k-Trees and Catalan Identities

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Abstract

In this paper we use k-trees to obtain generating function identities involving generalizations of Catalan Numbers, Central Binomial Numbers, and Fine Numbers. We give examples to show possible applications of these identities and show that the ratio of generalized Fine numbers to Catalan numbers is asymptotic to $\frac{2k}{(k+1)^2}$.

Keywords: Catalan, central binomial, and Fine numbers; k-trees; generating function identities; odd degree; odd outdegree

1 Introduction

In their recent paper Deutsch and Shapiro [1] listed functional identities involving the generating functions $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ of the Catalan numbers, $B(z) = \frac{1}{\sqrt{1-4z}}$ of the central binomial numbers, and $F(z) = \frac{1}{z}\frac{1-\sqrt{1-4z}}{3-\sqrt{1-4z}}$ of Fine numbers. In this paper we obtain similar identities involving the generating function $C = C(z)$ of k-trees [3], $B = B(z)$ of the analog of central binomial numbers, and $F = F(z)$ of generalized Fine numbers.

A k-tree is constructed from a single distinguished k-cycle by repeatedly gluing other k-cycles to existing ones along an edge. More than one cycle can be glued to a non-terminal or internal edge. For example, we obtain the three 3-trees shown in Figure 1 using two 3-cycles and the twelve 3-trees shown in Figure 2 using three 3-cycles.

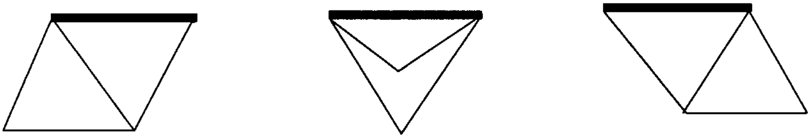

Figure 1: The three 3-trees consisting of two 3-cycles.

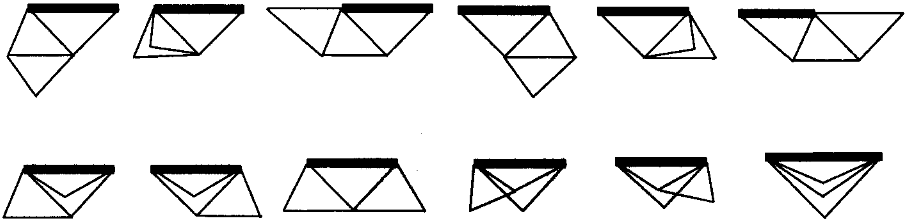

Figure 2: The twelve 3-trees on three cycles.

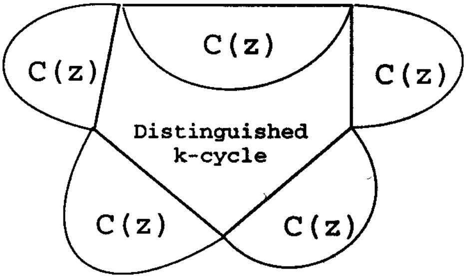

Figure 3: Recursive construction of *k*-trees.

Remark: *k*-trees generalize ordered trees (rooted plane trees) in the sense that ordered trees are 2-trees in which edges between nodes are drawn as digons or double edges.

1.1 Generalized Catalan Numbers

Let the generating function of *k*-trees be

$$C(z) = \sum_{n=0}^{\infty} C_{n,k} z^n$$

where

$C_{n,k}$ = the number of k-trees with exactly n k-cycles.

If we begin with a distinguished k-cycle and construct an ordered k-tree recursively by attaching another k-cycle to one of the k edges of the distinguished k-cycle as shown in Figure 3, we obtain a functional relation

$$C(z) = 1 + zC^k(z) = \frac{1}{1 - zC^{k-1}(z)}$$

where 1 counts the empty tree consisting of only the distinguished edge.

Rewriting

$$C(z) = 1 + zC^k(z)$$

as

$$u(z) = z(u(z)+1)^k, \text{ where } u(z) = C(z) - 1,$$

and applying Lagrange Inversion Formula [7] one obtains:

$$[z^n]C^s(z) = \frac{s}{kn+s}\binom{kn+s}{n}.$$

If we let $s = 1$, we obtain sequences of numbers given by

$$C_{n,k} = \frac{1}{kn+1}\binom{kn+1}{n} = \frac{1}{(k-1)n+1}\binom{kn}{n}$$

and these sequences of numbers count the number of *h*omogeneous ordered k-trees consisting of n k-cycles, which we refer to as the generalized Catalan numbers. It is shown in [4] that $C_{n,k}$ also counts the number of lattice paths from $(0,0)$ to $(n,(k-1)n)$ with unit east and north steps not crossing the line $y = (k-1)x$. This fact will be used later in this paper.

1.2 Analog of Central Binomial Numbers

Let

$$B(z) = \sum_{n=0}^{\infty} B_{n,k} z^n$$

where

$B_{n,k}$ = the number of k-trees with n k-cycles in which exactly one edge in the whole tree is colored red.

A functional relation

$$B(z) = 1 + kzB(z)C^{k-1}(z)$$

can be obtained recursively as shown below.

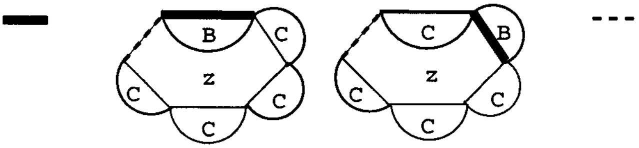

Figure 4: Recursive construction of B(z)

It is very easy to see that $B_{n,k}$ is the total number of edges in k-trees with n k-cycles and thus $B_{n,k} = \binom{kn}{n}$.

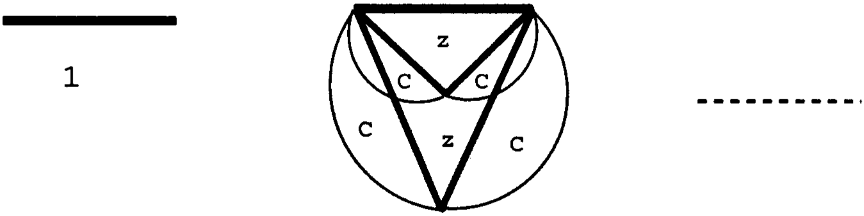

Figure 5: Recursive construction of Fine trees.

1.3 Generalized Fine Numbers

The Fine numbers F_n are the the coefficients of z^n in the Maclaurin series of

$$F(z) = \frac{1-\sqrt{1-4z}}{z(3-\sqrt{1-4z})}.$$

The first few terms of the Fine numbers are $1, 0, 1, 2, 6, 18, 57, \cdots$. The Fine numbers have many combinatorial interpretations [2] and they enumerate, among other things, the number of ordered trees with n edges that have even degree at the root. Using this analogy with k-trees, let $F_{n,k}$ be the number of k-trees on n k-cycles with even number of cycles glued to the distinguished edge. A functional relation for its generating function

$$F(z) = \sum_{n=0}^{\infty} F_{n,k} z^n$$

is obtained recursively as shown in Figure 5.

$$\begin{aligned} F(z) &= 1 + z^2 C^{2(k-1)} + z^4 C^{4(k-1)} + z^6 C^{6(k-1)} + \cdots \\ &= \frac{1}{1 - z^2 C^{2(k-1)}} = \frac{C}{1 + zC^{k-1}} \end{aligned}$$

2 Identities Involving C(z), B(z) and F(z)

Using the functional relations,

(1) $C = 1 + zC^k = \frac{1}{1-zC^{k-1}}$

(2) $B = 1 + kzBC^{k-1}$

(3) $F = \frac{1}{1-z^2C^{2(k-1)}}$

(4) $F = \frac{C}{1+zC^{k-1}}$; $C = \frac{F}{1-zFC^{k-2}}$; $C - F = zFC^{k-1}$

and simple algebraic manipulations, we obtain the following additional generating function identities:

$$(5) \quad B = \frac{C}{1-(k-1)zC^k} = \frac{C}{k-(k-1)C}$$

$$(6) \quad C^2 = \frac{1}{1-2zC^{k-2}}\left(1+z^2C^{2k}\right)$$

$$(7) \quad C' = BC^k$$

$$(8) \quad (zC)' = C(1+zBC^{k-1})$$

$$= B(2C-B)$$

$$(9) \quad \frac{B}{C} = 1+(k-1)zBC^{k-1}$$

$$(10) \quad kB = C(1+(k-1)B)$$

$$(11) \quad 1+C = (zC^{k-2}+2)F$$

$$(12) \quad z(zC^{k-2}+2)C^{k-2}F^2$$

$$-(1+2zC^{k-2})F+1=0$$

$$(13) \quad BC = (k+1)zFBC^{k-1}+F$$

$$(14) \quad F+kBC = (k+1)BF$$

$$(15) \quad B' = kB^3C^{k-2}$$

$$(16) \quad F' = 2zBC^{2k-3}F^2$$

For example, identity (7) is obtained by taking the derivative of (1) and rewriting identity (2) as $B = \frac{1}{1-kzC^{k-1}}$.

$$C' = C^k + zkC^{k-1}C'$$

$$\Leftrightarrow \quad C' = \frac{1}{1-kzC^{k-1}}C^k$$

$$\Leftrightarrow \quad C' = BC^k.$$

Below we give supplementary results that will be helpful in the application of the identities listed above.

Lemma 1.

$$[z^n]C^s(z)B(z) = \binom{kn+s}{n}$$

proof: We know that the total number of paths from $(0,0)$ to (a,b) is $\binom{a+b}{a}$. To verify this lemma, consider lattice paths from $(0,0)$ to $(n,(k-1)n+s)$ in the plane. Given any such path we can split it into two subpaths at the point where the original path visits the line $y=(k-1)x$ for the last time as shown below.

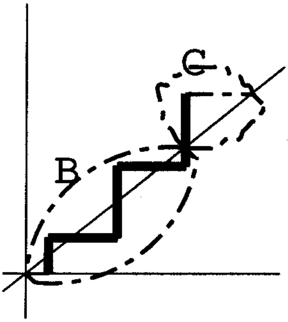

Figure 6: An example of partition of a path into B(z) and copies of C(z)

The first part is counted by B(z) [6] and the second part is counted by $C(z)$ [4]. The path that is counted by $C(z)$ can be further subdivided up to s Catalan paths. Hence the generating function of the number of lattice paths from $(0,0)$ to $(n,(k-1)n+s)$ is $B(z)C^s(z)$ and

$$[z^n](B(z)C^s(z)) = \binom{kn+s}{n}_\square$$

Lemma 2.

$$[z^n]C^s(z)F(z) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{s+2(k-1)i}{nk+s-2i} \binom{nk+s-2i}{n-2i}$$

proof:

$$\begin{aligned} [z^n]C^s(z)F(z) &= [z^n]\left(C^s(z)\frac{1}{1-z^2C^{2(k-1)}}\right) \\ &= [z^n]\sum_{i=0}^{\infty} z^{2i}C^{s+2(k-1)i} = [z^{n-2i}]\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} C^{s+2(k-1)i}(z) \end{aligned}$$

and the lemma follows from the result in section 1.1$\square$

3 Applications

3.1 Enumerating Edges of Odd Degree

Let $O_{n,k}$ be the total number of edges of odd degree among all k-trees with n k-cycles, and let $O(z)$ be the corresponding generating function, i.e.,

$$O(z) = \sum_{n=0}^{\infty} O_{n,k} z^n.$$

Observation: All non-distinguished edges of odd degree in k-trees have even outdegree.

- The generating function for the total number of edges of odd degree at level $l = 0$ among all k-trees is

$$\begin{aligned} & zC^{k-1}(z) + z^3 C^{3(k-1)}(z) + \cdots \\ = & \frac{zC^{k-1}(z)}{1 - z^2 C^{2(k-1)}} \\ = & zC^{k-1}(z)F(z) \end{aligned}$$

- The generating function for the total number of non-distinguished edges of odd degree at level 5 among all k-trees is obtained recursively as shown in Figure 7. In addition to the five Cs shown in the figure, we can attach two Cs at each of the five non-terminal edges of the unique path. Hence, the generating function for the total number of non-distinguished edges of odd degree at level 5 among all k-trees is $2^5 z^5 C^{15}(z)F(z)$. In general, the generating function for the total number of non-distinguished edges of odd degree at level $l \geq 1$ among all k-trees is

$$(k-1)^l z^l C^{kl}(z)F(z).$$

Hence the generating function for the total number of non-distinguished edges of odd degree among all k-trees is

$$\begin{aligned} & \sum_{l=1}^{\infty} (k-1)^l z^l C^{kl}(z)F(z) \\ = & \frac{(k-1)zC^k(z)}{1-(k-1)zC^k(z)} F(z) \\ = & (k-1)C^{k-1}(z)B(z)F(z). \end{aligned}$$

Combining these two cases we obtain:

$$\begin{aligned} O(z) &= zC^{k-1}(z)F(z) + (k-1)C^{k-1}(z)B(z)F(z) \\ &= zC^{k-1}(z)F(z)(1+(k-1)B(z)) \\ &= kzC^{k-2}(z)B(z)F(z) \\ &= \frac{k}{k+1} zC^{k-2}(z)F(z) + \frac{k^2}{k+1} zB(z)C^{k-1}(z) \end{aligned}$$

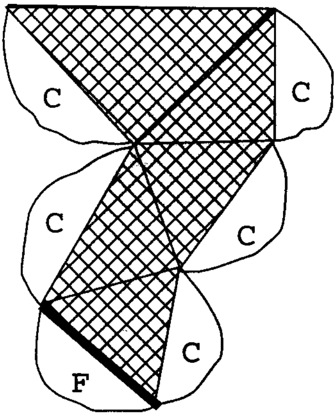

Figure 7: The unique path of 3-cycles to a terminal cycle at level 5.

Therefore, one obtains the total number $O_{n,k}$ of edges of odd degree among all k-trees with n k-cycles very easily using lemma 1 and lemma 2 to be:

$$O_{n,k} = [z^{n-1}] \left(\frac{k}{k+1} C^{k-2}(z) F(z) + \frac{k^2}{k+1} B(z) C^{k-1}(z) \right).$$

For example, the total number of edges of odd degree among the twelve 3-trees consisting of three 3-cycles in Figure 2 is:

$$O_{3,3} = \frac{3}{4}\left(\frac{1}{7}\binom{7}{2} + \frac{5}{5}\binom{5}{0}\right) + \frac{9}{4}\binom{8}{2} = 66$$

3.2 Enumerating Edges of Odd Outdegree

Let $U_{n,k}$ be the total number of edges of odd outdegree among all k-trees with n k-cycles and let

$$U(z) = \sum_{n=0}^{\infty} U_{n,k} z^n$$

be the corresponding generating function.

It can be seen very easily that the total number of edges of outdegree 1 among all k-trees with n k-cycles is

$$n[z^n] \left(z C^{k-1}(z) \right)$$

Formal Power Series Fact: If the generating function of a sequence $\{a_n\}_{n \geq 0}$ is $g(z)$, then the generating function of the sequence $\{n a_n\}_{n \geq 0}$ is $z g'(z)$.

Hence:

$$\begin{aligned} & n[z^n]\left(zC^{k-1}(z)\right) \\ = \; & [z^n]\left(z\left(zC^{k-1}(z)\right)'\right) \end{aligned}$$

Similarly, it can be shown that the total number of edges of outdegree $2m+1$ ($m \geq 0$) among all k-trees with n k-cycles is

$$\begin{aligned} & \frac{n}{2m+1}[z^n]\left(z^{2m+1}C^{(2m+1)(k-1)}(z)\right) \\ = \; & [z^n]\left(z\frac{\left(z^{2m+1}C^{(2m+1)(k-1)}(z)\right)'}{2m+1}\right) \end{aligned}$$

Taking derivative and applying identities (5) and (8) we arrive at

$$[z^n]\left(z^{2m+1}B(z)C^{(2m+1)k-(2m+2)}(z)\right)$$

Therefore,

$$\begin{aligned} U(z) &= zB(z)C^{k-2}(z) + z^3B(z)C^{3k-4}(z) + \cdots \\ &= zB(z)C^{k-2}(z)\left(1 + z^2C^{2k-2}(z) + \cdots\right) \\ &= zB(z)C^{k-2}(z)\left(\frac{1}{1 - z^2C^{2(k-1)}(z)}\right) \\ &= zB(z)C^{k-2}(z)F(z) \\ &= \frac{1}{k+1}zC^{k-2}(z)F(z) + \frac{k}{k+1}zB(z)C^{k-1}(z) \end{aligned}$$

4 Asymptotic Result

Deutsch and Shapiro showed in [2] that

$$\lim_{n\to\infty}\frac{F_n}{C_n} = \frac{4}{9}.$$

This follows from the identity

$$1 + C = (2+z)F \text{ or } C_n = 2F_n + F_{n-1}.$$

In our case, identity (11) has extra term for $k \geq 3$ that prevents direct application of the Deutsch/Shapiro technique to obtain the expected asymptotic result for $\frac{F_{n,k}}{C_{n,k}}$ as $n \to \infty$. However, we obtain the asymptotic result using the expression

$$F_{n,k} = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \frac{2(k-1)i}{nk-2i}\binom{nk-2i}{n-2i}$$

which follows from Lemma 2 by letting $s = 0$ and

$$C_{n,k} = \frac{1}{(k-1)n+1}\binom{nk}{n}$$

Now,

$$\lim_{n\to\infty}\frac{F_{n,k}}{C_{n,k}} = \lim_{n\to\infty}\left[\frac{1}{C_{n,k}}\frac{2(k-1)}{nk-2}\binom{nk-2}{n-2} + \frac{1}{C_{n,k}}\frac{4(k-1)}{nk-4}\binom{nk-4}{n-4} + \cdots\right]$$

and it can easily be shown that

$$\lim_{n\to\infty}\frac{1}{C_{n,k}}\frac{2i(k-1)}{nk-2i}\binom{nk-2i}{n-2i} = \frac{2i(k-1)^2}{k^{2i+1}}.$$

Hence

$$\begin{aligned}\lim_{n\to\infty}\frac{F_{n,k}}{C_{n,k}} &= \sum_{i=1}^{\infty}\frac{2i(k-1)^2}{k^{2i+1}}\\ &= \frac{2(k-1)^2}{k^3}\sum_{i=1}^{\infty}\frac{i}{k^{2i-2}}\\ &= \frac{2(k-1)^2}{k^3}\frac{k^4}{(k^2-1)^2}\\ &= \frac{2k}{(k+1)^2}.\end{aligned}$$

Thus we have the following result.
Theorem 3.

$$\lim_{n\to\infty}\frac{F_{n,k}}{C_{n,k}} = \frac{2k}{(k+1)^2} \textit{ for } k \geq 2.$$

5 Remarks

1. For a fixed k one can also use Zeilberger's creative telescoping algorithm [5, 8] to obtain a recurrence relation for $F_{n,k}$ that leads to a quick evaluation of the limit. For example, if $k = 3$

$$F_{n,3} = \sum_{i=0}^{\lfloor\frac{n}{2}\rfloor}\frac{4i}{3n-2i}\binom{3n-2i}{n-2i}$$

   satisfies the recurrence relation

$$F_{n,3} + 4F_{n+1,3} = \frac{(7n^2+10n+3)(3n)}{(2n+3)(n+1)^2}\frac{1}{2n+1}\binom{3n}{n}.$$

   Using this relation and the fact that

$$\lim_{n\to\infty}\frac{C_{n+1,k}}{C_{n,k}} = \frac{k^k}{(k-1)^{k-1}}$$

we obtain

$$\lim_{n\to\infty} \frac{F_{n,3}}{C_{n,3}} = \frac{3}{8}$$

confirming the above result for $k = 3$.

2. We showed in section 3 that the number of edges of odd degree among all k-trees is a multiple of edges of odd outdegree, specifically, $O_{n,k} = kU_{n,k}$. It is interesting to obtain a direct combinatorial 1-to-k mapping from edges of odd degree to edges of odd outdegree among all k-trees.
3. We were able to obtain analogs to sixteen of the seventeen identities listed in [1] with the exception of

$$\frac{1}{1-4z} = BC + z(BC)^2.$$

An attempt to generalize this particular identity using k-trees, the analogs of central binomial and Fine numbers introduced in this paper leads to a complicated equation. We leave it to the reader as an open problem to obtain a concise analog using suitable generalizations of Catalan numbers, central binomial numbers, and Fine numbers.

Acknowledgements

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