Combinatorial Expansions for Paths and Chung-Feller Theorem

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Abstract

This paper introduces a unified notion of combinatorial expansions of generating functions of various classes of restricted planar lattice paths (specifically the excursions and bridges defined by Banderier and Flajolet). The plane is colored in various ways and the lattice paths are decomposed into colored sections accordingly. The identities between the coefficients of these combinatorial expansions and the Chung-Feller like phenomena are explained from this viewpoint using a cut-and-paste technique to establish bijections between certain type of bridges and excursions. The technique is illustrated on several families of familiar paths and can be applied to many others.

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1 Introduction

Lattice paths of various types have been extensively studied in combinatorics. They serve as an important model to describe many combinatorial structures such as trees, pattern-avoid permutations, symmetric functions, orthogonal polynomials [17], Hankel determinants [16], continued fractions [9], etc. They also appear in other fields of mathematics including probability theory, statistics, computer science and queuing theory, etc.

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In this paper, lattice paths are usually described in the northeast orientation for convenience. This has the advantage that the 'size' of a path not only coincides with its span along the main diagonal but also coincides with its projective length upon the y-axis (or x-axis). Using the terminologies in [1], a *bridge* is a path in the plane $\mathbb{Z}\times\mathbb{Z}$ with the *north* $(0,a)$, *east* $(b,0)$ and *diagonal* (c,c) steps (a,b,c are positive integers) that goes from the origin to a point lying on the main diagonal $y=x$ and an *excursion* is a bridge that never passes below the main diagonal. For example, here are some families of excursions and their allowed steps.

excursions	allowed steps
Catalan paths	$(0,1),(1,0)$
Catalan paths of order d, $d\geq 1$	$(0,1),(d,0)$
Motzkin paths	$(0,2),(2,0),(1,1)$
Schröder paths	$(0,1),(1,0),(1,1)$

The Catalan paths (or called Dyck paths) are counted by the well-known *Catalan numbers* $\{c_n\}_{n\geq 0}=\{1,1,2,5,14,42,132,\ldots\}$. It is known that the generating functions $C=C(x)=\sum_{n\geq 0}c_nx^n$ for the Catalan numbers satisfies the functional equation $C=1+xC^2$. In [8], Eu *et al.* introduced the concept of Taylor expansion of generating functions, where the cases of Catalan numbers and Motzkin numbers are considered. By a *Taylor expansion*, the generating function is expanded in a form the remainder of which is expressed as a function of the generating function itself. For example, by iteration, the initial expansions of C are

$$\begin{aligned} C &= 1+xC^2 \\ &= 1+x+x^2(C^2+C^3) \\ &= 1+x+2x^2+x^3(2C^2+2C^3+C^4). \end{aligned}$$

From this algebraic manipulation, we are interested to know how to determine the functions for the remainders and what their combinatorial interpretations are. Indeed, the n-th Taylor expansion of C, first derived in [8], can be expressed in the form

$$C=\sum_{i=0}^{n-1}c_ix^i+x^nF_n(C),$$

where $F_n(z)=\sum_{p=0}^{n-1}\frac{n-p}{n+p}\binom{n+p}{p}z^{n-p+1}$ (cf. Theorem 3.3). Moreover, based on the expansion, a simplified proof for the classic Chung-Feller theorem was given in [8] (cf. Theorem

4.2). A study on the meanings of the remainders leads to the notion of combinatorial expansions of the generating functions.

Our main idea is that the plane $\mathbb{Z} \times \mathbb{Z}$ is partitioned into blue and red regions in various ways (e.g., by lines, curves, or multiple boundaries) and the lattice paths in the plane are decomposed into colored sections correspondingly. Some statistics (e.g., the number of paths of a given length and the number of blocks, etc.) of the paths under consideration can be obtained from the counting results of their blue and red sections. Since the enumeration of the colored sections can be formulated as generating functions by factorizations into building blocks of the paths, it admits an expansion of the generating functions for the paths. Moreover, the relations among expansions, based on different partitions of the plane, provide interesting information about the statistics of the paths. For ease of exposition, we focus our attention on the Taylor expansions for excursions and the flaw expansions for bridges, each of which relies on a two-color bisection of the plane $\mathbb{Z} \times \mathbb{Z}$ with respect to a line (e.g., horizontal, vertical and the main diagonal, etc.). Several families of familiar paths are discussed in detail for illustration. Consequently, the Chung-Feller and non-Chung-Feller phenomena for lattice paths are explained from this point of view and a universal cut-and-paste technique is developed to establish bijections between excursions and specific classes of bridges. In the illustrative examples, the plane is always halved by a line (usually a horizontal line or the main diagonal), but curved boundaries and more colors are obviously possible.

This paper is organized as follows. The theory of combinatorial expansions is given in Section 2. The Taylor expansions of the generating functions for some families of excursions (e.g., Catalan paths, Catalan paths of order d, Motzkin paths, and large and small Schröder paths) are derived in Section 3. The flaw expansions for some families of bridges are given in Section 4. A universal cut-and-paste technique is developed to establish bijections between excursions and bridges for the cases with Chung-Feller phenomena in Section 5. Analogues of Motzkin paths are discussed in Section 6. In particular, an expansion of the generating function according the absolute minimum of paths is given (Theorem 6.4). We shall show the Chung-Feller properties regarding the absolute minimum for Motzkin paths (Theorem 6.5). Some results for the Schröder paths are given in Section 7. In particular, a surprisingly neat formula for enumerating bridges (i.e., Schröder paths with flaws) is obtained (Theorem 7.1) and some asymptotic results regarding average number of returns of paths are given (Theorem 7.8).

We shall denote $[x^n]F(x)$ the coefficient of x^n of a power series (or the expansion of a function) $F(x)$.

2 The theory of combinatorial expansions

Let G be the generating functions for the number of lattice paths under consideration. Given a two-color partition of the plane $\mathbb{Z}\times\mathbb{Z}$, say blue and red, the paths are decomposed into blue and red sections correspondingly. The number of paths can be enumerated from the counting results of their blue and red sections. Suppose that the generating functions for the possibilities of the blue and red sections are formulated as G_1 and G_2, respectively. Then $[x^n]G = ([x^p]G_1) \star ([x^q]G_2)$, where $\star$ is some suitable operator and $p+q=n$, p,q depend the number of marked steps in blue and red colors, respectively. For ease of exposition, we restrict attention on the bisection of the plane with respect to a line (e.g., horizontal, vertical or the main diagonal line $y=x$, etc.).

In the plane $\mathbb{Z}\times\mathbb{Z}$, let $\mathcal{T}_m$ denote the set of paths from $(0,0)$ to (m,m) with the set $\mathsf{T}=\{(0,a),(b,0),(c,c)\}$ of allowed steps that never pass below the line $y=x$ and let $t_m=|\mathcal{T}_m|$. Members of $\mathcal{T}_m$ are called the T-*excursions of semilength* m. In addition to the semilength of a T-excursion π, the generating function for $\{t_m\}_{m\geq 0}$, given by $T=T(x)=\sum_{m\geq 0} t_m x^m$, can be equivalently formulated in terms of the length of the projective image of π upon the y-axis (or the x-axis). To establish an expansion, the plane $\mathbb{Z}\times\mathbb{Z}$ is partitioned into two colored half-planes with respect to a horizontal line, say $y=n$ for some $n\geq 1$. Given an excursion $\pi\in\mathcal{T}_m$, the section of π from the beginning to the first step that meets (i.e., either reaches or intersects) the line $y=n$, denoted by $\gamma_n^-(\pi)$, is called the *blue section* and the remaining section of π, denoted by $\gamma_n^+(\pi)$, is called the *red section*. For $m\geq n$, $\gamma_n^+(\pi)$ (resp. $\gamma_n^-(\pi)$) has a projection of length at most $m-n$ (resp. at least n) upon the y-axis. Suppose that the excursions in $\mathcal{T}_m$ are partitioned into a disjoint union of subsets $\mathcal{T}_m(1),\dots,\mathcal{T}_m(k)$ for some k so that the generating function for the possibilities of all red sections for the excursions $\pi\in\mathcal{T}_m(p)$ $(1\leq p\leq k)$ is formulated as a function $F_p(T,x)$ in T and x. This is certainly possible since $\gamma_n^+(\pi)$ can be factorized into building blocks of T-excursions and some permitted steps, but we do have a complication depending on the given set of allowed steps. The number of excursions in $\mathcal{T}_m$ $(m\geq n)$ is thus enumerated by the linear combination $\sum_{p=1}^{k}\alpha_p F_p(T,x)$

of the functions $F_p(T,x)$ $(1 \leq p \leq k)$, where α_p is the number of all blue sections for $\pi \in \mathcal{T}_m(p)$, i.e, $t_m = \sum_{p=1}^{k} \alpha_p([x^{m-n}]F_p(T,x))$. Hence T can be expressed in the form

$$T = \sum_{i=0}^{n-1} t_i x^i + x^n(\sum_{p=1}^{k} \alpha_p F_p(T,x)). \tag{1}$$

This is called the *n-th Taylor expansion of* T, in which $x^n(\sum_{p=1}^{k} \alpha_p F_p(T,x))$ is called the *remainder.* For illustration, the Taylor expansions of the generating functions for the Catalan paths (Theorem 3.3), the Catalan paths of order d (Theorem 3.7), the Motzkin paths (Theorem 3.10) and the large and small Schröder paths (Theorem 3.15) are derived.

For the paths with the statistics concerned indexed by the x-coordinates, the Taylor expansion along the x-axis is derived in a similar manner, in which case the plane $\mathbb{Z} \times \mathbb{Z}$ is bisected by a vertical line. As illustrated in Theorem 6.3, an analogue of Motzkin paths is discussed.

For bridges, a two-color bisection of the plane $\mathbb{Z} \times \mathbb{Z}$ with respect to the line $y = x$ is considered. By *flaws* of a bridge, we mean the steps of the bridge that pass below the line $y = x$. The set of all bridges of a given length can be partitioned according to the 'sizes' of the flaws. For $m \geq n$, let $\mathcal{T}_m^*(n)$ denote the set of bridges from $(0,0)$ to (m,m) with the set $\mathsf{T} = \{(0,a),(b,0),(c,c)\}$ of allowed steps the flaws of which have a projection of length n upon the y-axis. Let $t_{m,n} = |\mathcal{T}_m^*(n)|$. Members of $\mathcal{T}_m^*(n)$ are called the T-*bridges of semilength m with n flaws.* For the two-color bisection of the plane $\mathbb{Z} \times \mathbb{Z}$ with respect to the line $y = x$ and a bridge $\pi \in \mathcal{T}_m^*(n)$, let $\delta^-(\pi)$ denote the flaws of π, called the *blue part*, and let $\delta^+(\pi)$ denote the remaining part of π, called the *red part.* For $m \geq n$, $\delta^+(\pi)$ (resp. $\delta^-(\pi)$) has a projection of length $m - n$ (resp. n) upon the y-axis. Suppose that $\mathcal{T}_m^*(n)$ is partitioned into a disjoint union of subsets $\mathcal{T}_m^*(n,1), \ldots, \mathcal{T}_m^*(n,\ell)$ for some ℓ so that the generating function for the possibilities of all red parts for the bridges $\pi \in \mathcal{T}_m^*(n,p)$ $(1 \leq p \leq \ell)$ is formulated as a function $G_p(T,x)$ in T and x. Then the number of bridges in $\mathcal{T}_m^*(n)$ is enumerated by the linear combination $\sum_{p=1}^{\ell} \beta_p G_p(T,x)$ of the functions $G_p(T,x)$ $(1 \leq p \leq \ell)$, where β_p is the number of all blue parts for $\pi \in \mathcal{T}_m^*(n,p)$, i.e., $t_{m,n} = \sum_{p=1}^{\ell} \beta_p([x^{m-n}]G_p(T,x))$. Hence the generating function for $t_{m,n}$ is

$$x^n(\sum_{p=1}^{\ell} \beta_p G_p(T,x)). \tag{2}$$

This is called the *n-flaw expansion of the bridges.*

If (2) coincides with the remainder of (1) for $n = 1, \ldots, m$, then there induces a bijection between the set of T-excursions $\mathcal{T}_m$ and the specified class of T-bridges $\mathcal{T}_m^*(n)$. Hence there exists a variation of the Chung-Feller theorem $t_{m,1} = t_{m,2} = \cdots = t_{m,m} = t_m$. It is worth mentioning that Chung-Feller theorem is striking for its implication in probability theory and statistics, as Chung and Feller stated in their paper [3] "These results should serve as a warning to statisticians who might assume that fluctuation phenomena always follow the bell-shaped pattern and who would easily discover secular trends". We shall show the Chung-Feller phenomena for the Catalan paths (Theorems 4.1 and 4.2), the Catalan paths of order d (Theorems 4.3 and 4.4) and the lifted Motzkin paths (Theorems 6.4 and 6.5).

3 The Taylor expansions for some excursions

For illustration, the Taylor expansions of the generating functions for some familiar paths are discussed in detail.

3.1 Catalan paths

The m-th Catalan number $c_m = \frac{1}{m+1}\binom{2m}{m}$ counts the number of paths in the plane $\mathbb{Z} \times \mathbb{Z}$ from $(0,0)$ to (m,m) with the north N step $(0,1)$ and east E step $(1,0)$ that never pass below the line $y = x$. Such paths are called the *Catalan paths of semilength m*. Marking each north step with an x, the generating function $C = C(x) = \sum_{m \geq 0} c_m x^m$ satisfies $C = 1 + xC^2 = \frac{1-\sqrt{1-4x}}{2x}$. Moreover, the Catalan numbers satisfy the recurrence relations $c_{n+1} = \sum_{k=0}^{n} c_k c_{n-k}$ and $(n+2)c_{n+1} = 2(2n+1)c_n$ for $n \geq 0$ and $c_0 = 1$. Another useful fact is that $[x^n]C^k = \frac{k}{2n+k}\binom{2n+k}{n}$, which are known as the *ballot numbers*.

The technique of generating function is empowered by the "symbolic" method, owing to Schützenberger, to enumerate combinatorial structures by decomposing the structures into smaller building blocks either of the same type or of similar types. Refer to Flajolet and Sedgewick [11, Chapter 1] or Stanley [15, Chapter 5] for systematic treatments. In the framework of paths, factorizations into building blocks admit direct translations into generating functions. As shown in the proof of the following result, the argument becomes elementary and is repeatedly used in this paper by using the "symbolic" method.

Lemma 3.1 *For $0 \le p \le q$, the number of paths from $(0,0)$ to (p,q) with the $(0,1)$ and $(1,0)$ steps that never pass below the line $y = x$ is equal to $[x^p]C^{q-p+1}$.*

Proof: Note that the point (p,q) is on the line $y = x + q - p$. Each path from $(0,0)$ to (p,q) has a factorization $\nu_1 \mathsf{N}_1 \nu_2 \mathsf{N}_2 \cdots \nu_{q-p} \mathsf{N}_{q-p} \nu_{q-p+1}$, where N_i $(1 \le i \le q-p)$ is the last north step that rises from line $y = x + i - 1$ to line $y = x + i$ and ν_j $(1 \le j \le q - p + 1)$ is a Catalan path (possibly trivial). Hence the generating function for the number of paths from $(0,0)$ to (p,q) is $x^{q-p}C^{q-p+1}$. Since there are q north steps contained in such a path, what we need is $[x^q]\{x^{q-p}C^{q-p+1}\}$, as required. □

For illustration, a path from $(0,0)$ to a point $P = (p, p+2)$ for some $p \ge 0$ is in the form shown in Figure 1.

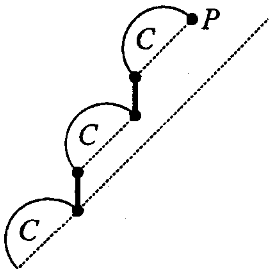

Figure 1: a path from $(0,0)$ to a point $P = (p, p+2)$

The Taylor expansion of C was first derived by Eu *et al.* in [8]. It is included here for an illustrative purpose.

Let $\mathsf{N}_{(x,y)}$ denote the north step from $(x, y-1)$ to (x,y). According to the first step that meets the line $y = n$ and $n \le m$, the paths in $\mathcal{C}_m$ are partitioned into a disjoint union of subsets $\mathcal{C}_m(p,n)$ for $p = 0, \ldots, n-1$, where

$$\mathcal{C}_m(p,n) = \{\pi \in \mathcal{C}_m | \mathsf{N}_{(p,n)} \in \pi\}.$$

For the two-color bisection of the plane $\mathbb{Z} \times \mathbb{Z}$ with respect to the line $y = n$, let $\mathcal{C}_m^-(p,n)$ and $\mathcal{C}_m^+(p,n)$ denote the sets of blue and red sections for the paths in $\mathcal{C}_m(p,n)$, respectively.

Lemma 3.2 *For $p = 0, \ldots, n-1$, we have*

(i) $|\mathcal{C}_m^-(p,n)| = [x^p]C^{n-p}$,

(ii) $|\mathcal{C}_m^+(p,n)| = [x^{m-n}]C^{n-p+1}$

Proof: For any path $\pi \in \mathcal{C}_m(p,n)$, the blue section $\gamma_n^-(\pi)$ goes from $(0,0)$ to $(p,n-1)$ and then rises from $(p,n-1)$ to (p,n). Hence $|\mathcal{C}_m^-(p,n)| = [x^p]C^{n-p}$ by Lemma 3.1.

On the other hand, the red section $\gamma_n^+(\pi)$ goes from (p,n) to (m,m). Observe that there is an immediate bijection between $\mathcal{C}_m^+(p,n)$ and the set of paths from $(0,0)$ to $(m-n, m-p)$, by which the paths are flipped over the line $x+y=m$ and traversed in the reverse order. By Lemma 3.1, (ii) follows. ☐

By a straightforward counting argument, we deduce the Taylor expansion of C.

Theorem 3.3 (Eu-Liu-Yeh [8]) *The n-th Taylor expansion of the C can be expressed in the form*

$$C = \sum_{i=0}^{n-1} c_i x^i + x^n \left(\sum_{p=0}^{n-1} \frac{n-p}{n+p} \binom{n+p}{p} C^{n-p+1} \right). \tag{3}$$

Proof: It suffices to show that $c_m = [x^{m-n}]\{\sum_{p=0}^{n-1} \frac{n-p}{n+p}\binom{n+p}{p} C^{n-p+1}\}$ for all $m \geq n$. By Lemma 3.2, we have

$$\begin{aligned} c_m &= |\mathcal{C}_m| \\ &= \sum_{p=0}^{n-1} |\mathcal{C}_m^-(p,n)| \cdot |\mathcal{C}_m^+(p,n)| \\ &= \sum_{p=0}^{n-1} ([x^p]C^{n-p})([x^{m-n}]C^{n-p+1}) \\ &= [x^{m-n}]\{\sum_{p=0}^{n-1} \frac{n-p}{n+p}\binom{n+p}{p} C^{n-p+1}\}, \end{aligned}$$

as required. ☐

Remark: There is an alternative representation for Catalan paths, also called Dyck paths, which goes from the origin to a point on the x-axis with the up step $(1,1)$ and down step $(1,-1)$. The two-color bisection of plane is therefore with respect to the line $x+y=n$.

We shall show in Theorem 4.1 that the number of Catalan paths with flaws are enumerated by the remainder of the Taylor expansions.

3.2 Catalan paths of order d

Let d be a positive integer. Let $\mathcal{T}_m^{(d)}$ denote the set of paths in the plane $d\mathbb{Z}\times\mathbb{Z}=\{(dx,y)|x,y\in\mathbb{Z}\}$ from $(0,0)$ to (dm,dm) with the unit north N step $(0,1)$ and the grand east E step $(d,0)$ that never pass below the line $y=x$ and let $t_m^{(d)}=|\mathcal{T}_m^{(d)}|$. Members of $\mathcal{T}_m^{(d)}$ are called the *Catalan path of order d of semilength dm* (or dm-CP(d) for short). Marking each north step by an x, the generating function for $\{t_m^{(d)}\}_{m\geq 0}$ is given by $T=T(x)=\sum_{m\geq 0}t_m^{(d)}x^{dm}$. For example, the ordinary Catalan paths are the examples of order $d=1$. For $d=2$, it is equivalent to the generalized Dyck paths considered by Cameron [2] and $T=1+x^2+3x^4+12x^6+55x^8+\cdots$. In fact, $t_m^{(d)}=\frac{1}{dm+1}\binom{dm+m}{m}$. Observe that each non-trivial CP(d) has a factorization $\mathsf{N}_1\nu_1\cdots\mathsf{N}_d\nu_d\mathsf{E}_1\nu_{d+1}$, where E_1 is the first east step that reaches the line $y=x$, N_i $(1\leq i\leq d)$ is the last north step that rises from line $y=x+i-1$ to line $y=x+i$ and ν_j $(1\leq j\leq d+1)$ is a CP(d) (possibly trivial). Hence T satisfies the functional equation $T=1+x^dT^{d+1}$. It is straightforward to prove the following result by a similar argument of Lemma 3.1.

Lemma 3.4 *For* $0\leq dp\leq q$, *the number of paths from* $(0,0)$ *to* (dp,q) *with the* $(0,1)$ *and* $(d,0)$ *steps that never pass below the line* $y=x$ *is equal to* $[x^{dp}]T^{q-dp+1}$.

By a two-color bisection of the plane $d\mathbb{Z}\times\mathbb{Z}$ with respect to the horizontal line $y=n$ and $n\leq dm$, we shall derive the Taylor expansions of T along the y-axis. The complication encountered is that the remainder is expressed as a function in $A_1,\ldots,A_d$ (defined in (4) below) due to the asymmetry of the permitted steps. Each of the A_i $(1\leq i\leq d)$ corresponds to one of the building blocks of a CP(d).

Let $\mathsf{N}_{(x,y)}$ denote the north step from $(x,y-1)$ to (x,y). According to the first step that meets the line $y=n$ and $n\leq dm$, the paths in $\mathcal{T}_m^{(d)}$ are partitioned into subsets $\mathcal{T}_m^{(d)}(dp,n)$ for $p=0,\ldots,\lfloor\frac{n-1}{d}\rfloor$, where

$$\mathcal{T}_m^{(d)}(dp,n)=\{\pi\in\mathcal{T}_m^{(d)}|\mathsf{N}_{(dp,n)}\in\pi\}.$$

Let $\mathcal{T}_m^{(d)-}(dp,n)$ and $\mathcal{T}_m^{(d)+}(dp,n)$ denote the sets of blue and red sections of $\mathcal{T}_m^{(d)}(dp,n)$, respectively. Observe that each blue section in $\mathcal{T}_m^{(d)-}(dp,n)$ goes from $(0,0)$ to $(dp,n-1)$ and then rises from $(dp,n-1)$ to (dp,n). By Lemma 3.4, we have the following result.

Lemma 3.5 *For $0 \leq dp < n$, $|\mathcal{T}_m^{(d)-}(dp,n)| = [x^{dp}]T^{n-dp}$.*

To enumerate $\mathcal{T}_m^{(d)+}(dp,n)$, consider a section of a path of the form $\mu\mathsf{E}$, where μ starts from and remains above a line, say $y = x + h$ for some $h \geq 0$ and E is the first east step that passes below the line $y = x + h$. μ is said to be of *type* A_i for $i \in \{1, \ldots, d\}$ if E goes from line $y = x + h + d - i$ to line $y = x + h - i$. Let A_i denote the generating function for the possibilities of a section of type A_i. It is easy to see that

$$A_i = x^{d-i}T^{d-i+1} \tag{4}$$

for $i = 1, \ldots, d$ since such a section μ has a factorization $\mu = \nu_0\mathsf{N}_1\nu_1 \cdots \mathsf{N}_{d-i}\nu_{d-i}$, where N_j $(1 \leq j \leq d-i)$ is the last north step that goes from line $y = x + h + j - 1$ to line $y = x + h + j$.

Lemma 3.6 *For $0 \leq dp < n \leq dm$,*

$$|\mathcal{T}_m^{(d)+}(dp,n)| = [x^{dm-n}]\{\sum_{a_1+2a_2+\cdots+da_d=n-dp} \binom{a_1+a_2+\cdots+a_d}{a_1, a_2, \cdots, a_d} A_1^{a_1}A_2^{a_2}\cdots A_d^{a_d}T\},$$

where A_i is defined in (4) for $i = 1, \ldots, d$.

Proof: Note that each red section $\gamma_n^+(\pi) \in \mathcal{T}_m^{(d)+}(dp,n)$ goes from (dp,n) to (dm,dm) and the point (dp,n) is on the line $y = x + n - dp$. Then $\gamma_n^+(\pi)$ has a factorization $\gamma_n^+(\pi) = \mu_k\mathsf{E}_k\mu_{k-1}\mathsf{E}_{k-1}\cdots\mu_1\mathsf{E}_1\mu_0$ for some $k \geq 1$, where E_k is the first east step that passes below the line $y = x+n-dp$, E_i $(k-1 \geq i \geq 1)$ is the first east step that goes lower (i.e., closer to the line $y = x$) than E_{i+1} does and E_1 is the first east step that reaches the $y = x$. Moreover, each section μ_i $(1 \leq i \leq k)$ is of certain type A_j for some $j \in \{1, \ldots, d\}$ and the terminal section μ_0 is a CP(d). Suppose that $\{\mu_1, \ldots, \mu_k\}$ consists of a_i sections of type A_i for some a_i and $i = 1, \ldots, d$. Then $a_1 + 2a_2 + \cdots + da_d = n - dp$ and the generating function for $|\mathcal{T}_m^{(d)+}(dp,n)|$ is $\sum_{a_1+2a_2+\cdots+da_d=n-dp}\binom{a_1+a_2+\cdots+a_d}{a_1,a_2,\cdots,a_d}A_1^{a_1}A_2^{a_2}\cdots A_d^{a_d}T$. What we need is the coefficient of x^{dm-n} of this function since the length of the projection of $\gamma_n^+(\pi)$ upon the y-axis is $dm - n$. □

In the following, let $\lambda_{dp,q} = [x^{dp}]T^{q-dp+1}$ and let $F_n(A_1, A_2, \ldots, A_d)$ be a function of the form

$$F_n(A_1, A_2, \ldots, A_d) = \sum_{a_1+2a_2+\cdots+da_d=n} \binom{a_1+a_2+\cdots+a_d}{a_1, a_2, \cdots, a_d} A_1^{a_1}A_2^{a_2}\cdots A_d^{a_d}.$$

Theorem 3.7 *The n-th Taylor expansion of T can be expressed in the form*

$$T = \sum_{i=0}^{\lfloor\frac{n-1}{d}\rfloor} t_i^{(d)} x^{di} + x^n \left(\sum_{p=0}^{\lfloor\frac{n-1}{d}\rfloor} \lambda_{dp,n-1} F_{n-dp}(A_1, A_2, \ldots, A_d)T \right), \tag{5}$$

where A_i is defined in (4) for $i = 1, \ldots, d$.

Proof: It suffices to show that $t_m^{(d)} = [x^{dm-n}]\{\sum_{p=0}^{\lfloor\frac{n-1}{d}\rfloor} \lambda_{dp,n-1} F_{n-dp}(A_1, A_2, \ldots, A_d)T\}$ for all $dm \geq n$. By Lemmas 3.5 and 3.6, we have

$$\begin{aligned}
t_m^{(d)} &= |\mathcal{T}_m^{(d)}| \\
&= \sum_{p=0}^{\lfloor\frac{n-1}{d}\rfloor} |\mathcal{T}_m^{(d)}(dp, n)| \\
&= \sum_{p=0}^{\lfloor\frac{n-1}{d}\rfloor} |\mathcal{T}_m^{(d)-}(dp, n)| \cdot |\mathcal{T}_m^{(d)+}(dp, n)| \\
&= \sum_{p=0}^{\lfloor\frac{n-1}{d}\rfloor} \lambda_{dp,n-1} \cdot [x^{dm-n}]\{ \sum_{a_1+2a_2+\cdots+da_d=n-dp} \binom{a_1+a_2+\cdots+a_d}{a_1, a_2, \cdots, a_d} A_1^{a_1} A_2^{a_2} \cdots A_d^{a_d} T\} \\
&= [x^{dm-n}]\{ \sum_{p=0}^{\lfloor\frac{n-1}{d}\rfloor} \lambda_{dp,n-1} F_{n-dp}(A_1, A_2, \ldots, A_d)T\},
\end{aligned}$$

as required. ☐

For example, let us take $d = 2$. The initial expansions of T are

$$\begin{aligned}
T &= 1 + xA_1T \\
&= 1 + x^2(A_1^2 + A_2)T \\
&= 1 + x^2 + x^3(A_1^3 + 2A_1A_2 + A_1)T \\
&= 1 + x^2 + x^4(A_1^4 + 3A_1^2A_2 + A_2^2 + 2A_1^2 + A_2)T,
\end{aligned}$$

where $A_1 = xT^2$ and $A_2 = T$. By substitution, we have

$$\begin{aligned}
T &= 1 + x^2T^3 \\
&= 1 + x^2T^2 + x^4T^5 \\
&= 1 + x^2 + x^4(T^3 + 2T^4) + x^6T^7 \\
&= 1 + x^2 + x^4(2T^2 + T^3) + x^6(2T^5 + 3T^6) + x^8T^9.
\end{aligned}$$

We shall show in Theorem 4.3 that the number of Catalan paths of order d with flaws are enumerated by the remainder of the Taylor expansions of T.

3.3 Motzkin paths

Let $\mathcal{M}_m$ denote the set of paths form $(0,0)$ to (m,m) with the north N step $(0,2)$, east E step $(2,0)$ and diagonal D step $(1,1)$ that never pass below the line $y=x$. Members of $\mathcal{M}_m$ are called the *Motzkin paths of semilength* m, which are enumerated by the well-known *Motzkin numbers*, denoted by $\{e_m\}_{m\geq 0}=\{1,1,2,4,9,21,51,\ldots\}$. Marking a unit projective length of a path upon the y-axis with an x, the generating function M for $\{e_m\}_{m\geq 0}$, given by $M=M(x)=\sum_{m\geq 0}e_m x^m$, satisfies $M=1+xM+x^2M^2=\frac{1-x-\sqrt{1-2x-3x^2}}{2x}$. Moreover, the Motzkin numbers satisfy the recurrence relations $e_{n+2}=e_{n+1}+\sum_{k=0}^{n}e_k e_{n-k}$ and $(n+4)e_{n+2}=(2n+5)e_{n+1}+(3n+3)e_n$ for $n\geq 0$ and $e_0=e_1=1$. Note that $x+y$ is even whenever there exists a Motzkin path visiting the point (x,y).

Lemma 3.8 *For* $0\leq p\leq q$, *the number of paths from* $(0,0)$ *to* $(p,2q-p)$ *with the* $(0,2)$, $(2,0)$ *and* $(1,1)$ *steps that never pass below the line* $y=x$ *is equal to* $[x^p]M^{q-p+1}$.

Proof: Note that the point $(p,2q-p)$ is on the line $y=x+2(q-p)$. Each path from $(0,0)$ to $(p,2q-p)$ has a factorization $\nu_1\mathsf{N}_1\nu_2\mathsf{N}_2\cdots\nu_{q-p}\mathsf{N}_{q-p}\nu_{q-p+1}$, where N_i $(1\leq i\leq q-p)$ is the last north step that rises from line $y=x+2(i-1)$ to line $y=x+2i$ and ν_j $(1\leq j\leq q-p+1)$ is a Motzkin path (possibly trivial). Hence the generating function for the number of paths from $(0,0)$ to $(p,2q)$ is $x^{2(q-p)}M^{q-p+1}$. Since the length of the projection of such a path upon the y-axis is $2q-p$, what we need is $[x^{2q-p}]\{x^{2(q-p)}M^{q-p+1}\}$, as required. $\square$

The Taylor expansion of M was first derived by Eu *et al.* in [8]. It is included here for completeness.

Let $\mathsf{D}_{(x,y)}$ denote the diagonal step from $(x-1,y-1)$ to (x,y) and let $\mathsf{N}_{(x,y)}$ denote the north step from $(x,y-2)$ to (x,y). According to the first step that meets the line $y=n$ and $n\leq m$, the paths in $\mathcal{M}_m$ are partitioned into subsets $U_m(n-2i,n)$ for $i=1,\ldots,\lfloor\frac{n}{2}\rfloor$, $V_m(n-2j,n)$ for $j=0,\ldots,\lfloor\frac{n-1}{2}\rfloor$ and $W_m(n-2k+1,n+1)$ for $k=1,\ldots,\lfloor\frac{n+1}{2}\rfloor$, where

$$\begin{aligned}
U_m(n-2i,n) &= \{\pi\in\mathcal{M}_m|\mathsf{N}_{(n-2i,n)}\in\pi\},\\
V_m(n-2j,n) &= \{\pi\in\mathcal{M}_m|\mathsf{D}_{(n-2j,n)}\in\pi\},\\
W_m(n-2k+1,n+1) &= \{\pi\in\mathcal{M}_m|\mathsf{N}_{(n-2k+1,n+1)}\in\pi\}.
\end{aligned}$$

For the two-color bisection of the plane $\mathbb{Z}\times\mathbb{Z}$ with respect to the line $y=n$ and $n\leq m$, let $U_m^-(n-2i,n)$, $V_m^-(n-2j,n)$ and $W_m^-(n-2k+1,n+1)$ (resp. $U_m^+(n-2i,n)$, $V_m^+(n-2j,n)$ and $W_m^+(n-2k+1,n+1)$) denote the sets of blue (resp. red) sections accordingly. Making use of Lemma 3.8, the following results are obtained by a similar argument of Lemma 3.2.

Lemma 3.9 *For* $i=1,\ldots,\lfloor\frac{n}{2}\rfloor$, $j=0,\ldots,\lfloor\frac{n-1}{2}\rfloor$ *and* $k=1,\ldots,\lfloor\frac{n+1}{2}\rfloor$, *we have*

(i) $|U_m^-(n-2i,n)|=[x^{n-2i}]M^i$,

(ii) $|V_m^-(n-2j,n)|=[x^{n-2j-1}]M^{j+1}$,

(iii) $|W_m^-(n-2k+1,n+1)|=[x^{n-2k+1}]M^k$,

(iv) $|U_m^+(n-2i,n)|=[x^{m-n}]M^{i+1}$,

(v) $|V_m^+(n-2j,n)|=[x^{m-n}]M^{j+1}$,

(vi) $|W_m^+(n-2k+1,n+1)|=[x^{m-n-1}]M^{k+1}$.

Theorem 3.10 (Eu-Liu-Yeh [8]) *The n-th Taylor expansion of the M can be expressed in the form*

$$M=\sum_{i=0}^{n-1}e_ix^i+x^n(G_n(M)+H_n(M))+x^{n+1}H_n(M)M,$$

where $G_n(z)=\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}g_{n,i}z^{i+1}$, $g_{n,i}=[x^{n-2i}]M^i$ *and* $H_n(z)=\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}h_{n,k}z^k$, $h_{n,k}=[x^{n-2k+1}]M^k$.

Proof: It suffices to show that $e_m=[x^{m-n}]\{G_n(M)+H_n(M)\}+[x^{m-n-1}]\{H_n(M)M\}$ for all $m\geq n$. By Lemma 3.9, we have

$$\begin{aligned}
e_m &= |\mathcal{M}_m| \\
&= \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}|U_m(n-2i,n)|+\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}|V_m(n-2j,n)|+\sum_{k=1}^{\lfloor\frac{n+1}{2}\rfloor}|W_m(n-2k+1,n+1)| \\
&= \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor}([x^{n-2i}]M^i)([x^{m-n}]M^{i+1})+\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}([x^{n-2j-1}]M^{j+1})([x^{m-n}]M^{j+1})
\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} ([x^{n-2k+1}]M^k)([x^{m-n-1}]M^{k+1}) \\
&= [x^{m-n}]\{\sum_{i=1}^{\lfloor \frac{n}{2} \rfloor} g_{n,k} M^{i+1} + \sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} h_{n,k} M^k\} + [x^{m-n-1}]\{\sum_{k=1}^{\lfloor \frac{n+1}{2} \rfloor} h_{n,k} M^{k+1}\},
\end{aligned}
$$

as required. ☐

We shall show in Theorem 4.5 that the number of Motzkin paths with flaws are enumerated by a partial remainder of the Taylor expansions.

3.4 Large and small Schröder paths

The *(large) Schröder numbers* $\{r_m\}_{m\geq 0} = \{1, 2, 6, 22, 90, 394, \ldots\}$ and the *small Schröder numbers* $\{s_m\}_{m\geq 0} = \{1, 1, 3, 11, 45, 197, \ldots\}$, first appeared in a paper of Ernst Schröder in 1870, have been found in many combinatorial configurations (see [15, Exercise 6.39], [4]). As noted by Stanley in [14], some of the numbers were known to Hipparchus in the second century B.C. The m-th Schröder number r_m counts the number of paths in the plane $\mathbb{Z}\times\mathbb{Z}$ from $(0,0)$ to (m,m) with the north N step $(0,1)$, east E step $(1,0)$ and diagonal D step $(1,1)$ that never pass below the line $y = x$. Such paths are called *(large) Schröder paths of semilength m* (or *m-Schröder paths* for short). Note that the terms of $\{r_m\}_{m\geq 1}$ are twice of those in $\{s_m\}_{m\geq 1}$. Consequently, the m-th small Schröder number s_m counts the number of m-Schröder paths without diagonal steps on the line $y = x$ (Proposition 3.11). It is known that the generating functions $R = R(x) = \sum_{m\geq 0} r_m x^m$ and $S = S(x) = \sum_{m\geq 0} s_m x^m$ of the large and small Schröder numbers satisfy the functional equations $R = 1 + xR + xR^2$ and $S = 1 - xS + 2xS^2$, respectively, and

$$
R = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x} \quad \text{and} \quad S = \frac{1 + x - \sqrt{1 - 6x + x^2}}{4x}. \tag{6}
$$

Moreover, the large Schröder numbers satisfy the recurrence relations $r_{n+1} = r_n + \sum_{k=0}^{n} r_k r_{n-k}$ and $(n+3)r_{n+2} = 3(2n+3)r_{n+1} - nr_n$ for $n \geq 0$ and $r_0 = 1$, $r_1 = 2$. The small Schröder numbers satisfy the recurrence relations $s_{n+1} = -s_n + 2\sum_{k=0}^{n} s_k s_{n-k}$ and $(n+3)s_{n+2} = 3(2n+3)s_{n+1} - ns_n$ for $n \geq 0$ and $s_0 = s_1 = 1$.

Among many other combinatorial structures counted by the small Schröder numbers, two interpretations of $\{s_m\}_{m\geq 1}$ are given in the following.

Proposition 3.11 *For $m \geq 1$, the following cases hold.*

(i) *The m-th small Schröder number s_m counts the number of m-Schröder paths with at least one diagonal step on the line $y = x$.*

(ii) *The m-th small Schröder number s_m counts the number of m-Schröder paths without diagonal steps on the line $y = x$.*

Proof: We shall establish a bijection ϕ between the above two sets. Given an m-Schröder path with at least one diagonal step on the line $y = x$, locate the first diagonal step D on the line $y = x$. Then π has a factorization of the form $\pi = \mu \mathsf{D} \nu$, where μ is a Schröder path without diagonal steps on the line $y = x$ and ν is a Schröder path (μ and ν are possibly trivial). The mapping ϕ carries π into $\phi(\pi) = \mu \mathsf{N} \nu \mathsf{E}$. On the other hand, given a path τ without diagonal step on the line $y = x$, locate the last north step N that rises from the line $y = x$ to the line $y = x + 1$. Then τ is factorized in the form $\tau = \mu' \mathsf{N} \nu' \mathsf{E}$. Then $\phi^{-1}(\tau) = \mu' \mathsf{D} \nu'$ is the required path. ☐

In particular, an m-Schröder path without diagonal steps on the line $y = x$ is called a *small m-Schröder path.* For $0 \leq p \leq q$, let $\Pi(p,q)$ denote the set of paths from $(0,0)$ to (p,q) with the $(0,1)$, $(1,0)$ and $(1,1)$ steps that never pass below the line $y = x$ and let $\Gamma(p,q) \subseteq \Pi(p,q)$ be the subset of paths without diagonal steps on the line $y = x$. By a similar argument of Lemma 3.1, we have the following results.

Lemma 3.12 *For $0 \leq p \leq q$, we have*

(i) $|\Pi(p,q)| = [x^p] R^{q-p+1}$,

(ii) $|\Gamma(p,q)| = [x^p] R^{q-p} S$.

Let $\mathsf{D}_{(x,y)}$ denote the diagonal step from $(x-1, y-1)$ to (x,y) and let $\mathsf{N}_{(x,y)}$ denote the north step from $(x, y-1)$ to (x,y). Let $\mathcal{R}_m$ denote the set of large m-Schröder paths. According to the first step that meets the line $y = n$ and $n \leq m$, the paths in $\mathcal{R}_m$ are partitioned into subsets $\mathcal{R}_m^N(i,n)$ for $i = 0, \ldots, n-1$ and $\mathcal{R}_m^D(j,n)$ for $j = 1, \ldots, n$, where

$$\mathcal{R}_m^N(i,n) = \{\pi \in \mathcal{R}_m | \mathsf{N}_{(i,n)} \in \pi\} \quad \text{and} \quad \mathcal{R}_m^D(j,n) = \{\pi \in \mathcal{R}_m | \mathsf{D}_{(j,n)} \in \pi\}.$$

For the two-color bisection of the plane $\mathbb{Z} \times \mathbb{Z}$ with respect to the line $y = n$ and $n \leq m$, let $\mathcal{R}_m^{N-}(i,n)$ and $\mathcal{R}_m^{N+}(i,n)$ denote the sets of blue and red sections of $\mathcal{R}_m^N(i,n)$, respectively and let $\mathcal{R}_m^{D-}(j,n)$ and $\mathcal{R}_m^{D+}(j,n)$ denote the sets of blue and red sections of $\mathcal{R}_m^D(j,n)$, respectively. By a similar argument of Lemma 3.2, it is easy to see that $|\mathcal{R}_m^{N-}(i,n)|$ coincides with $|\Pi(i,n-1)|$ for $i = 0,\ldots,n-1$ and $|\mathcal{R}_m^{D-}(j,n)|$ coincides with $|\Pi(j-1,n-1)|$ for $j = 1,\ldots,n$. Moreover, there is an immediate bijection between $\mathcal{R}_m^{N+}(i,n)$ and $\Pi(m-n,m-i)$ (resp. between $\mathcal{R}_m^{D+}(j,n)$ and $\Pi(m-n,m-j)$), by which the paths are flipped over the line $x+y=m$ and traversed backward. By Lemma 3.12(i), we have the following results.

Lemma 3.13 *For $i = 0,\ldots,n-1$ and $j = 1,\ldots,n$, we have*

(i) $|\mathcal{R}_m^{N-}(i,n)| = [x^i]R^{n-i}$,

(ii) $|\mathcal{R}_m^{D-}(j,n)| = [x^{j-1}]R^{n-j+1}$,

(iii) $|\mathcal{R}_m^{N+}(i,n)| = [x^{m-n}]R^{n-i+1}$,

(iv) $|\mathcal{R}_m^{D+}(j,n)| = [x^{m-n}]R^{n-j+1}$.

For the case of small Schröder paths, let $\mathcal{S}_m$ denote the set of small m-Schröder paths. $\mathcal{S}_m$ are partitioned into subsets $\mathcal{S}_m^N(i,n)$ for $i = 0,\ldots,n-1$ and $\mathcal{S}_m^D(j,n)$ for $j = 1,\ldots,n-1$, where

$$\mathcal{S}_m^N(i,n) = \{\pi \in \mathcal{S}_m | \mathsf{N}_{(i,n)} \in \pi\} \quad \text{and} \quad \mathcal{S}_m^D(j,n) = \{\pi \in \mathcal{S}_m | \mathsf{D}_{(j,n)} \in \pi\}.$$

The sets $\mathcal{S}_m^{N-}(i,n)$, $\mathcal{S}_m^{N+}(i,n)$, $\mathcal{S}_m^{D-}(j,n)$ and $\mathcal{S}_m^{D+}(j,n)$ are defined accordingly. Making use of Lemma 3.12(ii), the following results can be developed parallel to Lemma 3.13.

Lemma 3.14 *For $i = 0,\ldots,n-1$ and $j = 1,\ldots,n-1$, we have*

(i) $|\mathcal{S}_m^{N-}(i,n)| = [x^i]R^{n-i-1}S$,

(ii) $|\mathcal{S}_m^{D-}(j,n)| = [x^{j-1}]R^{n-j}S$,

(iii) $|\mathcal{S}_m^{N+}(i,n)| = [x^{m-n}]R^{n-i}S$,

(iv) $|\mathcal{S}_m^{D+}(j,n)| = [x^{m-n}]R^{n-j}S$.

The Taylor expansions of R and S are derived along the y-axis. For the case of S, the remainders of which are given as functions in R and S. One can obtain functions in S by substituting R with $2S-1$.

Theorem 3.15 *We have*

(i) *The n-th Taylor expansion of the generating function for large Schröder numbers can be expressed in the form*

$$R = \sum_{i=0}^{n-1} r_i x^i + x^n(F_n(R) + F_n(R)R),$$

where $F_n(z) = \sum_{k=1}^{n} f_{n,k} z^k$ *and* $f_{n,k} = [x^{n-k}]R^k$.

(ii) *The n-th Taylor expansion of the generating function for small Schröder numbers can be expressed in the form*

$$S = \sum_{i=0}^{n-1} s_i x^i + x^n(G_n(R)S + G_n^*(R)S).$$

where $G_n(z) = \sum_{k=1}^{n} g_{n,k} z^k$, $G_n^*(z) = \sum_{k=2}^{n} g_{n,k} z^{k-1}$ *and* $g_{n,k} = [x^{n-k}]\{R^{k-1}S\}$.

Proof: To prove (i), it suffices to show that $r_m = [x^{m-n}]\{F_n(R)+F_n(R)R\}$ for all $m \geq n$. By Lemma 3.13,

$$\begin{aligned} r_m &= |\mathcal{R}_m| \\ &= \sum_{j=1}^{n} |\mathcal{R}_m^D(j,n)| + \sum_{i=0}^{n-1} |\mathcal{R}_m^N(i,n)| \\ &= \sum_{j=1}^{n} |\mathcal{R}_m^{D-}(j,n)| \cdot |\mathcal{R}_m^{D+}(j,n)| + \sum_{i=0}^{n-1} |\mathcal{R}_m^{N-}(i,n)| \cdot |\mathcal{R}_m^{N+}(i,n)| \\ &= \sum_{i=0}^{n-1} ([x^i]R^{n-i})([x^{m-n}]\{R^{n-i} + R^{n-i+1}\}) \\ &= \sum_{k=1}^{n} ([x^{n-k}]R^k)([x^{m-n}]\{R^k + R^{k+1}\}), \end{aligned}$$

as required. The last equality is obtained by replacing $n-i$ with k. It is straightforward to prove (ii) by the same argument. □

We shall show in Theorem 4.6(i) that the number of large Schröder paths with flaws are enumerated by a partial remainder of the Taylor expansions of R. So there is no Chung-Feller type result in this case. However, a neat formula for enumerating large Schröder paths with flaws is obtained (Theorem 7.1).

4 The flaw expansions for some bridges

In this section, the flaw expansions for some familiar paths are derived.

4.1 Catalan paths with flaws

For $0 \le n \le m$, let $\mathcal{C}^*_m(n)$ denote the set of paths from $(0,0)$ to (m,m) with the north N $(0,1)$ and east E $(1,0)$ steps that contain n north steps under the line $y=x$. Members of $\mathcal{C}^*_m(n)$ are called the *Catalan paths of semilength m with n flaws*. For the two-color bisection of the plane $\mathbb{Z}\times\mathbb{Z}$ with respect to the line $y=x$, the blue part $\delta^-(\pi)$ of a path $\pi \in \mathcal{C}^*_m(n)$ (i.e., the flaws) is partitioned into nontrivial blocks. Each block is of the form $\mathsf{E}\nu\mathsf{N}$, where ν is a Catalan path flipped over the line $y=x$. So the generating function for the possibilities of a nontrivial block is xC. Moreover, if $\delta^-(\pi)$ has k nontrivial blocks then the red part $\delta^+(\pi)$ is a disjoint union of $k+1$ blocks (possibly trivial) of Catalan paths. According to the number of blocks in the blue part, the paths in $\mathcal{C}^*_m(n)$ are partitioned into $\mathcal{C}^*_m(n,k)$ for $k=1,\ldots,n$, where $\mathcal{C}^*_m(n,k)=\{\pi\in\mathcal{C}^*_m(n)|\delta^-(\pi)\text{ has }k\text{ nontrivial blocks}\}$. Since there are n and $m-n$ north steps contained in $\delta^-(\pi)$ and $\delta^+(\pi)$, respectively, we have $|\mathcal{C}^*_m(n,k)|=([x^n]x^kC^k)([x^{m-n}]C^{k+1})$. Then

$$\begin{aligned}
|\mathcal{C}^*_m(n)| &= \sum_{k=1}^{n}|\mathcal{C}^*_m(n,k)| \\
&= \sum_{k=1}^{n}([x^n]x^kC^k)([x^{m-n}]C^{k+1}) \\
&= [x^{m-n}]\{\sum_{k=1}^{n}\frac{k}{2n-k}\binom{2n-k}{n-k}C^{k+1}\}.
\end{aligned}$$

Hence the generating function for $|\mathcal{C}^*_m(n)|$ is $x^n(\sum_{k=1}^{n}\frac{k}{2n-k}\binom{2n-k}{n-k}C^{k+1})$, which coincides with the remainder of (3). This proves the following results.

Theorem 4.1 *The generating function for the number of Catalan paths of semilength m with n flaws coincides with the remainder of the n-th Taylor expansion of C (in Theorem 3.3).*

By Theorems 3.3 and 4.1, we deduce the Chung-Feller theorem anew.

Theorem 4.2 (Chung-Feller [3]) *The number of Catalan paths of semilength m with n flaws is c_m $(0 \leq n \leq m)$, independent of n.*

4.2 Catalan paths of order d with flaws

For $0 \leq n \leq dm$, let $\mathcal{T}_m^{(d)*}(n)$ denote the set of paths from $(0,0)$ to (dm, dm) with the unit north N step $(0,1)$ and the grand east E step $(d,0)$ that contain n north steps under the line $y = x$. Members of $\mathcal{T}_m^{(d)*}(n)$ are called the *Catalan paths of order d of semilength dm with n flaws.* For the two-color bisection of the plane $d\mathbb{Z} \times \mathbb{Z}$ with respect to the line $y = x$, the flaws of a path are partitioned into nontrivial blue blocks, each of which begins with an east step that passes below the line $y = x$. A blue block of the form $\mathsf{E}\tau$ is said to be of *type* B_i for $i \in \{1, \ldots, d\}$ if E goes from line $y = x + d - i$ to line $y = x - i$ and τ has a factorization $\tau = \nu_i \mathsf{N}_i \nu_{i-1} \mathsf{N}_{i-1} \ldots \nu_1 \mathsf{N}_1$, where N_j $(1 \leq j \leq i)$ is the first north step that rises from line $y = x - j$ to line $y = x - j + 1$. Note that ν_j $(1 \leq j \leq i)$ is simply a CP(d) flipped horizontally and vertically and traversed backward. Hence the generating function for the possibilities of a blue block of type B_i is $x^i T^i$. Observe that a blue block of type B_i is proceeded by a red section of type A_i and there are $dk + i$ north steps in a blue block of type B_i for some non-negative integer k.

For example, consider the case $d = 2$ and $n = 3$. Each path in $\mathcal{T}_m^{(2)*}(3)$ is in one of the four forms shown in Figure 2. Then the generating function for $|\mathcal{T}_m^{(2)*}(3)|$ is $x^3(A_1^3 + 2A_1A_2 + A_1)T$, which coincides with the remainder of the third expansion of T.

Theorem 4.3 *For $0 \leq n \leq dm$, the generating function for the number of Catalan paths of order d of semilength dm with n flaws coincides with the remainder of the n-th Taylor expansion of T (in Theorem 3.7).*

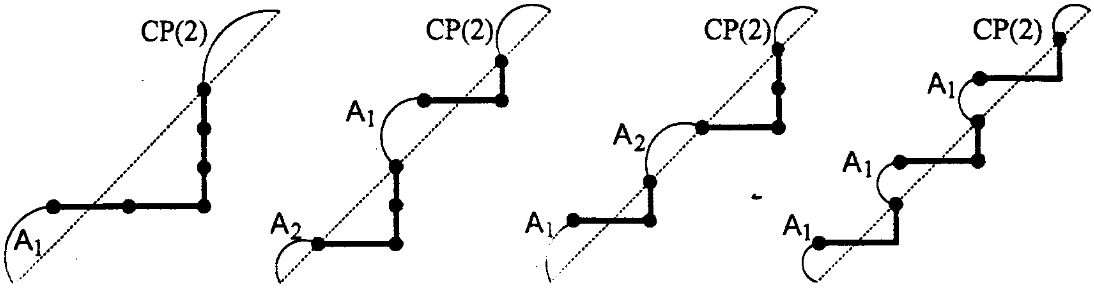

Figure 2: the 4 classes of paths in $\mathcal{T}_m^{(2)*}(3)$

Proof: Given a path $\pi \in \mathcal{T}_m^{(d)*}(n)$, suppose that the flaws of π consist of a_i blue blocks of type B_i for some a_i and $i = 1, \dots, d$. Note that $a_1 + 2a_2 + \cdots + da_d \equiv n \pmod{d}$. Assume that $n \equiv r \pmod{d}$ for some $r \in \{0, \dots, d-1\}$. Let $\mathcal{T}_m^{(d)*}(n, dk) \subseteq \mathcal{T}_m^{(d)*}(n)$ be the subset of paths with $a_1 + 2a_2 + \cdots + da_d = dk + r$ for $k = 0, \dots, \frac{n-r}{d}$. For each path $\pi \in \mathcal{T}_m^{(d)*}(n, dk)$, there are $dm - n$ north steps among the a_i $(1 \le i \le d)$ red sections of type A_i and a CP(d) (the terminal red section). Hence

$$\begin{aligned}
|\mathcal{T}_m^{(d)*}(n,dk)| &= \sum_{a_1+2a_2+\cdots+da_d=dk+r} [x^n]\{\binom{a_1+a_2+\cdots+a_d}{a_1,a_2,\cdots,a_d}\prod_{i=1}^{d}(x^iT^i)^{a_i}\}\cdot[x^{dm-n}]\{\prod_{i=1}^{d}A_i^{a_i}\cdot \\
&= \sum_{a_1+2a_2+\cdots+da_d=dk+r} \binom{a_1+a_2+\cdots+a_d}{a_1,a_2,\cdots,a_d}[x^{n-dk-r}]T^{dk+r}\cdot[x^{dm-n}]\{\prod_{i=1}^{d}A_i^{a_i}\cdot T \\
&= [x^{dm-n}]\{\lambda_{n-dk-r,n-1}\sum_{a_1+2a_2+\cdots+da_d=dk+r}\binom{a_1+a_2+\cdots+a_d}{a_1,a_2,\cdots,a_d}\prod_{i=1}^{d}A_i^{a_i}\cdot T\}.
\end{aligned}$$

Hence the generating function for $|\mathcal{T}_m^{(d)*}(n)|$ is

$$x^n\left(\sum_{k=0}^{\frac{n-r}{d}}\lambda_{n-dk-r,n-1}F_{dk+r}(A_1,A_2,\dots,A_d)T\right),$$

which coincides with the remainder of (5). Since the argument works for $n \equiv r \pmod{d}$ for all $r \in \{0, \dots, d-1\}$, the proof is complete. □

By Theorems 3.7 and 4.3, we have the following variation of the Chung-Feller theorem.

Theorem 4.4 *For $0 \le n \le dm$, the number of Catalan paths of order d of semilength dm with n flaws is $t_m^{(d)}$, independent of n.*

Remark: Cameron [2] used a standard manipulation of bivariate generating functions to obtain the case $d = 2$.

4.3 Motzkin paths with flaws

For $m \geq n$, let $\mathcal{M}^*_m(n)$ denote the set of paths form $(0,0)$ to (m,m) with the north N step $(0,2)$, east E step $(2,0)$ and diagonal D step $(1,1)$ the flaws of which have a projection of length n upon the y-axis. Members of $\mathcal{M}^*_m(n)$ are called the *Motzkin paths of semilength m with n flaws.* For the two-color bisection of the plane $\mathbb{Z} \times \mathbb{Z}$ with respect to the line $y = x$, note that each nontrivial block of the blue part $\delta^-(\pi)$ of a path $\pi \in \mathcal{M}^*_m(n)$ is of the form $\mathsf{E}\nu\mathsf{N}$ with a semilength at least 2, where ν is a Motzkin path flipped over the line $y = x$. Hence the generating function for the possibilities of a nontrivial block is x^2M. Let $\mathcal{M}^*_m(n)$ be partitioned into $\mathcal{M}^*_m(n,i)$ for $i = 1, \ldots, \lfloor\frac{n}{2}\rfloor$, where $\mathcal{M}^*_m(n,i) = \{\pi \in \mathcal{M}^*_m(n) | \delta^-(\pi)$ has i nontrivial blocks$\}$. Since $\delta^+(\pi)$ consists of $i+1$ blocks of Motzkin paths (possibly trivial) with a total semilength $m-n$ for $\pi \in \mathcal{M}^*_m(n,i)$, we have $|\mathcal{M}^*_m(n,i)| = ([x^n]x^{2i}M^i)([x^{m-n}]M^{i+1})$. Hence

$$\begin{aligned}
|\mathcal{M}^*_m(n)| &= \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} |\mathcal{M}^*_m(n,i)| \\
&= \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} ([x^n]x^{2i}M^i)([x^{m-n}]M^{i+1}) \\
&= [x^{m-n}]\{\sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} g_{n,i}M^{i+1}\},
\end{aligned}$$

where $g_{n,i} = [x^{n-2i}]M^i$ for $i = 1, \ldots, \lfloor\frac{n}{2}\rfloor$. This prove the following result.

Theorem 4.5 *Let $G_n(z) = \sum_{i=1}^{\lfloor\frac{n}{2}\rfloor} g_{n,i}z^{i+1}$ and $g_{n,i} = [x^{n-2i}]M^i$. The generating function for the number of Motzkin paths of semilength m with n flaws is $x^n(G_n(M))$.*

Remark: It follows from Theorems 3.10 and 4.5 that there is no Chung-Feller property in this case. A recurrence relation for the number of Motzkin paths with flaws was derived in [8].

4.4 Large and small Schröder paths with flaws

For $0 \le n \le m$, let $\mathcal{R}^*_m(n)$ denote the set of paths from $(0,0)$ to (m,m) with the north N step $(0,1)$, east E $(1,0)$ and diagonal D step $(1,1)$ such that the flaws of which have a projection of length n upon the y-axis. Members of $\mathcal{R}^*_m(n)$ are called the *(large) m-Schröder paths with n flaws.* Let $\mathcal{S}^*_m(n) \subseteq \mathcal{R}^*_m(n)$ be the subset of paths without diagonal steps on the line $y = x$, called the *small m-Schröder paths with n flaws.*

Theorem 4.6 *Let* $F_n(z) = \sum_{k=1}^{n} f_{n,k} z^k$ *and* $f_{n,k} = [x^{n-k}]R^k$. *The following cases hold.*

(i) *The generating function for the number of large m-Schröder paths with n flaws is* $x^n(F_n(R)R)$.

(ii) *The generating function for the number of small m-Schröder paths with n flaws is* $x^n(F_n(S)S)$.

Proof: For the two-color bisection of the plane $\mathbb{Z} \times \mathbb{Z}$ with respect to the line $y = x$, the blue part $\delta^-(\pi)$ of a path $\pi \in \mathcal{R}^*_m(n)$ is partitioned into nontrivial blocks of the from EνN, where ν is a large Schröder path flipped over the line $y = x$. Hence the generating function for the possibilities of a nontrivial block is xR. Let $\mathcal{R}^*_m(n)$ be partitioned into $\mathcal{R}^*_m(n,k)$ for $k = 1, \ldots, n$, where $\mathcal{R}^*_m(n,k) = \{\pi \in \mathcal{R}^*_m(n) | \delta^-(\pi)$ has k nontrivial blocks$\}$. Since the red part $\delta^+(\pi)$ of π consists of $k+1$ blocks of large Schröder paths (possibly trivial) with a total semilength $m-n$ for $\pi \in \mathcal{R}^*_m(n,k)$, we have $|\mathcal{R}^*_m(n,k)| = ([x^n]x^kR^k)([x^{m-n}]R^{k+1})$. Hence

$$
\begin{aligned}
|\mathcal{R}^*_m(n)| &= \sum_{k=1}^{n} |\mathcal{R}^*_m(n,k)| \\
&= \sum_{k=1}^{n} ([x^n]x^kR^k)([x^{m-n}]R^{k+1}) \\
&= [x^{m-n}]\{\sum_{k=1}^{n} f_{n,k}R^{k+1}\},
\end{aligned}
$$

as required.

(ii) Let $\mathcal{S}^*_m(n)$ be partitioned into $\mathcal{S}^*_m(n,k)$ for $k = 1, \ldots, n$, where $\mathcal{S}^*_m(n,k) = \{\pi \in \mathcal{S}^*_m(n) | \delta^-(\pi)$ has k nontrivial blocks$\}$. Following the above argument, note that the red part $\delta^+(\pi)$ of π consists of $k+1$ blocks of small Schröder paths (possibly trivial) for $\pi \in \mathcal{S}^*_m(n,k)$. Hence

$$\begin{aligned} |\mathcal{S}^*_m(n)| &= \sum_{k=1}^{n} |\mathcal{S}^*_m(n,k)| \\ &= \sum_{k=1}^{n} ([x^n]x^k R^k)([x^{m-n}]S^{k+1}) \\ &= [x^{m-n}]\{\sum_{k=1}^{n} f_{n,k} S^{k+1}\}. \end{aligned}$$

The proof is complete. □

Remark: By Theorem 3.15(i) and 4.6(i), there is no Chung-Feller property for the large paths with flaws in this case. However, if we assign two possible colors to the north step from $(m,m-1)$ to (m,m) then the generating function for the number of large m-Schröder paths with n flaws will become $x^n(F_n(R)(1+R))$, which coincides with the remainder of the n-th expansion of R. Although it is somewhat artificial, we have a variation of Chung-Feller theorem regarding the flaws with coloring for large Schröder paths.

5 The bijections and cut-and-paste technique

In this section, we introduce a universal cut-and-paste technique to establish bijections between excursions and specific classes of bridges.

5.1 Catalan paths

Inspired by the proofs of Theorems 3.3 and 4.1, a bijection $\phi : \mathcal{C}_m \to \mathcal{C}^*_m(n)$ is established, which carries a path $\pi \in \mathcal{C}_m$ into the image $\phi(\pi) \in \mathcal{C}^*_m(n)$ by cutting the blue section of π into slices and pasting them into blocks of the blue part of $\phi(\pi)$. The cut-and-paste procedure is repeatedly used and becomes a standard technique.

Recall that $\mathcal{C}_m = \cup_{p=0}^{n-1}\mathcal{C}_m(p,n)$, where $\mathcal{C}_m(p,n) = \{\pi \in \mathcal{C}(m) | \mathsf{N}_{(p,n)} \in \pi\}$. Given a path $\pi \in \mathcal{C}_m(p,n)$, the red section $\gamma^+_n(\pi)$ has a factorization $\gamma^+_n(\pi) = \mu_{n-p}\mathsf{E}_{n-p}\cdots\mu_1\mathsf{E}_1\mu_0$, where E_i $(1 \le i \le n-p)$ is the first east step that goes from line $y = x+i$ to line $y = x+i-1$. On the other hand, the blue section $\gamma^-_n(\pi)$ has a factorization $\gamma^-_n(\pi) =$

$\nu_1 \mathsf{N}_1 \nu_2 \mathsf{N}_2 \cdots \nu_{n-p} \mathsf{N}_{n-p}$, where N_i $(1 \le i \le n-p)$ is the last north step that rises from line to line $y = x + i - 1$ to line $y = x + i$. To establish a mapping, form a new section $\gamma_n^-(\pi)' = \mathsf{N}_0 \nu_1 \mathsf{N}_1 \nu_2 \cdots \mathsf{N}_{n-p-1} \nu_{n-p}$ from $\gamma_n^-(\pi)$ (with N_{n-p} moved to the beginning and denoted by N_0). Let $\widehat{\nu}_i$ be the section of path obtained from ν_i by flipping over the line $y = x$. Define a mapping ϕ_p which carries $\pi = \gamma_n^-(\pi)\gamma_n^+(\pi)$ into

$$\phi_p(\pi) = \mu_{n-p} \mathsf{E}_{n-p} \widehat{\nu}_{n-p} \mathsf{N}_{n-p-1} \cdots \mu_1 \mathsf{E}_1 \widehat{\nu}_1 \mathsf{N}_0 \mu_0.$$

Since the n north steps in $\gamma_n^-(\pi)$ are partitioned into the $n-p$ blocks $\mathsf{E}_{n-p} \nu_{n-p} \mathsf{N}_{n-p-1}$, $\ldots$, $\mathsf{E}_1 \nu_1 \mathsf{N}_0$ of the blue part of $\phi_p(\pi)$, $\phi_p(\pi) \in \mathcal{C}_m^*(n)$. Hence a mapping $\phi : \mathcal{C}_m \to \mathcal{C}_m^*(n)$ is established with $\phi|_{\mathcal{C}_m(p,n)} = \phi_p$ for $p = 0, \ldots, n-1$. In order to find the ϕ^{-1}, we only need to reverse the procedure.

For example, consider the case $m = 9$ and $n = 5$. In Figure 3, on the left is a Catalan path $\pi \in \mathcal{C}_9$, the red section $\gamma_5^+(\pi)$ (from $(2,5)$ to $(9,9)$) has a factorization $\gamma_5^+(\pi) = \mu_3 \mathsf{E}_3 \mu_2 \mathsf{E}_2 \mu_1 \mathsf{E}_1 \mu_0$, where $\mathsf{E}_3 = 10$, $\mathsf{E}_2 = 15$ and $\mathsf{E}_1 = 16$, and $\mu_3 = (8,9)$, $\mu_2 = (11,12,13,14)$, μ_1 is trivial and $\mu_0 = (17,18)$. On the other hand, the blue section $\gamma_5^-(\pi)$ (from $(0,0)$ to $(2,5)$) has a factorization $\gamma_5^-(\pi) = \nu_1 \mathsf{N}_1 \nu_2 \mathsf{N}_2 \nu_3 \mathsf{N}_3$, where $\mathsf{N}_1 = 3$, $\mathsf{N}_2 = 4$ and $\mathsf{N}_3 = 7$, and $\nu_1 = (1,2)$, ν_2 is trivial and $\nu_3 = (5,6)$. For $\gamma_5^-(\pi)' = \mathsf{N}_0 \nu_1 \mathsf{N}_1 \nu_2 \mathsf{N}_2 \nu_3$, $\mathsf{N}_0 = \mathsf{N}_3 = 7$. On the right is the path $\phi(\pi) = \mu_3 \mathsf{E}_3 \widehat{\nu}_3 \mathsf{N}_2 \mu_2 \mathsf{E}_2 \widehat{\nu}_2 \mathsf{N}_1 \mu_1 \mathsf{E}_1 \widehat{\nu}_1 \mathsf{N}_0 \mu_0 \in \mathcal{C}_9^*(5)$.

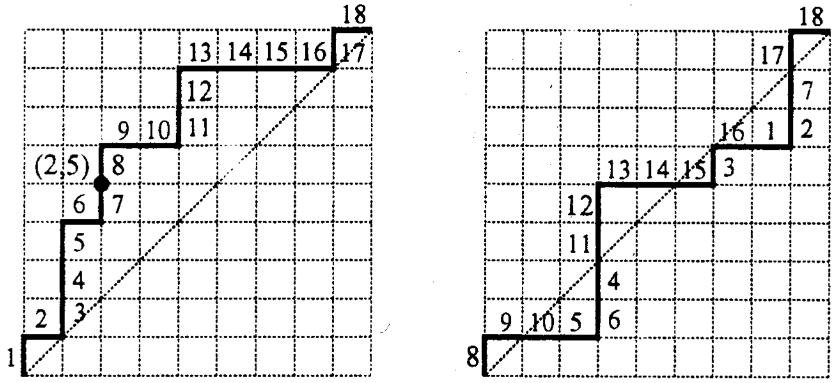

Figure 3: a path $\pi \in \mathcal{C}_9(2,5)$ and the $\phi(\pi) \in \mathcal{C}_9^*(5)$

5.2 Catalan paths of order d

In the following, we shall establish a bijection $\phi : \mathcal{T}_m^{(d)} \rightarrow \mathcal{T}_m^{(d)*}(n)$ by the cut-and-paste technique.

Recall that $\mathcal{T}_m^{(d)}(dp,n) = \{\pi \in \mathcal{T}_m^{(d)} | \mathsf{N}_{(dp,n)} \in \pi\}$ for $p = 0,\ldots,\lfloor\frac{n-1}{d}\rfloor$. Given a path $\pi \in \mathcal{T}_m^{(d)}(dp,n)$, the red section $\gamma_n^+(\pi)$ (from (dp,n) to (dm,dm)) has a factorization $\gamma_n^+(\pi) = \mu_k\mathsf{E}_k\mu_{k-1}\mathsf{E}_{k-1}\cdots\mu_1\mathsf{E}_1\mu_0$ for some $k \geq 1$, where E_k is the first east step passing below the line $y = x+n-dp$, E_i $(k-1 \geq i \geq 1)$ is the first east step going lower than E_{i+1} and E_1 is the first east step reaching the $y = x$. Suppose that the section μ_i is of type A_{j_i} for some $j_i \in \{1,\ldots,d\}$ and $i = 1,\ldots,k$. Then $j_1+\cdots+j_k = n-dp$. Let $b_i = j_1+\cdots+j_i$ $(1 \leq i \leq k)$ and let $b_0 = 0$. On the other hand, the blue section $\gamma_n^-(\pi)$ (from $(0,0)$ to (dp,n)) has a factorization $\gamma_n^-(\pi) = \nu_1\mathsf{N}_1\nu_2\mathsf{N}_2\cdots\nu_{n-dp}\mathsf{N}_{n-dp}$, where N_i $(1 \leq i \leq n-dp)$ is the last north step that rises from the line $y = x+i-1$ to the line $y = x+i$. To establish a mapping, form a new section $\gamma_n^-(\pi)' = \mathsf{N}_0\nu_1\mathsf{N}_1\nu_2\mathsf{N}_2\cdots\mathsf{N}_{n-dp-1}\nu_{n-dp}$ from $\gamma_n^-(\pi)$ (with N_{n-dp} moved to the beginning and denoted by N_0) and then factorize it into k sections $\gamma_n^-(\pi)' = \omega_1\ldots\omega_k$ such that ω_i is the section from $\mathsf{N}_{b_{i-1}}$ to ν_{b_i} for $i = 1,\ldots,k$. Moreover, let $\widehat{\omega}_i$ be the section obtained from ω_i by flipping horizontally and vertically. Define a mapping ϕ_p that carries π into

$$\phi_p(\pi) = \mu_k\mathsf{E}_k\widehat{\omega}_k\mu_{k-1}\mathsf{E}_{k-1}\widehat{\omega}_{k-1}\cdots\mu_1\mathsf{E}_1\widehat{\omega}_1\mu_0.$$

Since the n north steps of $\gamma_n^-(\pi)$ are partitioned into the k blocks $\widehat{\omega}_1,\ldots,\widehat{\omega}_k$ of flaws of $\phi_p(\pi)$, $\phi_p(\pi) \in \mathcal{T}_m^{(d)*}(n)$. Hence a mapping $\phi : \mathcal{T}_m^{(d)} \rightarrow \mathcal{T}_m^{(d)*}(n)$ is established with $\phi|_{\mathcal{T}_m^{(d)}(dp,n)} = \phi_p$ for $p = 0,\ldots,\lfloor\frac{n-1}{d}\rfloor$. To fine the ϕ^{-1}, just reverse the procedure. The bijection ϕ is established.

For example, consider the case $d = 2$, $m = 5$ and $n = 7$. In Figure 4, on the left is a path $\pi \in \mathcal{T}_5^{(2)}(4,7)$. The red section $\gamma_7^+(\pi)$ (from $(4,7)$ to $(10,10)$) has a factorization $\gamma_7^+(\pi) = \mu_2\mathsf{E}_2\mu_1\mathsf{E}_1\mu_0$, where $\mathsf{E}_2 = 11$, $\mathsf{E}_1 = 12$, $\mu_2 = (10)$ is of type A_1, μ_1 is a trivial block of type A_2 and $\mu_0 = (13,14,15)$. The blue section $\gamma_7^-(\pi)$ (from $(0,0)$ to $(4,7)$) has a factorization $\gamma_7^-(\pi) = \nu_1\mathsf{N}_1\nu_2\mathsf{N}_2\nu_3\mathsf{N}_3$, where $\mathsf{N}_1 = 4$, $\mathsf{N}_2 = 5$ and $\mathsf{N}_3 = 9$. For $\gamma_7^-(\pi)' = \mathsf{N}_0\nu_1\mathsf{N}_1\nu_2\mathsf{N}_2\nu_3 = \omega_1\omega_2$, $\mathsf{N}_0 = \mathsf{N}_3 = 9$, $\omega_1 = \mathsf{N}_0\nu_1\mathsf{N}_1\nu_2 = (9,1,2,3,4)$ and $\omega_2 = \mathsf{N}_2\nu_3 = (5,6,7,8)$. On the right is the path $\phi(\pi) = \mu_2\mathsf{E}_2\widehat{\omega}_2\mu_1\mathsf{E}_1\widehat{\omega}_1\mu_0 \in \mathcal{T}_5^{(2)*}(7)$.

Remark: The universal cut-and-paste technique can be further applied to the following situation. Let $\mathcal{G}$ denote the set of bridges with the unit north step $(0,1)$ and the allowed

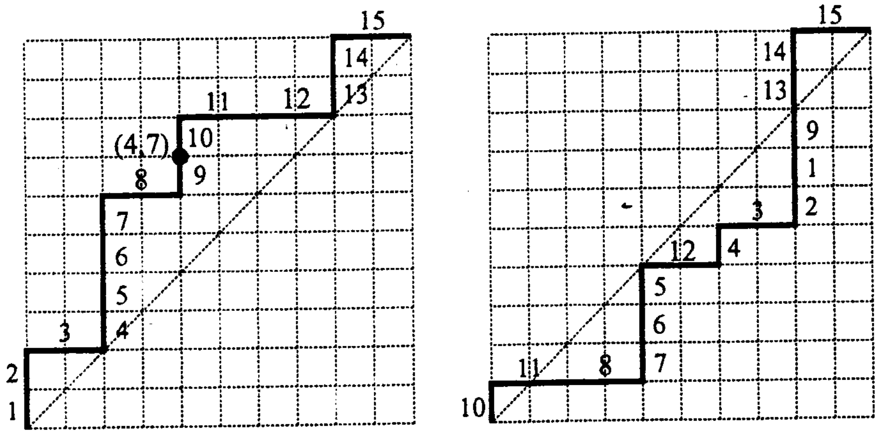

Figure 4: a path $\pi \in \mathcal{T}_5^{(2)}(4,7)$ and the $\phi(\pi) \in \mathcal{T}_5^{(2)*}(7)$

set of east steps $\{(d,0) | d \geq 1\}$. Let $\mathcal{G}_m \subseteq \mathcal{G}$ be the set of excursions from $(0,0)$ to (m,m). For $m \geq n$, let $\mathcal{G}_m^*(n) \subseteq \mathcal{G}$ be the set of bridges with n north steps under the line $y = x$. With a minor modification of the previous argument, an analogous bijection between $\mathcal{G}_m$ and $\mathcal{G}_m(n)$ is established by the same procedure.

To see this, let $\mathcal{G}_m(p,n) \subseteq \mathcal{G}_m$ be the set of paths that contains the north step from $(p, n-1)$ to (p,n). Given a path $\pi \in \mathcal{G}_m(p,n)$, the factorization $\gamma_n^+(\pi) = \mu_k \mathsf{E}_k \mu_{k-1} \mathsf{E}_{k-1} \cdots \mu_1 \mathsf{E}_1 \mu_0$ of the red section of π and the factorization $\gamma_n^-(\pi) = \nu_1 \mathsf{N}_1 \nu_2 \mathsf{N}_2 \cdots \nu_{n-p} \mathsf{N}_{n-p}$ of the blue section of π are defined in the same manner as the previous argument. With the assumption that E_i $(1 \leq i \leq k)$ goes to a line of j_i levels lower than the line μ_i starts from, the above argument works perfectly well to establish a bijection between $\mathcal{G}_m$ and $\mathcal{G}_m(p,n)$. □

6 Lifted Motzkin paths

In this section, we shall show an analogous paths, enumerated by the Motzkin numbers, the generating function of which is expanded along the x-axis and there is a variation of the Chung-Feller theorem regarding the index of the absolute minimum of the paths (see Theorem 6.5), as first indicated in [13] by L. Shapiro.

For convenience, the Motzkin paths are tilted so that the statistics concerned are indexed by the x-coordinates. Let $\mathcal{H}_m$ denote the set of paths from $(0,-1)$ to $(m,0)$ with the *up* U step $(1,1)$, the *down* D step $(1,-1)$, and the *level* L step $(1,0)$ that never pass below the x-axis except for the initial step and let $h_m = |\mathcal{H}_m|$. For example, the paths of $\mathcal{H}_4$ are shown in Figure 5. Members of $\mathcal{H}_m$ are called the *lifted Motzkin paths.* Note that the section from $(1,0)$ to $(m,0)$ of each path in $\mathcal{H}_m$ is a (tilted) Motzkin path of semilength $m-1$. So $\mathcal{H}_m$ is enumerated by the Motzkin numbers $h_m = e_{m-1}$ for $m \geq 1$. Marking a unit length along the x-axis with an x, the generating function H for $\{h_m\}_{m\geq 1}$, given by $H = H(x) = \sum_{m\geq 1} h_m x^m$, satisfies $H = xM = \frac{1-x-\sqrt{1-2x-3x^2}}{2}$.

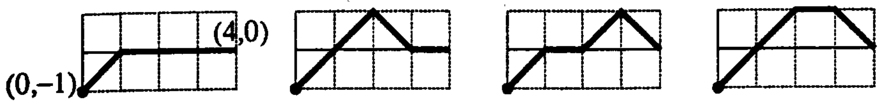

Figure 5: the paths of $\mathcal{H}_4$

Lemma 6.1 *For* $0 \leq q \leq p-1$, *the number of paths from* $(0,-1)$ *to* (p,q) *with the* $(1,1)$, $(1,-1)$ *and* $(1,0)$ *steps that never pass below the* x*-axis except for the initial step is equal to* $[x^p]H^{q+1}$.

Proof: Observe that each path from $(0,-1)$ to (p,q) has a factorization $\mathsf{U}_0\mu_0\mathsf{U}_1\mu_1\cdots\mathsf{U}_q\mu_q$, where U_i $(0 \leq i \leq q)$ is the last up step that rises from line $y = i-1$ to line $y = i$ and μ_i is a Motzkin path (possibly trivial). Hence the generating function for the number of paths from $(0,-1)$ to (p,q) is $x^{q+1}M^{q+1} = H^{q+1}$. Since the projective length of such a path upon the x-axis is p, the assertion follows. ☐

6.1 The Taylor expansion

For $1 \leq n \leq m$, note that a path in $\mathcal{H}_m$ intersects the line $x = n$ at a point (n,i) for some i, $0 \leq i \leq \min\{n, m-n+1\} - 1$. Let $t = \min\{n, m-n+1\}$. According to the intersections with the line $x = n$, the paths in $\mathcal{H}_m$ are partitioned into a disjoint union of subsets $\mathcal{H}_m(n,i)$ for $i = 0,\ldots,t-1$, where

$$\mathcal{H}_m(n,i) = \{\pi \in \mathcal{H}_m | \pi \text{ passes the point } (n,i)\}.$$

For the two-color bisection of the plane $\mathbb{Z}\times\mathbb{Z}$ with respect to the line $x = n$, let $\gamma_n^-(\pi)$ denote the section of π from the beginning to the line $x = n$, called the *blue section*, and let $\gamma_n^+(\pi)$ be the remaining section of π, called the *red section*. Let $\mathcal{H}_m^-(n,i)$ and $\mathcal{H}_m^+(n,i)$ denote the sets of blue and red sections for the paths in $\mathcal{H}_m(n,i)$, respectively.

Lemma 6.2 *Let $t = \min\{n, m-n+1\}$. For $i = 0, \dots, t-1$, we have*

(i) $|\mathcal{H}_m^-(n,i)| = [x^n]H^{i+1}$,

(ii) $|\mathcal{H}_m^+(n,i)| = [x^{m-n}]\{H^i M\}$.

Proof: (i) follows immediately from Lemma 6.1. To prove (ii), note that each red section $\gamma_n^+(\pi) \in \mathcal{H}_m^+(n,i)$ goes from (n,i) to $(m,0)$. If $\gamma_n^+(\pi)$ is attached with a down step from $(m,0)$ to $(m+1,-1)$ and flipped over the line $x = \frac{m+1}{2}$, then it corresponds to a path from $(0,-1)$ to $(m-n+1,i)$ traversed backward and such a correspondence is a bijection. By Lemma 6.1, $|\mathcal{H}_m^+(n,i)| = [x^{m-n+1}]H^{i+1} = [x^{m-n}]\{H^i M\}$. ☐

Theorem 6.3 *The n-th Taylor expansion of H can be expressed in the form*

$$H = \sum_{i=1}^{n-1} h_i x^i + x^n(F_n(H)M),$$

where $F_n(z) = \sum_{i=0}^{t-1} f_{n,i} z^i$, $f_{n,i} = [x^n]H^{i+1}$, and $t = \min\{n, m-n+1\}$ depends on the index m of the Motzkin numbers $\{h_m\}_{m \geq n}$.

Proof: It suffices to show that $h_m = [x^{m-n}]\{F_n(H)M\}$ for $m \geq n$. It is straightforward that

$$\begin{aligned} h_m &= |\mathcal{H}_m| \\ &= \sum_{i=0}^{t-1} |\mathcal{H}_m(n,i)| \\ &= \sum_{i=0}^{t-1} |\mathcal{H}_m^-(n,i)| \cdot |\mathcal{H}_m^+(n,i)| \\ &= \sum_{i=0}^{t-1} ([x^n]H^{i+1})([x^{m-n}]H^i M) \\ &= [x^{m-n}]\{\sum_{i=0}^{t-1} f_{n,i} H^i M\}, \end{aligned}$$

as required. ☐

6.2 A variation of Chung-Feller theorem for Motzkin paths

Consider all paths from $(0,-1)$ and $(m,0)$ with the $(1,1)$, $(1,-1)$ and $(1,0)$ steps. For each path locate the point of absolute minimum, say occurs at the line $x = k$. If there are at least 2 points of the absolute minimum choose the rightmost one. For instance, there are 16 paths from $(0,-1)$ to $(4,0)$. The four paths with $k = 2$ is illustrated below. Note that $\mathcal{H}_m$ is the set of paths with $k = 0$.

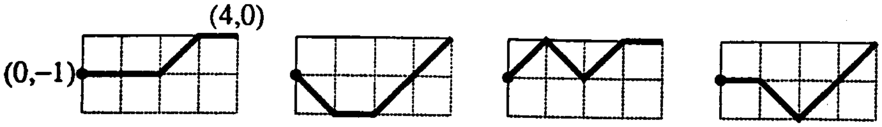

Figure 6: the paths from $(0,-1)$ to $(4,0)$ with absolute minimum at $x = 2$

Theorem 6.4 *The generating function for the number of paths from* $(0,-1)$ *to* $(m,0)$ *with the* $(1,1)$, $(1,-1)$ *and* $(1,0)$ *steps the absolute minimum of which occurs at* $x = m - n$ $(1 \le n \le m-1)$ *coincides with the remainder of the n-th Taylor expansion of H.*

Proof: Let $\mathcal{H}_m^*(m-n)$ denote the set of paths counted above. Note that the absolute minimum of a path in $\mathcal{H}_m^*(m-n)$ is at $(m-n,-j)$ for some j, $1 \le j \le \min\{n, m-n+1\}$. Let $t = min\{n, m-n+1\}$. According to the height of the absolute minimum (along the line $x = m-n$), the paths in $\mathcal{H}_m^*(m-n)$ is partitioned into subsets $\mathcal{H}_m^*(m-n,j)$ for $j = 1,\dots,t$, where

$$\mathcal{H}_m^*(m-n,j) = \{\pi \in \mathcal{H}_m^*(m-n) | \text{ the absolute minimum of } \pi \text{ is at } (m-n,-j)\}.$$

For each path $\pi \in \mathcal{H}_m^*(m-n,j)$, the section $\delta_{m-n}^-(\pi)$ from the beginning to the line $x = m-n$ either is a Motzkin path (for $j = 1$) or has a factorization $\delta_{m-n}^-(\pi) = \mu_0 \mathsf{D}_1 \mu_1 \cdots \mathsf{D}_{j-1}\mu_{j-1}$ (for $j \ge 2$), where D_i $(1 \le i \le j-1)$ is the first down step that goes from line $y = -i$ to line $y = -i-1$ and μ_k $(0 \le k \le j-1)$ is a Motzkin path. So the generating function for the possibilities of $\delta_{m-n}^-(\pi)$ is $x^{j-1}M^j$. On the other hand, since the absolute minimum of π occurs at $x = m-n$, the remaining section $\delta_{m-n}^+(\pi)$ of π begins with an up step and has a factorization $\delta_{m-n}^+(\pi) = \mathsf{U}_j\nu_j \cdots \mathsf{U}_1\nu_1$, where U_i is the last up step that goes from line $y = -i$ to line $y = -i+1$ and ν_i is a Motzkin path for

$i = 1, \ldots, j$. So the generating function for the possibilities of $\delta^{+}_{m-n}(\pi)$ is $x^j M^j$. Hence

$$\begin{aligned}
|\mathcal{H}^{*}_{m}(m-n)| &= \sum_{j=1}^{t} |\mathcal{H}^{*}_{m}(m-n, j)| \\
&= \sum_{j=1}^{t} ([x^{m-n}]x^{j-1}M^{j})([x^{n}]x^{j}M^{j}) \\
&= \sum_{j=1}^{t} ([x^{m-n}]H^{j-1}M)([x^{n}]H^{j}) \\
&= [x^{m-n}]\{\sum_{i=0}^{t-1} f_{n,i}H^{i}M\},
\end{aligned}$$

where $f_{n,i} = [x^n]H^{i+1}$ and the assertion follows. □

By Theorems 6.3 and 6.4, we have the following variation of Chung-Feller theorem for Motzkin paths.

Theorem 6.5 *The number of paths from* $(0, -1)$ *to* $(m, 0)$ *with the* $(1, 1)$, $(1, -1)$ *and* $(1, 0)$ *steps the absolute minimum of which occurs at* $x = n$ $(0 \leq n \leq m-1)$ *is equal to* h_m, *independent of* n.

Inspired by the proofs of Theorem 6.3 and 6.4, a bijection $\rho : \mathcal{H}_m \to \mathcal{H}^{*}_{m}(m-n)$ $(1 \leq n \leq m)$ is established immediately, which carries a path $\pi = \gamma^{-}_{n}(\pi)\gamma^{+}_{n}(\pi) \in \mathcal{H}_m$ into $\rho(\pi) = \gamma^{+}_{n}(\pi)\gamma^{-}_{n}(\pi) \in \mathcal{H}^{*}_{m}(m-n)$. For illustration, the four paths of $\mathcal{H}_4$ shown in Figure 5 are carried into the four paths of $\mathcal{H}^{*}_{4}(2)$ in Figure 6 from left to right accordingly.

7 Some results for Schröder paths with flaws

As shown in Theorems 3.15 and 4.6, there are no Chung-Feller properties for large and small Schröder paths with flaws. In this section, a neat formula for enumerating large Schröder paths with flaws is obtained and some asymptotic results regarding average number of returns of paths are given.

7.1 A fast enumeration of large Schröder paths with flaws

For enumerating large Schröder paths with flaws, we have the following formula which is evaluated simply in terms of $\{r_m\}_{m\geq 0}$ and $\{s_m\}_{m\geq 0}$.

Theorem 7.1 *The number of m-Schröder paths with n flaws is* $r_m - \sum_{k=1}^{n} s_{k-1}r_{m-k}$, *or equivalently* $r_m - \frac{1}{2}\sum_{k=1}^{n} r_{k-1}r_{m-k}$.

To prove this theorem, some interesting bijections between certain sets of paths are established. Recall that $\mathcal{R}_m$ is the set of all m-Schröder paths and $\mathcal{R}_m^D(n) = \bigcup_{j=1}^{n}\mathcal{R}_m^D(j,n)$, where $\mathcal{R}_m^D(j,n) = \{\pi \in \mathcal{R}_m | \mathsf{D}_{(j,n)} \in \pi\}$. Let $A_m(n) \subseteq \mathcal{R}_m$ be the subset of paths that contain at least one diagonal step in the segment from $(0,0)$ to (n,n) of the line $y = x$.

Lemma 7.2 *We have*

(i) $|A_m(n)| = \sum_{k=1}^{n} s_{k-1}r_{m-k}$.

(ii) $|\mathcal{R}_m^D(n)| = [x^{m-n}]\{F_n(R)\}$, *where* $F_n(z) = \sum_{k=1}^{n} f_{n,k}z^k$ *and* $f_{n,k} = [x^{n-k}]R^k$.

Proof: (i) Note that each path in $A_m(n)$ contains at least one member of the set $\{\mathsf{D}_{(k,k)} | k = 1,\ldots,n\}$. For $1 \leq k \leq n$, let $\alpha(k) \subseteq A_m(n)$ be the subset of paths the first diagonal step on the line $y = x$ of which is $\mathsf{D}_{(k,k)}$. Clearly, $\{\alpha(k) | k = 1,\ldots,n\}$ forms a partition of $A_m(n)$. Observe that each path $\pi \in \alpha(k)$ starts with a small $(k-1)$-Schröder path, followed by $\mathsf{D}_{(k,k)}$ and then another large $(m-k)$-Schröder path. Hence $|\alpha(k)| = s_{k-1}r_{m-k}$ and $|A_m(n)| = \sum_{k=1}^{n} |\alpha(k)| = \sum_{k=1}^{n} s_{k-1}r_{m-k}$.

(ii) It follows from Lemma 3.13 and the argument of Theorem 3.15(i). □

That the sizes of $A_m(n)$ and $\mathcal{R}_m^D(n)$ coincide is proved in the following.

Theorem 7.3 *The number of m-Schröder paths that contain at least one diagonal step in the segment from* $(0,0)$ *to* (n,n) *of the line* $y = x$ *is equal to the number of m-Schröder paths that rise from the line* $y = n-1$ *to the line* $y = n$ *by a diagonal step.*

Proof: Let $A_m(n)$ be partitioned into $A_m(k,n)$ for $1 \le k \le n$, where

$$A_m(k,n) = \{\pi \in A_m(n) | \text{ either } \mathsf{N}_{(k,n)} \in \pi \text{ or } \mathsf{D}_{(k,n)} \in \pi\}.$$

We shall establish a bijection ϕ_k between $A_m(k,n)$ and $\mathcal{R}_m^D(k,n)$ for $k = 1, \ldots, n$.

For $k = n$, note that $A_m(n,n)$ and $\mathcal{R}_m^D(n,n)$ are identical. So ϕ_n is the identity map. For $1 \le k \le n-1$ and any path $\pi \in A_m(k,n)$, let D_1 denote the first diagonal step on the line $y = x$ of the blue section $\gamma_n^-(\pi)$ of π. Then $\gamma_n^-(\pi)$ has a factorization $\gamma_n^-(\pi) = \mu \mathsf{D}_1 \nu$. Define a mapping ϕ_k which carries $\pi = \gamma_n^-(\pi)\gamma_n^+(\pi)$ into $\phi_k(\pi) = \nu\mu\mathsf{D}_1\gamma_n^+(\pi)$. Since the blue section $\nu\mu\mathsf{D}_1$ of $\phi_k(\pi)$ reaches (k,n) by a diagonal step, $\phi_k(\pi) \in \mathcal{R}_m^D(k,n)$.

To show the ϕ_k^{-1}, observe that the blue section $\gamma_n^-(\tau)$ of a path $\tau \in \mathcal{R}_m^D(k,n)$ can be factorized as $\gamma_n^-(\tau) = \nu'\mu'\mathsf{D}_{(k,n)}$, where μ' is the section of maximum length remained above the line $y = x + n - k$ (without diagonal steps on the line $y = x + n - k$). Then $\phi_k^{-1}(\tau) = \mu'\mathsf{D}\nu'\gamma_n^+(\tau)$ is the required path. The assertion follows. ☐

For example, consider the case $m = 8$ and $n = 6$. In Figure 7, on the left is a path $\pi \in A_8(4,6)$. The first diagonal step D_1 on the line $y = x$ of π is labelled by 3 and the blue section $\gamma_6^-(\pi)$ (from $(0,0)$ to $(4,6)$) is factorized as $\gamma_6^-(\pi) = \mu\mathsf{D}_1\nu$, where $\mu = (1,2)$ and $\nu = (4,5,6,7,8)$. On the right is the path $\phi(\pi) = \nu\mu\mathsf{D}_1\gamma_6^+(\pi) \in \mathcal{R}_8^D(4,6)$.

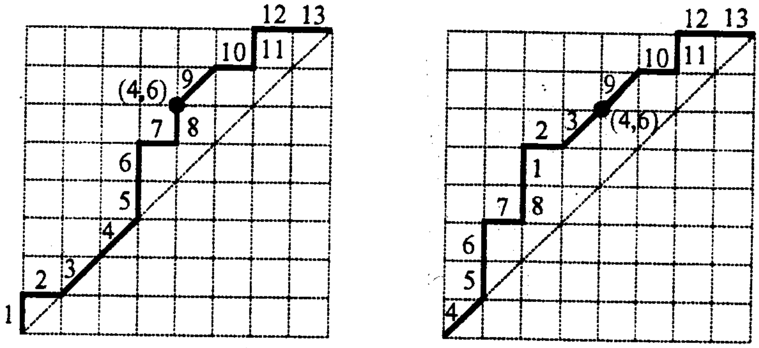

Figure 7: a path $\pi \in A_8(4,6)$ and the $\phi(\pi) \in \mathcal{R}_8^D(4,6)$

Proof of Theorem 7.1: Let $\mathcal{R}_m^*(n)$ be the set of m-Schröder paths with n flaws. By Lemma 7.2 and Theorems 3.15(i), 4.6(i) and 7.3, we have

$$\begin{aligned} |\mathcal{R}_m^*(n)| &= [x^{m-n}]\{F_n(R)R\} && \text{(Theorem 4.6(i))} \\ &= r_m - [x^{m-n}]\{F_n(R)\} && \text{(Theorem 3.15(i))} \end{aligned}$$

$$\begin{aligned} &= r_m - |\mathcal{R}_m^D(n)| && \text{(Lemma 7.2(ii))} \\ &= r_m - |A_m(n)| && \text{(Theorem 7.3)} \\ &= r_m - \sum_{k=1}^{n} s_{k-1} r_{m-k}, && \text{(Lemma 7.2(i))} \end{aligned}$$

where $F_n(z)$ is given in Theorem 3.15(i). This completes the proof of Theorem 7.1. $\square$

By Lemma 7.2(i) and Theorem 7.1, we have the following result. A bijective proof is also given.

Theorem 7.4 *The number of m-Schröder paths with an n-flaw is equal to the number of m-Schröder paths without diagonal steps in the segment from* $(0,0)$ *to* (n,n) *of the line* $y = x$.

7.2 The bijection

We shall give a bijective proof of Theorem 7.4, which is established in two stages.

Let $\mathcal{R}_m^N(n) = \cup_{k=0}^{n-1} \mathcal{R}_m^N(k,n)$, where $\mathcal{R}_m^N(k,n) = \{\pi \in \mathcal{R}_m | \mathsf{N}_{(k,n)} \in \pi\}$ and let $B_m(n) \subseteq \mathcal{R}_m$ be the subset of paths without diagonal steps in the segment from $(0,0)$ to (n,n) of the line $y = x$.

First, we shall establish a bijection $\varphi : B_m(n) \to \mathcal{R}_m^N(n)$. For $0 \le k \le n-1$, let $B_m(k,n) = \{\pi \in B_m(n) |$ either $\mathsf{N}_{(k,n)} \in \pi$ or $\mathsf{D}_{(k,n)} \in \pi\}$. Given a path $\pi \in B_m(k,n)$, the blue section $\gamma_n^-(\pi)$ has a factorization $\gamma_n^-(\pi) = \mu \mathsf{N}_1 \nu$, where N_1 is the last north step that rises from the line $y = x + n - k - 1$ to the line $y = x + n - k$. Define a mapping φ_k which carries $\pi = \gamma_n^-(\pi)\gamma_n^+(\pi)$ into $\varphi_k(\pi) = \nu\mu\mathsf{N}_1\gamma_n^+(\pi)$. Since $\nu\mu\mathsf{N}_1$ is a blue section that reaches (k,n) by a north step, $\varphi_k(\pi) \in \mathcal{R}_m^N(k,n)$.

To show the φ_k^{-1}, observe that the blue section $\gamma_n^-(\tau)$ of a path $\tau \in \mathcal{R}_m^N(k,n)$ can be factorized as $\gamma_n^-(\tau) = \nu'\mu'\mathsf{N}_{(k,n)}$, where μ' is the section of maximum length without diagonal steps on the line $y = x$. Then $\varphi_k^{-1}(\tau) = \mu'\mathsf{N}\nu'\gamma_n^+(\tau)$ is the required path. So the bijection $\varphi : B_m(n) \to \mathcal{R}_m^N(n)$ is established with $\varphi|_{B_m(k,n)} = \varphi_k$ for $k = 0, \ldots, n-1$.

For example, let $m = 10$ and $n = 6$. In Figure 8, on the left is a path $\pi \in B_{10}^*(3,6)$. Note that the point $(3,6)$ is on the line $y = x + 3$ and that the last north step N that rises from line $y = x + 2$ to line $y = x + 3$ is labelled by 5. The blue section $\gamma_6^-(\pi)$ (from

$(0,0)$ to $(3,6)$) has a factorization $\gamma_6^-(\pi) = \mu\mathsf{N}\nu$, where $\mu = (1,2,3,4)$ and $\nu = (6,7,8)$. On the right is the path $\varphi(\pi) = \nu\mu\mathsf{N}\gamma_6^+(\pi) \in \mathcal{R}_{10}^N(3,6)$.

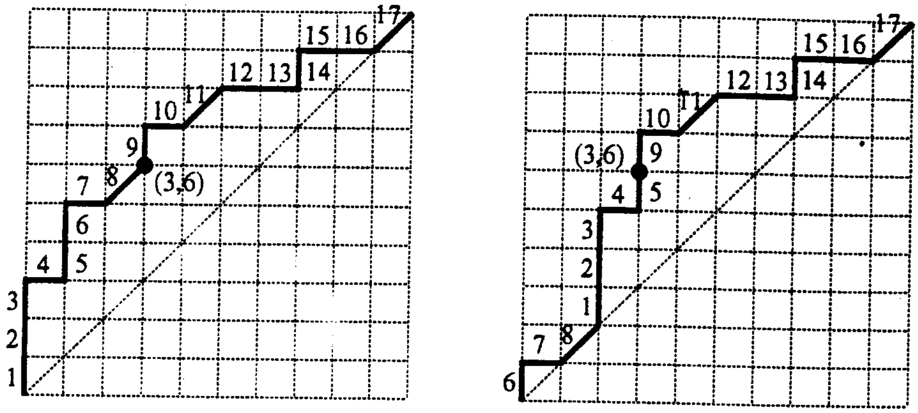

Figure 8: a path $\pi \in B_{10}^*(3,6)$ and $\varphi(\pi) \in \mathcal{R}_{10}^N(3,6)$

Next, we shall establish a bijection $\sigma : \mathcal{R}_m^N(n) \to \mathcal{R}_m^*(n)$. Given a path $\pi \in \mathcal{R}_m^N(k,n)$, the red section $\gamma_n^+(\pi)$ (from (k,n) to (m,m)) has a factorization $\gamma_n^+(\pi) = \mu_{n-k}\mathsf{E}_{n-k}\cdots\mu_1\mathsf{E}_1\mu_0$, where E_i ($n-k \geq i \geq 1$) is the first east step that goes from line $y = x+i$ to line $y = x+i-1$. On the other hand, the blue section $\gamma_n^-(\pi)$ (from $(0,0)$ to (k,n)) has a factorization $\gamma_n^-(\pi) = \nu_1\mathsf{N}_1\cdots\nu_{n-k}\mathsf{N}_{n-k}$, where N_i ($1 \leq i \leq n-k$) is the last north step that rises from line $y = x+i-1$ to line $y = x+i$. In order to establish the bijection, form a new section $\gamma_n^-(\pi)' = \mathsf{N}_0\nu_1\mathsf{N}_1\nu_2\cdots\mathsf{N}_{n-k-1}\nu_{n-k}$ from $\gamma_n^-(\pi)$ (with N_{n-k} moved to the beginning and denoted by N_0). Moreover, let $\widehat{\nu}_i$ be the section of path obtained from ν_i by flipping over the line $y = x$. Define a mapping σ_k which carries $\pi = \gamma_n^-(\pi)\gamma_n^+(\pi)$ into

$$\sigma_k(\pi) = \mu_{n-k}\mathsf{E}_{n-k}\widehat{\nu}_{n-k}\mathsf{N}_{n-k-1}\cdots\mu_1\mathsf{E}_1\widehat{\nu}_1\mathsf{N}_0\mu_0.$$

Note that the blue section $\gamma_n^-(\pi)$ of π are partitioned into the blocks $\mathsf{E}_{n-k}\widehat{\nu}_{n-k}\mathsf{N}_{n-k-1}$, $\ldots$, $\mathsf{E}_1\widehat{\nu}_1\mathsf{N}_0$ (under the line $y = x$) of $\sigma_k(\pi)$. Hence $\sigma_k(\pi) \in \mathcal{R}_m^*(n)$ and a mapping $\sigma : \mathcal{R}_m^N(n) \to \mathcal{R}_m^*(n)$ is established with $\sigma|_{\mathcal{R}_m^N(k,n)} = \sigma_k$ for $k = 0,\ldots,n-1$. In order to show the σ^{-1}, we only need to reverse the procedure. With the composition $\sigma \circ \varphi$, a bijection between $B_m(n)$ and $\mathcal{R}_m^*(n)$ is established.

As an example, for the path π on the left of Figure 9, the red section $\gamma_6^+(\pi)$ (from $(3,6)$ to $(10,10)$) has a factorization $\gamma_6^+(\pi) = \mu_3\mathsf{E}_3\mu_2\mathsf{E}_2\mu_1\mathsf{E}_1\mu_0$, where $\mathsf{E}_3 = 12$, $\mathsf{E}_2 = 13$ and $\mathsf{E}_1 = 16$, and $\mu_3 = (9,10,11)$, μ_2 is trivial, $\mu_1 = (14,15)$ and $\mu_0 = (17)$. On the other hand, the blue section $\gamma_6^-(\pi)$ (from (0,0) to (3,6)) has a factorization $\gamma_6^-(\pi) = \nu_1\mathsf{N}_1\nu_2\mathsf{N}_2\nu_3\mathsf{N}_3$, where $\mathsf{N}_1 = 1$, $\mathsf{N}_2 = 2$ and $\mathsf{N}_3 = 5$, $\nu_1 = (6,7,8)$, ν_2 is trivial and $\nu_3 = (3,4)$. For $\gamma_6^-(\pi)' = \mathsf{N}_0\nu_1\mathsf{N}_1\nu_2\mathsf{N}_2\nu_3$, $\mathsf{N}_0 = \mathsf{N}_3 = 5$. On the right is the path $\sigma(\pi) = \mu_3\mathsf{E}_3\widehat{\nu}_3\mathsf{N}_2\mu_2\mathsf{E}_2\widehat{\nu}_2\mathsf{N}_1\mu_1\mathsf{E}_1\widehat{\nu}_1\mathsf{N}_0\mu_0 \in \mathcal{R}^*_{10}(6)$.

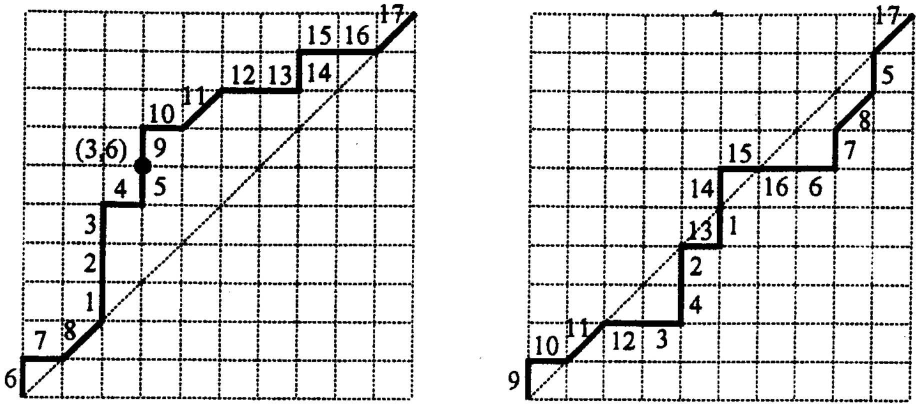

Figure 9: a path $\pi \in \mathcal{R}^N_{10}(3,6)$ and the $\sigma(\pi) \in \mathcal{R}^*_{10}(6)$

7.3 Statistic numbers of returns

By Proposition 3.11, the number of large n-Schröder paths with at least one diagonal step on the line $y = x$ and the number of small n-Schröder paths share the same generating function S. For the former case, such a path must either be empty, or start with a small Schröder path, followed by a diagonal step on the line $y = x$ and then another large Schröder path. Hence S satisfies the functional equation

$$S = 1 + xRS. \tag{7}$$

Moreover, by $R = 2S - 1$, we have

$$S = R - xRS. \tag{8}$$

In the following, we consider the statistic *number of returns* of large and small Schröder paths by a similar argument used in [6, Section 7]. A *return* (to the line $y = x$) of a path occurs whenever there is an east step that goes from line $y = x + 1$ to line $y = x$. Define $X_n^S = \{(P, \pi) | P$ is a point on the line $y = x$ and $\pi \in \mathcal{S}_n$ passes $P\}$. Counting the set X_n^S, note that π has a factorization $\pi = \mu_1\mathsf{N}\nu\mathsf{E}\mu_2$, where E reaches P, ν is a large Schröder path and μ_1, μ_2 are small Schröder paths (possibly trivial). So the generating function for the total number of returns of all small n-Schröder paths is

$$S(xR)S = S(S-1) = S^2 - S = \frac{S + xS - 1}{2x} - S,$$

where the identity $2xS^2 = S + xS - 1$ is obtained by substituting R with $2S-1$ in (7). Since $|X_n^S| = [x^n]\{\frac{S+xS-1}{2x} - S\} = \frac{1}{2}(s_{n+1} - s_n)$, we have the following result.

Proposition 7.5 *The total number of returns of all small n-Schröder paths is* $\frac{1}{2}(s_{n+1} - s_n)$.

For the case of large Schröder paths, count the set X_n^R of ordered pairs (P, π), where P is a point on the line $y = x$ and $\pi \in \mathcal{R}_n$ has a factorization $\pi = \mu_1 \mathsf{N} \mu_2 \mathsf{E} \mu_3$ such that E reaches P and μ_1, μ_2, μ_3 are large Schröder paths (possibly trivial). So the generating function for the total number of returns of all large n-Schröder paths is

$$R(xR)R = R(R-1-xR) = R^2 - R - xR^2 = \frac{R - xR - 1}{x} - R - (R - xR - 1),$$

where the identity $xR^2 = R - xR - 1$ is obtained by substituting S with $\frac{1}{2}(R+1)$ in (7). Since $|X_n^R| = [x^n]\{\frac{R-xR-1}{x} - R - (R - xR - 1)\} = r_{n+1} - 3r_n + r_{n-1}$, we have the following result.

Proposition 7.6 *The total number of returns of all large n-Schröder paths is* $r_{n+1} - 3r_n + r_{n-1}$.

As a generalization, an *extended return* of a large Schröder path is either a diagonal step on the line $y = x$ or an east step that goes from line $y = x + 1$ to line $y = x$. For example, the path shown in Figure 10 has 4 extended returns.

Let $X_n^{\hat{R}}$ be the set of ordered pairs (P, π), where P is a point on the line $y = x$ and $\pi \in \mathcal{R}_n$ passes P. Since such a path π is factorized either of the form $\pi = \mu_1 \mathsf{D} \mu_2$ or $\pi = \mu_1 \mathsf{N} \mu_2 \mathsf{E} \mu_3$, where P is the end point of D or E and μ_1, μ_2, μ_3 are large Schröder paths (possibly trivial), the generating function for the total number of extended returns of all large n-Schröder paths is

$$R(x + xR)R = R^2 - R = \frac{R - xR - 1}{x} - R.$$

Hence we have the following result.

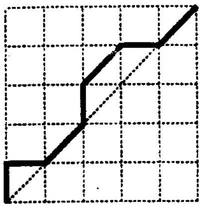

Figure 10: a Schröder path with 2 returns and 4 extended returns

Proposition 7.7 *The total number of extended returns of all large n-Schröder paths is* $r_{n+1} - 2r_n$.

The average behavior of the statistics $|X_n^S|$, $|X_n^R|$ and $|X_n^{\hat{R}}|$ can be estimated by a result of Flajolet and Sedgewick [10, Theorem 4.4]. From (6), note that R and S have the same dominant singular point $3-2\sqrt{2}$ and the asymptotic growing rate of the large (or small) Schröder numbers is

$$\lim_{n\to\infty}\frac{r_{n+1}}{r_n}=\lim_{n\to\infty}\frac{s_{n+1}}{s_n}=\frac{1}{3-2\sqrt{2}}=3+2\sqrt{2}.$$

To find the expected number of returns, divide the total number of returns by the total number of paths. It follows that

$$\begin{aligned}
E(|X_n^S|) &= \frac{s_{n+1}-s_n}{2s_n}\to\frac{(3+2\sqrt{2})-1}{2}=1+\sqrt{2},\\
E(|X_n^R|) &= \frac{r_{n+1}-3r_n+r_{n-1}}{r_n}\to(3+2\sqrt{2})-3+(3-2\sqrt{2})=3,\\
E(|X_n^{\hat{R}}|) &= \frac{r_{n+1}-2r_n}{r_n}\to(3+2\sqrt{2})-2=1+2\sqrt{2}.
\end{aligned}$$

In [6, Section 7], Deutsch and Shapiro obtained that the expected number of returns of a Catalan path of semilength n is $\frac{3n}{n+1}$, which approaches to 3 as n gets large. This result goes back to Dershowitz and Zaks [5] who dealt with plane trees. Note that a Catalan path is simply a small Schröder path without any diagonal steps at all. The average behavior of the various statistic number of returns of paths with/without diagonal steps are summarized as follows.

Theorem 7.8 *The following asymptotic results hold.*

(i) *The expected numbers of returns of a Catalan path and a small Schröder path approach to* 3 *and* $1+\sqrt{2}$, *respectively.*

(ii) *The expected numbers of returns and extended returns of a large Schröder path approach to* 3 *and* $1+2\sqrt{2}$, *respectively.*

8 Conclusion

Based on various partitions of the plane $\mathbb{Z}\times\mathbb{Z}$, we introduce the concept of combinatorial expansions of generating functions for lattice paths, and cut short our discussion to a two-color bisection of the plane with respect to a line. In the illustrative examples, we restrict attention to the Taylor expansions and flaw expansions for some familiar paths. However, the expansions do not have to be confined to the two-coloring or the bisections of the plane with respect to a line. It admits generalizations of the expansion to multi-coloring of the plane with respect to parallel lines or curves (e.g., elliptic or hyperbolic, etc.).

The cut-and-paste technique is helpful in establishing bijective relations between excursions and bridges. In similar fashion, corresponding results can be found for other combinatorial structures, e.g., trees, pattern-avoid permutations, non-crossing partitions, tableaux, etc. For example, in the two-color modelling, the Taylor expansions for plane trees can be interpreted as an assignment to the alternative color afterward when the tree is searched to a given number of marked elements (e.g., edges or vertices, etc.) and the flaw expansions for plane trees can be simply interpreted as a two-coloring of the branches of the root. Moreover, the bijections between excursions and bridges not only induce identical distributions of certain pair of statistics for excursions and bridges but also provide evidence to the identical distributions of the corresponding pair of statistics in other combinatorial structures.

As a striking example, the Chung-Feller phenomena for lattice paths can be explained in terms of expansions. Even for classes of paths (or other combinatorial structures) without Chung-Feller property, we can make adjusted version of important statistics equal (by assigning appropriate weights as shown in the remark after Theorem 4.6). This slightly artificial maneuver has the effect of bringing together the Taylor expansions and flaw expansions.

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