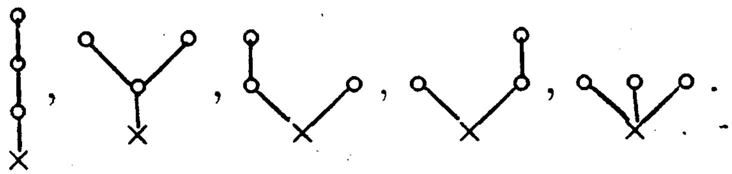
Trees. The counting of rooted plane trees is the Catalan sequence,

1, 1, 2, 5, 14, 42, ...



There are as many internal vertices as the leaves.(10)

Among the internal vertices, there are as many with multiple outdegrees as single outdegree(excluding those at the roots).(4)

There are as many trees with single outdegree at the root as outdegree 2.

Partition a tree by the leftmost branch at the root, there are subtrees above the branch and to the right of the branch,

$c_n = \sum_{k=0} c_k c_{n-1-k}$.

$C = C(x) = \sum c_i x^i = 1 + xC^2 = \frac{1-\sqrt{1-4x}}{2x}$

$= 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + 429x^7 + 1430x^8 + O\left(x^9\right)$

the ordinary generating function of the Catalan sequence.

Partition the trees by the number of branches at the root, we have the following Catalan triangle:

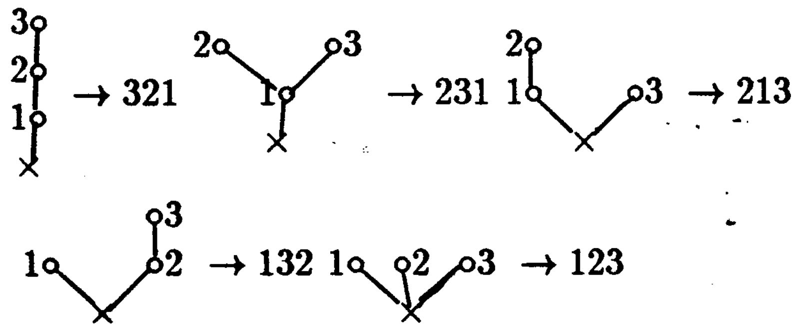
$$\begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ 42 & 42 & 28 & 14 & 5 & 1 \end{pmatrix}$$

$c_{n,k} = \sum_{j=k-1} c_{n-1,j}$

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ 42 & 42 & 28 & 14 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 9 \\ 28 \\ 90 \\ 297 \end{pmatrix}$$

$\sum_{k=1}^{n} c_k c_{n-k}$.The total number of branches at the root is $c_{n+1} - c_n$.

Label vertices of a tree in preorder, read permutation in postorder.



avoid (312)

Permutations avoid (123). Contain no increasing sequence of length 3. Partition by the position k of the largest number n.

312, 321, 132, 231, 213

$$L = \begin{pmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 5 & 5 & 3 & 1 \\ 14 & 14 & 9 & 4 & 1 \\ 42 & 42 & 28 & 14 & 5 & 1 \end{pmatrix}$$

Scheme of inserting next largest number. By induction

$P \in A(n-1, j)$ for $j > k-1$, insert n into the k^{th} position.

$P \in A(n-1, k-1)$, insert n into the k^{th} position and move $(n-1)$ to the first position.

5 4 3 8 2 7 6 1

Inserting 9 in front of 8: 5 4 9 3 8 2 7 6 1

Inserting 9 after 8: 5 4 3 8 9 2 7 6 1

8 5 4 3 9 2 7 6 1

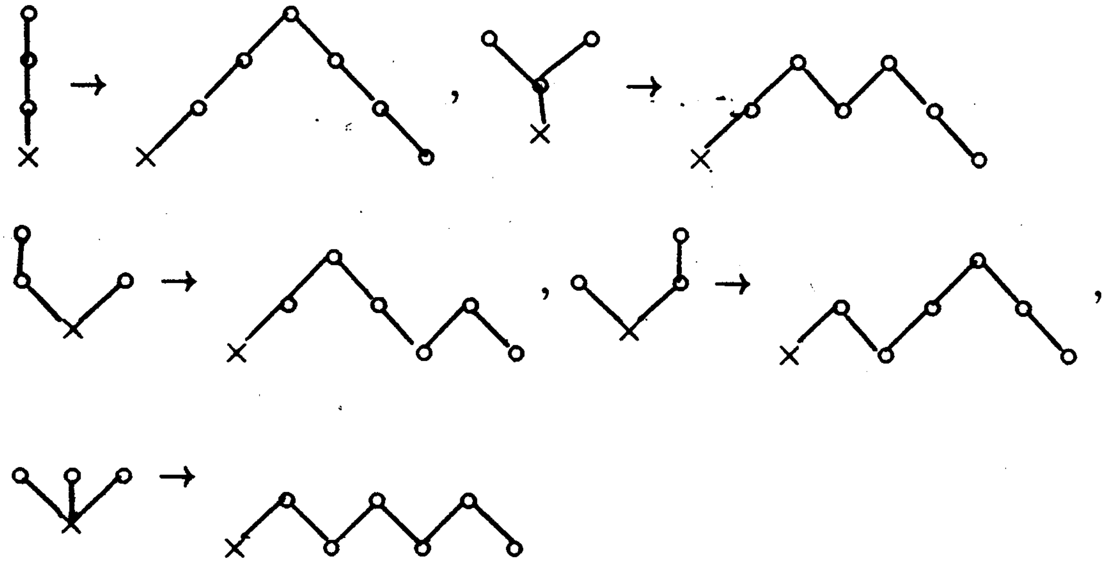
$c_{n,k} = \sum_{j=k-1} c_{n-1,j}$. For $k = 1$, $c_{n,1} = \sum_{j=1} c_{n-1,j}$.

Catalan partitions: Let P $\in$avoid(123) , let P_1 be set of all numbers before n together with n and P_2 be the set of number before next largest number including that number. $\{P_i\}$ form a partition of $\{1, 2, 3, .., , n\}$ and it has the property that $P_i^* > P_{i+1}^*$, where P_i^* is the subset of P_i with the largest element removed.

Example,

8 5 ⑨ 4 ⑦ 3 1 ⑥ ②, 9 8 5 , 7 4 , 6 3 1 , 2

Lattice Paths. Traverse the trees by going up on the left and coming down ~~from~~ on the right of the branch, we have Dyck paths with steps $U = (1,1)$ and $D = (1,-1)$,

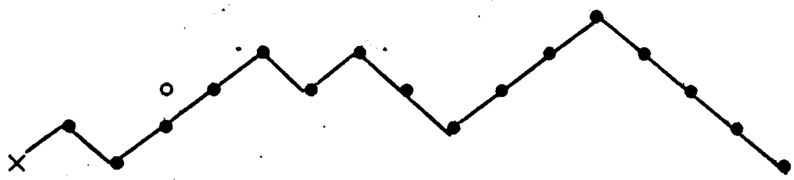


$$(c_{i,j}) = \begin{pmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 1&0&1&0&0&0\\ 0&2&0&1&0&0\\ 2&0&3&0&1&0\\ 0&5&0&4&0&1 \end{pmatrix} \begin{pmatrix} 0&1&&&&\\ 1&0&1&&&\\ &1&0&1&&\\ &&1&0&1&\\ &&&1&0&1\\ &&&&1&0 \end{pmatrix} = \begin{pmatrix} 0&1&&&&\\ 1&0&1&&&\\ 0&2&0&1&&\\ 2&0&3&0&1&\\ 0&5&0&4&0&1\\ 5&0&9&0&5&0 \end{pmatrix}$$

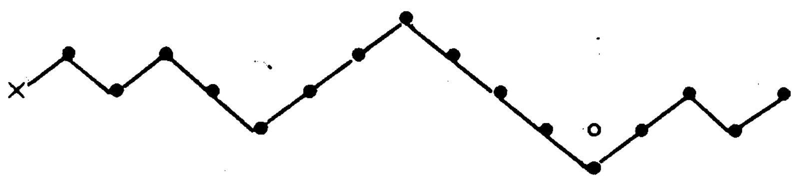
$$c_{i,j} = c_{i-1,j-1} + c_{i-1,j+1}$$

The Catalan number $c_n = \frac{1}{n+1}\binom{2n}{n}$, for each U step in a Dyck path→Binomial path

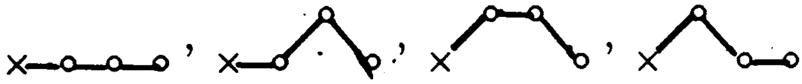
$P =$



$Q =$



Add a level step $L = (1, 0)$, we have **Motzkin** paths,



Partition by the first step L or U

$m_n = m_{n-1} + \sum m_{k-2} m_{n-k}$

The ordinary generating function is

$M = M(x) = 1 + xM + x^2M^2 = \frac{1-x-\sqrt{1-2x-3x^2}}{2x^2}$

$= 1 + x + 2x^2 + 4x^3 + 9x^4 + 21x^5 + 51x^6 + 127x^7 + O\left(x^8\right)$

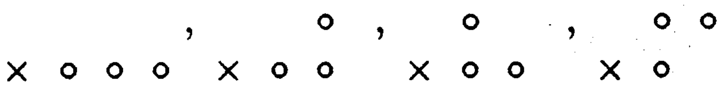
$$(m_{i,j}) = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 2 & 2 & 1 & & \\ 4 & 5 & 3 & 1 & \\ 9 & 12 & 9 & 4 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & & & \\ 1 & 1 & 1 & & \\ & 1 & 1 & 1 & \\ & & 1 & 1 & 1 \\ & & & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & & & \\ 2 & 2 & 1 & & \\ 4 & 5 & 3 & 1 & \\ 9 & 12 & 9 & 4 & 1 \\ 21 & 30 & 25 & 14 & 5 \end{pmatrix}$$

$m_{i,j} = m_{i-1,j-1} + m_{i-1,j} + m_{i-1,j+1}.$

A directed animal S is a set of points in the first octant such that

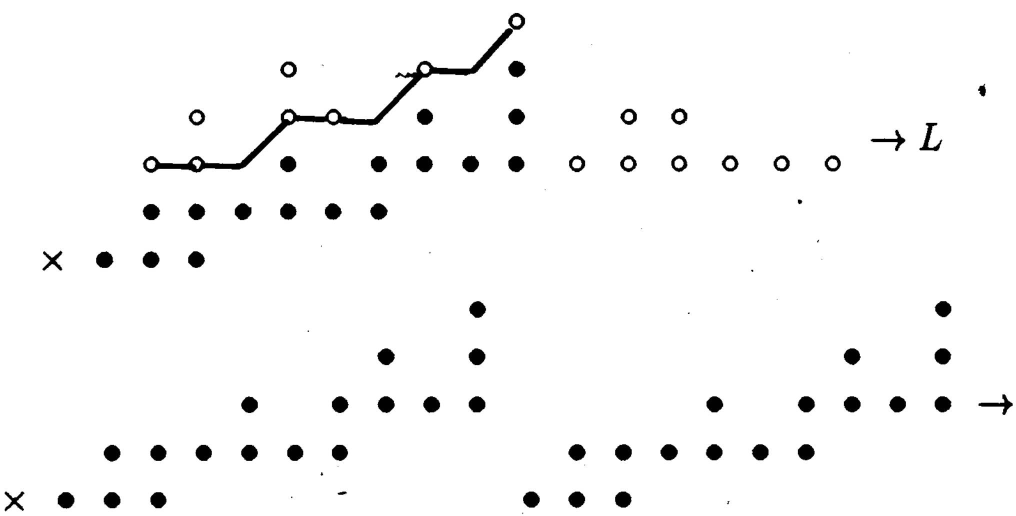
1. $(0,0) \in S$
2. $(0,0) \neq (a,b) \in S$, then either $(a-1,b) \in S$ or $(a,b-1) \in S$.

For $n = 3$,



The counting of DA was discovered by Gouyou-Beauchamps and Viennot in 1988.

Bijection . The counting of directed animals is the Motzkin sequence, $1, 1, 2, 4, 9, 21, 51, \ldots$



R

The partition path of animal: Let S be an animal and start the path P_S at $(a_0, b_0) = (a_0, a_0)$ in S with the smallest $a_0 > 0$, if $(a_i, b_i) \in S$, go $E = (1,0)$, else take $D = (1,1)$. Until there are no more points in S above or below the point. Let $L \subset S$ be the set of points on or above the path and let $R \subset S$ denote the set of points below the path. For lattice paths we may assign weight to the steps, if we assign weight 2 to the level step in the Motzkin paths.

$$\begin{pmatrix} 1 & & & & \\ 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 14 & 14 & 6 & 1 & \\ 42 & 48 & 27 & 8 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & 2 & 1 & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & & & \\ 5 & 4 & 1 & & \\ 14 & 14 & 6 & 1 & \\ 42 & 48 & 27 & 8 & 1 \\ 132 & 165 & 110 & 44 & 10 \end{pmatrix}$$

Labelling Trees.

We label the vertices(except the root) of trees with n edges by $\{1, 2, 3, 4, .., n\}$ in increasing manner as follows:

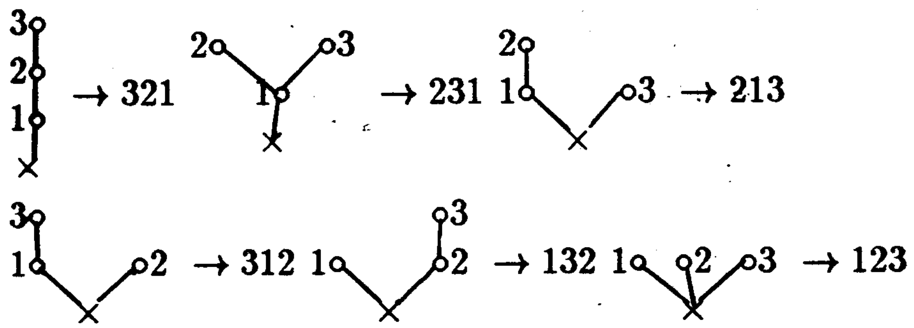
(1) Increasing on any path going up,

(2) Increasing from left to right of the immediate siblings of a vertex.

Labeled increasing rooted plane trees, the counting is the n!

For example, traverse the tree in postorder.

$n = 3$



Let a_n be the number of labeled trees of order n, then by induction the recurrence relation is

$$a_n = \sum_{k=0}^{n-1} \binom{n-1}{k} a_k a_{n-k-1} = \sum \frac{(n-1)!}{(k)!(n-k-1)!}(k)!(n-k-1)! = n((n-1)!) = n!$$

$$\frac{dy}{dx} = y^2 \ , \ y = \frac{1}{1-x} = \sum x^n = \sum n! \frac{x^n}{n!}$$

Let P be a permutation. Partition $P = LsR$ by the smallest element s, by induction, Left branch start with s and the subtree above s corresponding to L and to the right of s corresponding to R. Conversely, traverse the labeled tree by postorder.

Partition the labeled trees by the number of leaves, we have the following Eulerian numbers.

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 4 & 1 \\ 1 & 11 & 11 & 1 \\ 1 & 26 & 66 & 26 & 1 \\ 1 & 57 & 302 & 302 & 57 & 1 \end{pmatrix}$$

Inductively, attach an edge with end vertex labeled n above any leaf, the number of leaves remain the same.

Otherwise, to the rightmost of the subtree at internal vertex, the count increase by one.

Use post-order to traverse the tree:

Partition by the number of branches at the root, the only case that would increase the number of branches at the root, is to attach the n^{th} labeled edge at the root.

The nth row is the coefficients of the expansion of $(x+1)(x+2)...(x+n)$, which are the Sterling numbers of the first kind.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 \\ 6 & 11 & 6 & 1 & 0 & 0 \\ 24 & 50 & 35 & 10 & 1 & 0 \\ 120 & 274 & 225 & 85 & 15 & 1 \end{pmatrix}$$

(1) Consider a permutation avoiding (123), for any vertex the partition LsR, the smallest element in R is at the end and R is in decreasing order. Thus the corresponding labeled tree is of one-two tree. The right branch labeled vertices is also in decreasing order.

(2) One-two trees with right branches in single chain,

$a_n = \sum \binom{n-1}{k} a_k(1)$,

$\frac{dy}{dx} = y(e^x)$,

$y = e^{e^x - 1} = \exp(\exp(x) - 1) = 1 + x + x^2 + \frac{5}{6}x^3 + \frac{5}{8}x^4 + \frac{13}{30}x^5 + \frac{203}{720}x^6 \ ...$

The Bell number sequence, $1, 1, 2, 5, 15, 52, 203, ...$

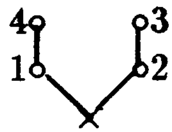
(3) **Labeled increasing one-two trees, the generating function of the counting is the** $secx+tanx-1$**, the counting of zig-zag permutations, i.e.,** $a_1 > a_2 < a_3 > a_4 < a_5 \ldots$ **The sequence is** $1, 1, 2, 5, 16, 61, 272, \ldots$

(4) Trees of height at most two

$a_n = \sum \binom{n-1}{k} 1 * a_k$ **the same as (2).**

(5) Trees avoid321 is the subset of (4), the vertices of height 2 are in increasing order.

the only Bell tree is not avoid321 is



$\rightarrow 4132$

Riordan Matrices. An infinite lower triangular matrix of the form

$$(a_{i,j})_{i,j\geq 0} = L = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & . \\ a_{1,0} & 1 & 0 & 0 & 0 & 0 & . \\ a_{2,0} & a_{2,1} & 1 & 0 & 0 & 0 & . \\ a_{3,0} & a_{3,1} & a_{3,2} & 1 & 0 & 0 & . \\ a_{4,0} & a_{4,1} & a_{4,2} & a_{4,3} & 1 & 0 & . \\ a_{5,0} & a_{5,1} & a_{5,2} & a_{5,3} & a_{5,4} & 1 & . \\ . & . & . & . & . & . & . \end{pmatrix} = (g, f)$$

is said to be a Riordan matrix if the ordinary generating function for the k^{th} **column** $C_k = g(x)f(x)^k$ **for some fixed**

$g = g(x) = 1 + g_1 x + g_2 x^2 + \ldots$

$f = f(x) = x + f_2 x^2 + f_3 x^3 + \ldots$

The multiplication of two matrices as follows:

$(g, f)(h, l) = (gh(f), l(f))$.

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \\ 9 \\ 28 \\ 90 \\ 297 \end{pmatrix}$$

$h(x) = \frac{1}{(1-x)^2}, g = C = C(x)$ **and** $f = xC, C = 1 + xC^2 = \frac{1}{1-xC}$

$gh(f) = C\frac{1}{(1-xC)^2} = C\frac{C-1}{x} = \frac{C^2-C}{x}.$

$C = \frac{1-\sqrt{1-4x}}{2x} = 1 + x + 2x^2 + 5x^3 + 14x^4 + 42x^5 + 132x^6 + \dots.$

$C^2 = (\frac{1-\sqrt{1-4x}}{2x})^2 = 1 + 2x + 5x^2 + 14x^3 + 42x^4 + 132x^5 + 429x^6 + \dots.$

The row sum

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 2 & 2 & 1 & & & \\ 5 & 5 & 3 & 1 & & \\ 14 & 14 & 9 & 4 & 1 & \\ 42 & 42 & 28 & 14 & 5 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 5 \\ 14 \\ 42 \\ 132 \end{pmatrix}$$

$gh(f) = C\frac{1}{(1-xC)} = CC = C^2.$

For a Riordan matrix L, we define a Stiltjes matrix $S_L = L^{-1}\overline{L}$, where $\overline{L}$ is obtained from L by deleting the first row of L.

Entries in n^{th} row of L is linear combinations of $(n-1)^{th}$ row with coefficients from S_L.

$$S_L = \begin{pmatrix} 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ 1 & 1 & 1 & 1 & & \\ 1 & 1 & 1 & 1 & 1 & \\ 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & 1 & & & \\ & 1 & 0 & 1 & & \\ & & 1 & 0 & 1 & \\ & & & 1 & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 & & & & \\ 1 & 1 & 1 & & & \\ & 1 & 1 & 1 & & \\ & & 1 & 1 & 1 & \\ & & & 1 & 1 & 1 \\ & & & & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 1 & & & \\ 1 & 2 & 1 & & \\ & 1 & 2 & 1 & \\ & & 1 & 2 & 1 \\ & & & 1 & 2 \end{pmatrix}$$

Note that the sequences in the columns are the same except the first for Riordan matrix.

Hankel Matrices

Let $S = \{a_0 = 1, a_1, a_2, a_3, ...\}$ be a sequence of real numbers. The Hankel matrix generated by S is given by the infinite matrix

$$H = \begin{bmatrix} 1 & a_1 & a_2 & a_3 & a_4 & . \\ a_1 & a_2 & a_3 & a_4 & a_5 & . \\ a_2 & a_3 & a_4 & a_5 & a_6 & . \\ a_3 & a_4 & a_5 & a_6 & a_7 & . \\ a_4 & a_5 & a_6 & a_7 & a_8 & . \\ . & . & . & . & . & . \end{bmatrix}.$$

Perform Gaussian elimination, factor diagonal matrix D, then the transport L,

$$S_L = L^{-1}\overline{L} = \begin{bmatrix} b_0 & 1 & 0 & 0 & 0 & . \\ c_0 & b_1 & 1 & 0 & 0 & . \\ 0 & c_1 & b_2 & 1 & 0 & . \\ 0 & 0 & c_2 & b_3 & 1 & . \\ 0 & 0 & 0 & c_3 & b_4 & . \\ . & . & . & . & . & . \end{bmatrix}$$

If L is a Riordan matrix with ordinary generating function, $b = b_i, c = c_i$ for $i > 0$.

For exponential generation function $\{b_i\}$ and $\{\frac{c_i}{i+1}\}$ form arithmetic sequences.

Big Schroder sequence $1, 2, 6, 22, 90, 394, 1806, 8558, 41586, 206098, 1037718,$ 5293446. $g = 1 + xg + xg^2 = \frac{1-x-\sqrt{1-6x+x^2}}{2x} = 1 + 2x + 6x^2 + 22x^3 + 90x^4 + 394x^5 + 1806x^6 + \ldots$

$$\begin{bmatrix} 1 & 2 & 6 & 22 & 90 \\ 2 & 6 & 22 & 90 & 394 \\ 6 & 22 & 90 & 394 & 1806 \\ 22 & 90 & 394 & 1806 & 8558 \\ 90 & 394 & 1806 & 8558 & 41586 \end{bmatrix}, \textbf{ Gaussian elimination: } \begin{bmatrix} 1 & 2 & 6 & 22 & 90 \\ 0 & 2 & 10 & 46 & 214 \\ 0 & 0 & 4 & 32 & 196 \\ 0 & 0 & 0 & 8 & 88 \\ 0 & 0 & 0 & 0 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 6 & 22 & 90 \\ 0 & 1 & 5 & 23 & 107 \\ 0 & 0 & 1 & 8 & 49 \\ 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \textbf{ transpose: } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 5 & 1 & 0 & 0 \\ 22 & 23 & 8 & 1 & 0 \\ 90 & 107 & 49 & 11 & 1 \end{bmatrix},$$

$$\textbf{inverse: } \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 4 & -5 & 1 & 0 & 0 \\ -8 & 17 & -8 & 1 & 0 \\ 16 & -49 & 39 & -11 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 4 & -5 & 1 & 0 \\ -8 & 17 & -8 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 6 & 5 & 1 & 0 & 0 \\ 22 & 23 & 8 & 1 & 0 \\ 90 & 107 & 49 & 11 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 \\ 0 & 2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 3 & 1 \end{bmatrix}$$

$f = x(1 + 3f + 2f^2) = \frac{1-3x-\sqrt{1-6x+x^2}}{4x}$,

$g = \frac{1}{1-xb_1-xc_1 f} = \frac{1}{(1-2x-2x(\frac{1-3x-\sqrt{1-6x+x^2}}{4x})} = \frac{2}{1-x+\sqrt{(1-6x+x^2)}} = \frac{1-x-\sqrt{1-6x+x^2}}{2x}$.

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 6 & 5 & 1 & 0 & 0 \\ 22 & 23 & 8 & 1 & 0 \\ 90 & 107 & 49 & 11 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 0 & 16 \end{pmatrix} \begin{pmatrix} 1 & 2 & 6 & 22 & 90 \\ 0 & 1 & 5 & 23 & 107 \\ 0 & 0 & 1 & 8 & 49 \\ 0 & 0 & 0 & 1 & 11 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$LDU = \begin{pmatrix} 1 & 2 & 6 & 22 & 90 \\ 2 & 6 & 22 & 90 & 394 \\ 6 & 22 & 90 & 394 & 1806 \\ 22 & 90 & 394 & 1806 & 8558 \\ 90 & 394 & 1806 & 8558 & 41586 \end{pmatrix}$$

For one-two trees, exponential generating function

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 5 & 11 & 6 & 1 & 0 & 0 & 0 \\ 16 & 45 & 35 & 10 & 1 & 0 & 0 \\ 61 & 211 & 210 & 85 & 15 & 1 & 0 \\ 272 & 1113 & 1351 & 700 & 175 & 21 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 6 & 4 & 1 & 0 & 0 \\ 0 & 0 & 0 & 10 & 5 & 1 & 0 \\ 0 & 0 & 0 & 0 & 15 & 6 & 1 \end{pmatrix}.$$

$f' = 1 + f + \frac{1}{2}f^2$, $ln(g) = \int 1 + f dx$

$f = \tan\frac{2x+\pi}{4} - 1 = x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{2}{15}x^5 + \frac{61}{720}x^6 + \frac{17}{315}x^7 + \frac{277}{8064}x^8$

$g = \frac{1}{2}(\sec\frac{2x+\pi}{4})^2 = 1+x+x^2+\frac{5}{6}x^3+\frac{2}{3}x^4+\frac{61}{120}x^5+\frac{17}{45}x^6+\frac{277}{1008}x^7$