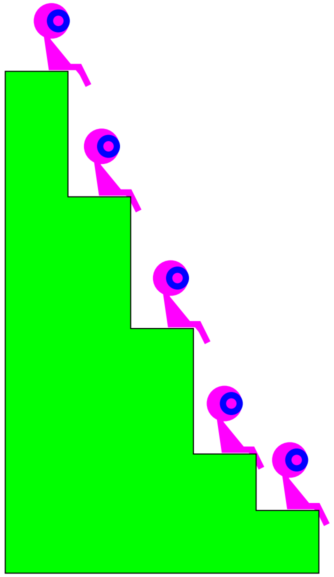
$\leftrightarrow$ Partitions into odd parts

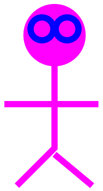
$$\{(7),(6,1),(5,2),(4,3),(4,2,1)\} \leftrightarrow \{(7),(5,1,1),(3,3,1),(3,1^4),(1^7)\}$$

LHP in L_n

$\leftrightarrow$ Partitions into odd parts $\leq 2n-1$

$n=2$ $\{(7),(6,1),(5,2)\} \leftrightarrow \{(3,3,1),(3,1^4),(1^7)\}$





Odd/Even LHP Theorem

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_{n-1}}{2} \geq \frac{\lambda_n}{1} \geq 0,$$

$\lambda_o = (\lambda_1, \lambda_3, \ldots)$

$\lambda_e = (\lambda_2, \lambda_4, \ldots)$.

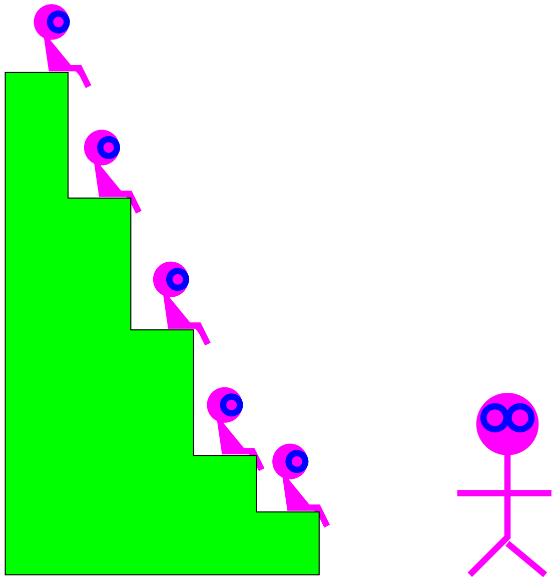
Ex: $\lambda = (8, 6, 4, 2, 1)$

$\lambda_o = (8, 4, 1)$ $\lambda_e = (6, 2)$

$$L_n(x, y) \triangleq \sum_{\lambda \in L_n} x^{|\lambda_o|} y^{|\lambda_e|} = \frac{1}{(x; xy)_n}$$

Recursive/combinatorial proof, bijection

Bousquet-Mélou - Eriksson /Yee



Refined LHP theorem

$\lceil \lambda \rceil = (\lceil \lambda_1/n \rceil, \lceil \lambda_2/(n-1) \rceil, \ldots, \lceil \lambda_n/1 \rceil)$

Ex: $\lambda = (8,6,4,2,1)$ $\lceil \lambda \rceil = (2,2,2,1,1)$

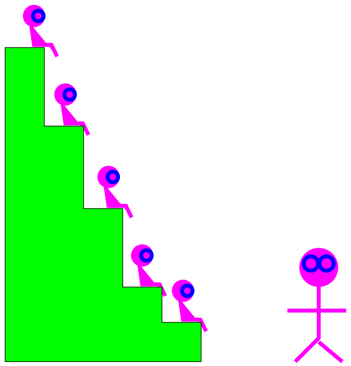
Nb of odd parts of $\lceil \lambda \rceil$: $o(\lceil \lambda \rceil)$

$o(\lceil \lambda \rceil) = 2$

$$L_n(u,v,q) \triangleq \sum_{\lambda \in L_n} q^{|\lambda|} u^{|\lceil \lambda \rceil|} v^{o(\lceil \lambda \rceil)} = \frac{(-uvq;q)_n}{(u^2q^{n+1};q)_n}$$

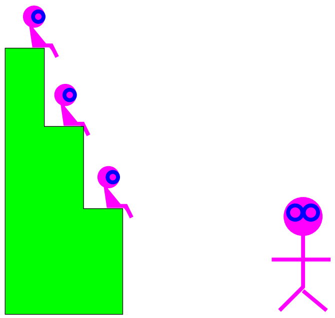
Complicated gf proof/ Bijective proofs

Bousquet-Mélou, Eriksson, Yee



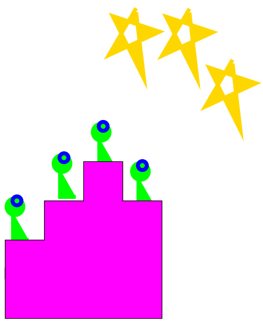
Refined LHP Theorem and q-series

$$\frac{a^n (c/a;q)_n}{(c;q)_n} = \sum_{m=0}^{n} \frac{(a;q)_m (q^{-n};q)_m}{(c;q)_m (q;q)_m} q^m$$



Truncated Lecture Hall partitions

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n-k+1} > 0.$$



(Trunc.) Anti-Lecture Hall comp.

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \ldots \geq \frac{\lambda_k}{n} \geq 0.$$

Refined LHP Theorem and q-series

$$L_n(u,v,q) = \sum_{\mu \in P_n} u^{|\mu|} v^{o(\mu)} L_\mu(q); \quad L_\mu(q) \triangleq \sum_{\substack{\lambda \in L_n \\ \lceil \lambda \rceil = \mu}} q^{|\lambda|}$$

$\lambda \in L_n \, : \, \lambda_i = (n-i+1)\mu_i - r_i; \; 0 \le r_i \le n-i$

$\Leftrightarrow \mu_i > \mu_{i+1}$ or $\mu_i = \mu_{i+1}$ and $r_i \le r_{i+1}$

$$L_\mu(q) = q^{\sum_{i=1}^n (n-i+1)\mu_i} \sum_{(r_1,\ldots,r_n)} q^{\sum_{i=1}^n r_i} = q^{\sum_{i=1}^n (n-i+1)\mu_i} \begin{bmatrix} n \\ m_0, m_1, \ldots, m_{\mu_1} \end{bmatrix}_{1/q}.$$

m_i multiplicity of i in μ

Refined LHP Theorem and q-series

$m = n - m_0$; $\tilde{\mu}_i = \mu_i - 1$; $1 \leq i \leq m$

$$L_{\tilde{\mu}}(q) = q^{\sum_{i=1}^{m}(m-i+1)(\mu_i-1)} \begin{bmatrix} m \\ m_1, \ldots, m_{\mu_1} \end{bmatrix}_{1/q}$$

$$L_{\mu}(q) = q^{(n-m)|\tilde{\mu}| + \binom{m+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q L_{\tilde{\mu}}(q),$$

$$L_n(u,v,q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} L_m(uq^{n-m}, 1/v, q).$$

Refined LHP Theorem and q-series

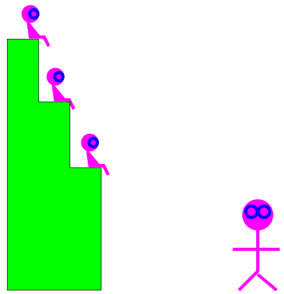
$$L_n(u,v,q) = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} L_m(uq^{n-m}, 1/v, q).$$

$$q - \text{Chu Vandermonde} \quad \frac{a^n (c/a;q)_n}{(c;q)_n} = \sum_{m=0}^{n} \frac{(a;q)_m (q^{-n};q)_m}{(c;q)_m (q;q)_m} q^m$$

Set $a = -vq^{-n}/u$, $c = q^{-2n}/u^2$,

$$\frac{(-uvq;q)_n}{(u^2q^{n+1};q)_n} = \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix}_q (uv)^m q^{\binom{m+1}{2}} \frac{(-(u/v)q^{n-m+1};q)_m}{(u^2q^{2n-m+1};q)_m}.$$

$$\Rightarrow L_n(u,v,q) = \frac{(-uvq;q)_n}{(u^2q^{n+1};q)_n}$$



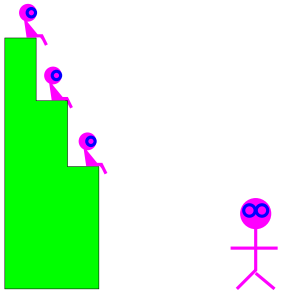
Truncated Lecture Hall partitions

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n-k+1} > 0.$$

Refined Truncated Lecture Hall Theorem

$$L_{n,k}(u,v,q) \triangleq \sum_{\lambda \in L_{n,k}} q^{|\lambda|} u^{|\lceil \lambda \rceil|} v^{o(\lfloor \lambda \rfloor)} = (uv)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ m \end{bmatrix}_q \frac{(-(u/v)q^{n-k+1};q)_k}{(u^2 q^{2n-k+1};q)_k}.$$

$$L_{n,k}(u,v,q) = (uv)^k q^{\binom{k+1}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q L_k(uq^{n-k}, 1/v, q)$$



Truncated Lecture Hall partitions $k \leq n$

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \ldots \geq \frac{\lambda_k}{n-k+1} > 0.$$

Odd/Even Truncated Lecture Hall Theorem

$$L_{n,k}(x,y) \triangleq \sum_{\lambda \in L_{n,k}} x^{|\lambda_o|} y^{|\lambda_e|} = \frac{(x^{\lfloor k/2 \rfloor + 1} y^{\lfloor k/2 \rfloor})^{\lceil k/2 \rceil} \begin{bmatrix} n - \lceil k/2 \rceil \\ \lfloor k/2 \rfloor \end{bmatrix}_{xy}}{(x; xy)_{\lceil k/2 \rceil} (x^n y^{n-1}; (xy)^{-1})_{\lfloor k/2 \rfloor}}.$$

$\lambda_o = (\lambda_1, \lambda_3, \lambda_5, \ldots)$; $\lambda_e = (\lambda_2, \lambda_4, \lambda_6, \ldots)$

Odd/Even Truncated LHP (Sketch of proof)

Define a bijection

$\mathrm{BME}_{n,k} : L_{n-1,k-1} \times \mathbb{N} \to L_{n,k}$

such that if $\mu = \mathrm{BME}_{n,k}(\lambda, s)$

$|\mu_e| = |\lambda_o|$

k odd then $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s + 1.$

$$L_{n,2k+1}(x,y) = \frac{x}{1-x} L_{n-1,2k}(x^2y, x^{-1})$$

k even $|\mu_o| = 2|\lambda_o| - |\lambda_e| + s - \lfloor \frac{(n-2k)\lambda_{2k-1}}{n-2k+1} \rfloor$

$$L_{n,2k}(x,y) = L_{n-1,2k}(x^2y, 1/x) + \frac{1}{1-x} L_{n-1,2k-1}(x^2y, 1/x)$$

Odd/Even Truncated LHP

$$L_{n,2k+1}(x,y) = \frac{x}{1-x} L_{n-1,2k}(x^2y, x^{-1})$$

$$L_{n,2k}(x,y) = L_{n-1,2k}(x^2y, 1/x) + \frac{1}{1-x} L_{n-1,2k-1}(x^2y, 1/x)$$

$$L_{n,k}(x,y) = \frac{\left(x^{\lfloor k/2 \rfloor + 1} y^{\lfloor k/2 \rfloor}\right)^{\lceil k/2 \rceil} \begin{bmatrix} n - \lceil k/2 \rceil \\ \lfloor k/2 \rfloor \end{bmatrix}_{xy}}{(x; xy)_{\lceil k/2 \rceil} \left(x^n y^{n-1}; (xy)^{-1}\right)_{\lfloor k/2 \rfloor}}.$$

Combinatorial characterization

$$TL_{n,k}: \qquad \frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_k}{n-k+1} \geq 0.$$

$TL_{n,k}$ λ where $k(\lambda) = |\lambda_o| - |\lambda_e| = j$ $\longleftrightarrow$

Partitions into j odd parts $\leq 2n-1$ with at most $\lfloor k/2 \rfloor$ parts from $\{2\lceil k/2 \rceil + 1, \ldots, 2(n - \lfloor k/2 \rfloor) - 1\}$.

Example: $n = 6, \ k = 2$

$TL_{6,2}$ with $k(\lambda) = j \leftrightarrow$ Partitions into j odd parts ≤ 11 with at most 1 part in $\{3, 5, 7, 9\}$.

$(11) \leftrightarrow (1^{11})$; $(10, 1) \leftrightarrow (3, 1^8)$; $(9, 2) \leftrightarrow (5, 1^6)$;
$(8, 3) \leftrightarrow (7, 1^4)$; $(7, 4) \leftrightarrow (9, 1^2)$; $(6, 5) \leftrightarrow (11)$

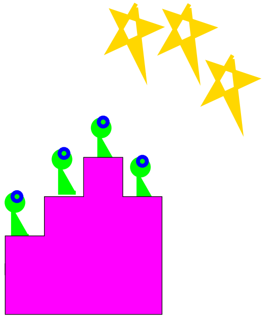
Sketch of Proof

$R_{n,k}(z,q)$ generating function for partitions into odd parts $\leq 2n-1$ with at most $\lfloor k/2 \rfloor$ parts from $\{2\lceil k/2\rceil+1,\ldots,2(n-\lfloor k/2\rfloor)-1\}$ where z counts the number of parts and q the weight.

$$R_{n,k}(z,q) = \frac{\sum_{i=0}^{\lfloor k/2\rfloor} z^i q^{i(2\lceil k/2\rceil+1)} \begin{bmatrix} n-k-1+i \\ i \end{bmatrix}_{q^2}}{(zq;q^2)_{\lceil k/2\rceil}(zq^{2n-1};q^{-2})_{\lfloor k/2\rfloor}}$$

$$R_{n,k}(z,q) - R_{n,k-1}(z,q) = L_{n,k}(zq,q/z)$$

Bijection?



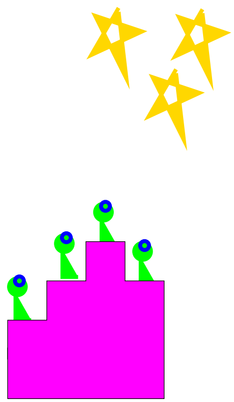
Anti-Lecture Hall compositions A_n

$$\frac{\lambda_1}{1} \geq \frac{\lambda_2}{2} \geq \ldots \geq \frac{\lambda_n}{n} \geq 0.$$

Ex : Compositions of 4 in A_3 $(4),\ (3,1),\ (2,2),\ (2,1,1),\ (1,2,1)$

$$A_n(u,v,q) \triangleq \sum_{\lambda \in A_n} q^{|\lambda|} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} = \frac{(-uvq;q)_n}{(u^2q^2;q)_n}$$

$$A_n(x,y) \triangleq \sum_{\lambda \in A_n} x^{|\lambda_o|} y^{|\lambda_e|} = \frac{\begin{bmatrix} n \\ \lfloor n/2 \rfloor \end{bmatrix}_{xy}}{(x;xy)_{\lceil n/2 \rceil}(xy^2;xy)_{\lfloor n/2 \rfloor}}.$$



Truncated Anti-Lecture compositions

$$\frac{\lambda_1}{n-k+1} \geq \frac{\lambda_2}{n-k+2} \geq \ldots \geq \frac{\lambda_{k-1}}{n-1} \geq \frac{\lambda_k}{n} \geq 0.$$

$$A_{n,k}(u,v,q) \triangleq \sum_{\lambda \in A_{n,k}} q^{|\lambda|} u^{|\lfloor \lambda \rfloor|} v^{o(\lfloor \lambda \rfloor)} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-uvq^{n-k+1};q)_k}{(u^2 q^{2(n-k+1)};q)_k}.$$

$$A_{n,k}(x,y) \triangleq \sum_{\lambda \in A_{n,k}} x^{|\lambda_o|} y^{|\lambda_o|} = \frac{\begin{bmatrix} n \\ \lfloor k/2 \rfloor \end{bmatrix}_{xy}}{(x;xy)_{\lceil k/2 \rceil} (x^{n-k+1} y^{n-k+2}; xy)_{\lfloor k/2 \rfloor}}.$$

More?

Finitization of Refinements of Euler's Theorem : Partitions λ into k distinct parts with $k(\lambda) = j \leftrightarrow$ Partitions into j odd parts such that $\lfloor k/2 \rfloor$ of the parts $\geq 2\lceil k/2 \rceil + 1$ (Sylvester, Lascoux)

New identity?

$$\sum_{m=0}^{n} \left(aq^{\lfloor m/2 \rfloor}\right)^{\lceil m/2 \rceil} \frac{\begin{bmatrix} n - \lceil m/2 \rceil \\ \lfloor m/2 \rfloor \end{bmatrix}_q}{(a;q)_{\lceil m/2 \rceil}(aq^{n-1};q^{-1})_{\lfloor m/2 \rfloor}} = 1/(a;q)_n.$$

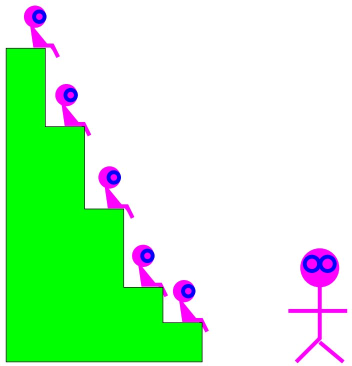
More to do... Simple bijections?

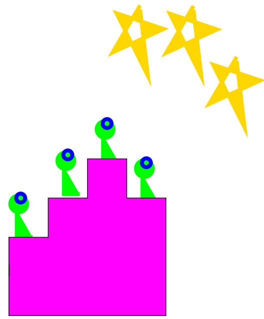
Direct bijections for Truncated Refined Theorems?

Simplified proofs for the Truncated Odd/even Theorems?

Links with Coxeter groups?

Easy argument to understand why all these are products?





Thanks!