A Brief Review of $q$-Series

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Abstract

We sample certain results from the theory of $q$-series including summation and transformation formulas, as well as some recent results which are not available in book form. Our approach is systematic and uses the Askey–Wilson calculus and Rodrigues type formulas.

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1 Introduction

The subject of $q$-series is very diverse and the present lecture notes represent the author’s philosophy on how to approach the subject. Other points of view are available in [4], [16], [35]. We have used our own approach, which was jointly developed over the years with several coauthors, and we owe a great deal to our coauthors, especially Dennis Stanton.

Central to our approach is the use of a calculus for the Askey–Wilson operator, and the identity theorem for analytic functions.

We built our approach to $q$-series on the theory of the Askey–Wilson polynomials and related topics. Other approaches use the theory of $q$-series to treat the Askey–Wilson polynomials but we develop both theories simultaneously and we believe it is this interaction that best motivates
and explains the subject. We hope we have succeeded in demonstrating the power of the Askey–Wilson calculus which started in [7] and later developed in a series of papers by the present author and his collaborators.

One essential tool is the theory of \( q \)-Taylor series for the Askey–Wilson operator. The first attempt to use this tool was in [18]. This was followed by [24] and [25] where the theory was applied and a \( q \)-Taylor series was developed for expansions of entire functions. The latter theory requires further development. Another ingredient is the Rodrigues formulas and the usage of an explicit expression for the powers of the Askey–Wilson operator.

Recall the identity theorem for analytic functions.

**Theorem 1.1.** [30] Let \( f(z) \) and \( g(z) \) be analytic in a domain \( \Omega \) and assume that \( f(z_n) = g(z_n) \) for a sequence \( \{z_n\} \) converging to an interior point of \( \Omega \). Then \( f(z) = g(z) \) at all points of \( \Omega \).

Theorem 1.1 will be used to derive summation theorems and identities throughout this work by identifying a special parameter in the identity to be proven. Let \( b \) be the special parameter. Usually one can prove the desired identity when \( b = \lambda q^n, n = 0, 1, \ldots \). When both sides of the desired formula are analytic in the special parameter \( b \), we invoke Theorem 1.1 and conclude its validity in the domain of analyticity in \( b \). We must note that one has to be careful with the use of this technique as George Gasper explains some of the pitfalls of misapplying it in his interesting article [15].

## 2 Notation and \( q \)-Operators

This section contains all the notation used in the rest of these notes. The \( q \)-shifted factorials are

\[
(a; q)_0 := 1, \quad (a; q)_n := \prod_{k=1}^{n} (1 - aq^{k-1}), \quad n = 1, 2, \ldots, \text{ or } \infty,
\]

\[
(2.1)
\]

\[
(a_1, a_2, \ldots, a_k; q)_n := \prod_{j=1}^{k} (a_j; q)_n.
\]

The \( q \)-analogue of the binomial coefficient is

\[
\binom{n}{k}_q := \frac{(q; q)_n}{(q)_k (q; q)_{n-k}}.
\]

Usually it is referred to as the \( q \)-binomial coefficient or the Gaussian binomial coefficient. It is clear from (2.1) that

\[
(a; q)_n = (a; q)_{\infty} / (aq^n; q)_{\infty}, \quad n = 0, 1, \ldots,
\]

which suggests the following definition for \( q \)-shifted factorials of any \( n \)

\[
(2.3)
\]

\[
(a; q)_z := (a; q)_{\infty} / (aq^z; q)_{\infty}.
\]
It is easy to see that

\[(a; q)_{-n} = 1/(aq^{-n}; q)_n, \quad (a; q)_m(aq^m; q)_n = (a; q)_{m+n}, \quad m, n = 0, \pm 1, \pm 2, \ldots \]

Some useful identities which follow from the definitions (2.1) and (2.3) are

\[(a q^{-n}; q)_{nk} = \frac{(a; q)_k(q/a; q)_n}{(q^{1-k}/a; q)_n} q^{-nk}, \quad (a; q^{-1})_n = (1/a; q)_n(-a)^n q^{-\binom{n}{2}}, \]

\[(aq^{-n}; q)_n = (q/a; q)_n(-a)^n q^{-n(n+1)/2}, \quad (a; q)_n = (-1)^n q^{\binom{n+1}{2}} (q/a; q)_n a^n, \quad n = 0, 1, 2, \ldots , \]

\[(a; q)_{n-k} = \frac{(a; q)_n}{(q^{1-n}/a; q)_k} (-a)^{-k} q^{\binom{k+1}{2}-nk}, \quad (a; q)_{n-k} = \frac{(a; q)_n(q^{1-n}/b; q)_k}{(b; q)_n(a/q)_k} \left( \frac{b}{a} \right)^k. \]

Unless we say otherwise we shall always assume that

\[0 < q < 1. \]

The \(q\)-gamma function is

\[\Gamma_q(z) = \frac{(q; q)_{\infty}}{(q^z; q)_{\infty}} (1 - q)^{1-z}. \]

One can show that \([3] \lim_{q \to 1^-} \Gamma_q(z) = \Gamma(z). \) It is easy to see that

\[\Gamma_q(z + 1) = \frac{1 - q^z}{1 - q} \Gamma_q(z), \quad \Gamma_q(1) = \Gamma(2) = 1. \]

A basic \textbf{hypergeometric series} is

\[r \phi_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \right| q, z \right) = r \phi_s(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z)
\]

\[= \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(q, b_1, \ldots, b_s; q)_n} z^n \left( -q^{n-1}/2 \right)^n (s+1-r). \]

Note that \((q^{-k}; q)_n = 0\) for \(n = k+1, k+2, \ldots, \) when \(k\) is a nonnegative integer. If one of the numerator parameters is of the form \(q^{-k}\) then the sum on the right-hand side of (2.11) is a finite sum and we say that the series in (2.11) is \textbf{terminating}. A series that does not terminate is called \textbf{nonterminating}.

The radius of convergence of the series in (2.11) is \(1, 0 \text{ or } \infty\) when \(r = s + 1, r > s + 1\) or \(r < s + 1, \) as can be seen from the ratio test.

These notions extend the familiar notions of shifted and multishifted factorials

\[(a)_0 := 1, \quad (a)_n := a(a + 1) \ldots (a + n - 1), \quad (a_1, \ldots, a_k)_n = \prod_{j=1}^{k} (a_j)_n, n = 1, 2, \ldots, \]
and the generalized hypergeometric functions

\[ _rF_s\left(\begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| z \right) = _rF_s(a_1, \ldots, a_r; b_1, \ldots, b_s; z) = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r)_n}{(1, b_1, \ldots, b_s)_n} z^n. \]

It is clear that \( \lim_{q \to 1^{-}} (q^a; q)_n/(1-q)^n = (a)_n \), hence

\[ \lim_{q \to 1^{-}} \phi_s \left( \begin{array}{c} q^{a_1}, \ldots, q^{a_r} \\ q^{b_1}, \ldots, q^{b_s} \end{array} \middle| q, z(1-q)^{-r+1} \right) = _rF_s \left( \begin{array}{c} a_1, \ldots, a_r \\ b_1, \ldots, b_s \end{array} \middle| (1-q)^{-s+1} \right), \quad r \leq s + 1. \]

We shall also use the Bailey notation

\[ _{r+3}W_{r+2}(a^2; a_1, \ldots, a_r; q, z) := _{r+3}\phi_{r+2} \left( \begin{array}{c} a^2, qa, -qa, a_1, \ldots, a_r \\ a, -a, qa^2/a_1, \ldots, qa^2/a_r \end{array} \middle| q, z \right). \]

The \( \phi \) function in (2.15) is called very well-poised.

An important tool in our treatment is the Askey–Wilson operator \( D_q \), which will be defined below. Given a polynomial \( f \) we set \( \tilde{f}(e^{i\theta}) := f(x), \quad x = \cos \theta \), that is

\[ (D_q f)(x) := \frac{\tilde{f}(q^{1/2}z) - \tilde{f}(q^{-1/2}z)}{\tilde{e}(q^{1/2}z) - \tilde{e}(q^{-1/2}z)}, \quad x = (z+1/z)/2, \]

with

\[ e(x) = x. \]

The definition (2.17) easily reduces to

\[ (D_q f)(x) = \frac{\tilde{f}(q^{1/2}z) - \tilde{f}(q^{-1/2}z)}{(q^{1/2} - q^{-1/2}) [z - 1/z]/2}, \quad x = [z+1/z]/2. \]

Note that \( D_q = D_{q^{-1}} \).

It is important to note that although we use \( x = \cos \theta \), \( \theta \) is not necessarily real. In fact \( e^{i\theta} \) is defined as

\[ e^{i\theta} = x + \sqrt{x^2 - 1}, \]

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and the branch of the square root is taken such that \( \sqrt{x^2 - 1} \approx x \) as \( x \to \infty \). It is clear that the Askey–Wilson operator is linear.

The Chebyshev polynomials of the first and second kind are

\[
T_n(\cos \theta) = \cos(n\theta), \quad U_n(\cos \theta) = \frac{\sin((n + 1)\theta)}{\sin \theta}.
\]

It is clear that both \( T_n \) and \( U_n \) have degree \( n \).

As an example we apply \( D_q \) to a Chebyshev polynomial. It is clear that \( \ddot{T}_n(z) = \frac{z^n + z^{-n}}{2} \).

It is easy to see that

\[
(D_q f)(x) = \pi q^{1/2} \left[ \dot{f}(q^{1/2}) \dot{g}(1) - \ddot{f}(q^{1/2}) \ddot{g}(-1) \right]
- \langle f(x), (1 - x^2)^{1/2} D_q((1 - x^2)^{-1/2} g(x)) \rangle.
\]

Let \( \eta_q \) denote the shift operator

\[
(\eta_q f)(x) = (\eta_q \ddot{f})(z) = \ddot{f}(q^{1/2} z),
\]

The Askey–Wilson operator satisfies the product rule

\[
(D_q(fg))(x) = (\eta_q g)(x)(D_q f)(x) + (\eta_q - 1 f)(x)(D_q g)(x).
\]
The Leibniz rule for the operator $D_q$ is [18, (1.22)]

$$
D^n_q(fg) = \sum_{k=0}^{n} q^{k(n-k)/2} \left[ \begin{array}{c} n \\ k \end{array} \right] (\eta q^k D^n_q f)(\eta^{-k} D^k_q g)
$$

and follows from (2.26) by induction. A more symmetric form of (2.26) is

$$
(D_q(fg))(x) = (A_q g)(x)(D_q f)(x) + (A_q f)(x)(D_q g)(x), \quad \mathcal{A} := \frac{1}{2} [\eta q + \eta^{-1}].
$$

Another Leibniz rule due to S. Cooper is [11]

$$
(D^n_q f)(x) = \frac{2^n q^n(1-n)/4}{(q^{1/2} - q^{-1/2})^n} \sum_{k=0}^{n} \left[ \begin{array}{c} n \\ k \end{array} \right] \frac{q^{k(n-k)} z^{2k-n} j(q^{n/2-k} z)}{q^{k-2k+1/2} z^2; q_k(2q^{k-n+1} z - 2; q)_{n-k}}.
$$

This can also be proved by induction.

## 3 $q$-Taylor Series

We shall base our development of the theory of $q$-series on an Askey–Wilson calculus. In the Askey–Wilson calculus we will have a product rule, a Leibniz rule, and $q$-analogues of integration by parts, as well as $q$-Taylor series. The Askey–Wilson calculus has natural polynomial bases which play the role of $\{ (x - a)^n : n = 0, 1, \ldots \}$ in calculus. With $x = \cos \theta$ they are

$$
\phi_n(x; a) = (ae^{i\theta}, ae^{-i\theta}; q)_n = \prod_{k=0}^{n-1} [1 - 2axq^k + a^2 q^{2k}],
$$

$$
\phi_n(x) = (q^{1/4} e^{i\theta}, q^{1/4} e^{-i\theta}; q^{1/2})_n = \prod_{k=0}^{n-1} [1 - 2xq^{1/4 + k/2} + q^{1/2 + k}]
$$

$$
\rho_n(x) = (1 + e^{2i\theta})e^{-i\theta}(-q^{2-n} e^{2i\theta}; q^2)_{n-1}, \quad n > 0, \quad \rho_0(x) := 1.
$$

It is worth noting that

$$
\rho_{2n}(x) = q^n(1-n)(-e^{2i\theta}, -e^{-2i\theta}; q^2)_n, \quad \rho_{2n+1}(x) = 2q^{-n} \cos \theta(qe^{2i\theta}, -qe^{-2i\theta}; q^2)_n.
$$

**Theorem 3.1.** The action of the Askey–Wilson operator on the bases $\{ \phi_n(x; a) \}$, $\{ \phi_n(x) \}$, and $\{ \rho_n(x; a) \}$ is given by

$$
D_q \phi_n(x; a) = \frac{2a(1 - q^n)}{1 - q} \phi_{n-1}(x; a q^{1/2}),
$$

$$
D_q \phi_n(x) = -2q^{1/4} \frac{1 - q^n}{1 - q} \phi_{n-1}(x),
$$

$$
D_q \rho_n(x) = 2q^{(1-n)/2} \frac{1 - q^n}{1 - q} \rho_{n-1}(x).
$$

The proof follows from the definitions (3.1)–(3.3).

We next present a $q$-Taylor series for polynomials.
Theorem 3.2. (Ismail [18]) Let $f$ be a polynomial of degree $n$. Then

\[ f(x) = \sum_{k=0}^{n} f_k(a e^{i \theta}, a e^{-i \theta}; q)_k, \]

where

\[ f_k = \frac{(q - 1)^k}{(2a)^k q^{k(k-1)/4}} \left( \mathcal{D}_q^k f \right)(x_k) \]

with

\[ x_k := \frac{1}{2} \left( a q^{k/2} + q^{-k/2} / a \right). \]

Proof. It is clear that the expansion (3.8) exists, so we now compute the $f_k$’s. Formula (3.5) yields

\[ \mathcal{D}_q^k(a e^{i \theta}, a e^{-i \theta}; q)_{n|x=x_k} = (2a)^k q^{(0+1+\cdots+k-1)/2} (q; q)_n \left( a q^{k/2} e^{i \theta}, a q^{k/2} e^{-i \theta}; q \right)_{n-k}|_{x=ae^{\theta}} \]

\[ = \frac{(q; q)_k}{(q-1)^k} (2a)^k q^{k(k-1)/4} \delta_{k,n}. \]

The theorem now follows by applying $\mathcal{D}_q^j$ to both sides of (3.8) then setting $x = x_j$. \quad \Box

Similarly one can prove the following $q$-Taylor expansions from [24].

Theorem 3.3. If $f$ is a polynomial of degree $n$ then

\[ f(x) = \sum_{k=0}^{n} f_k(\phi) \phi_k(x), \quad f_k(\phi) := \frac{(q - 1)^k q^{-k/4}}{2^k (q; q)_k} \left( \mathcal{D}_q^k f \right)(\zeta_0), \]

where

\[ \zeta_0 := \frac{1}{2} (q^{1/4} + q^{-1/4}). \]

Elliptic analogues of the $q$-Taylor series have been introduced by Schlosser and Schlosser and Yoo in the very interesting papers [39], [40].

Another $q$-analogue of the derivative is the $q$-difference operator

\[ (D_q f)(x) = (D_{q,x} f)(x) = \frac{f(x) - f(qx)}{(1 - q)x}. \]

It is clear that

\[ D_{q,x} (x^n) = \frac{1 - q^n}{1 - q} x^{n-1}, \]
and for differentiable functions

\[
\lim_{q \to 1^-} (D_qf)(x) = f'(x).
\]

The operator \( D_q \) was employed systematically by Jackson who was not the first to use it. Interestingly, while the \( q \)-derivative was apparently anticipated by Euler and by Heine, it does not seem to appear explicitly in their work. The first explicit occurrence of the \( q \)-derivative, together with applications such as \( q \)-Taylor series expansions, probably appeared in the 1877 work of Leopold Schendel [38].

The evaluation of a \( q \)-integral is a restatement of a series identity, however its interpretation as a \( q \)-integral may give insight to its origin and where it fits conceptually.

For finite \( a \) and \( b \) the \( q \)-integral is

\[
\int_a^b f(x) \, dq_x := \int_0^b f(x) \, dq_x - \int_0^a f(x) \, dq_x.
\]

It is clear from (3.16) that the \( q \)-integral is an infinite Riemann sum with the division points in a geometric progression. We would then expect \( \int_a^b f(x) \, dq_x \to \int_a^b f(x) \, dx \) as \( q \to 1 \) for continuous functions. The \( q \)-integral over \([0, \infty)\) uses the division points \( \{q^n : -\infty < n < \infty\} \) and is

\[
\int_0^{\infty} f(x) \, dq_x := (1 - q) \sum_{n=-\infty}^{\infty} q^n f(q^n).
\]

The relationship

\[
\int_a^b f(x)g(qx) \, dq_x = q^{-1} \int_a^b g(x)f(x/q) \, dq_x + q^{-1}(1 - q)[ag(a)f(a/q) - bg(b)f(b/q)]
\]

follows from series rearrangements. The proof is straightforward and will be omitted.

Consider the weighted inner product

\[
\langle f, g \rangle_q := \int_a^b f(t)g(t) \, dq_t
\]

(3.19)

\[
= (1 - q) \sum_{k=0}^{\infty} f(y_k)g(y_k) y_k w(y_k) - (1 - q) \sum_{k=0}^{\infty} f(x_k)g(x_k) x_k w(x_k),
\]

where

\[
x_k := aq^k, \quad y_k := bq^k,
\]

and \( w(x_k) > 0 \) and \( w(y_k) > 0 \) for \( k = 0, 1, \ldots \). We will take \( a \leq 0 \leq b \).
Theorem 3.4. An analogue of integration by parts for $D_{q,x}$ is

$$
\langle D_{q,x}f, g \rangle_q = -f(x_0)g(x_1)w(x_1) + f(y_0)g(y_1)w(y_1)
$$

$$
-q^{-1}\left( f, \frac{1}{w(x)}D_{q^{-1},x}(g(x)w(x)) \right)_q,
$$

provided that the series on both sides of (3.21) converge absolutely and

$$
\lim_{n \to \infty} w(x_n)f(x_{n+1})g(x_n) = \lim_{n \to \infty} w(y_n)f(y_{n+1})g(y_n) = 0.
$$

The proof is left as an exercise.

The product rule for $D_q$ is

$$
(D_q(fg))(x) = f(x)(D_qg)(x) + g(qx)(D_qf)(x).
$$

In this work we do not develop applications of $q$-integrals for two reasons. The first is that they are treated in some detail in the books [16] and [4]. The second is that we have concentrated on integrals with respect to measures which are absolutely continuous with respect to the Lebesgue measure in order to include new material not available in book form. We note that infinite $q$-Taylor series expansions involving $D_q$ has been rigorously developed by Annaby and Mansour in [5]. For earlier or formal treatments see the references in [5].

When studying an operator it is helpful to describe its null space. For differentiation the null space is the constant functions. For $D_q$, the null space is the set of functions satisfying $f(x) = f(qx)$. If we write $x = q^y$, then the null space is the set of functions periodic in $y$ with period 1. For the Askey–Wilson operator we use $x = \cos \theta$. As functions of $\theta$ the null space of the Askey–Wilson operator is clearly the doubly periodic functions with periods $2\pi$ and $i \log q$.

4 Summation Theorems

We will use the $q$-Taylor expansion for polynomials to derive some of the summation theorems in the $q$-calculus. The idea is to expand polynomials in one of the bases in (3.1), (3.2), or (3.3). The explicit $q$-Taylor coefficients are found either by computing the action of $D_q^k$ on the function to be expanded, or using the Cooper formula (2.29). It must be noted that Zhi-Guo Liu [32–34] used the operator $D_{q,x}$ and his characterization of functions satisfying $D_{q,x}f(x,y) = D_{q,y}f(x,y)$

to obtain summation theorems and $q$-series identities. His approach is noteworthy.

A basic hypergeometric function (2.11) is called balanced if

$$
r = s + 1 \quad \text{and} \quad qa_1a_2 \cdots a_{s+1} = b_1b_2 \cdots b_s.
$$
Theorem 4.1. (q-Pfaff–Saalschütz). The sum of a terminating balanced \(3\phi_2\) is given by

\[
3\phi_2 \left( \frac{q^{-n}, a, b}{c, d} \bigg| q, q \right) = \frac{(d/a, d/b; q)_n}{(d, d/ab; q)_n}, \quad \text{with } cd = abq^{1-n}.
\]

Proof. Apply Theorem 3.2 to \(f(\cos \theta) = (be^{i\theta}, be^{-i\theta}; q)_n\) and use (3.8) and (3.9) to obtain

\[
f_k = \frac{(q; q)_n (b/a)^k}{(q; q)_{k} (q; q)_{n-k}} \frac{(bq^{k/2}e^{i\theta}, bq^{k/2}e^{-i\theta}; q)_{n-k} e^{i\theta} = aq^{k/2}}{(q; q)_n (b/a)^k} \frac{(bq^{k}, b/a; q)_{n-k}}{(q; q)_n (q; q)_{n-k}}.
\]

Therefore (3.8) becomes

\[
\left( \frac{be^{i\theta}, be^{-i\theta}; q)_n}{(ab, b/a; q)_n} = \frac{b^k (ae^{i\theta}, ae^{-i\theta}; q)_{k} (abq^{k}, b/a; q)_{n-k}}{a^k (q; q)_{k} (q; q)_{n-k}},
\]

Using (2.7) we can rewrite the above equation in the form

\[
\left( \frac{be^{i\theta}, be^{-i\theta}; q)_n}{(ab, b/a; q)_n} = 3\phi_2 \left( \frac{q^{-n}, ae^{i\theta}, ae^{-i\theta}}{ab, q^{1-n}a/b} \bigg| q, q \right),
\]

which is equivalent to the desired result.

Theorem 4.2. We have the q-analogue of the Chu-Vandermonde sum

\[
2\phi_1(q^{-n}, a; c; q, q) = \frac{(c/a; q)_n}{(c; q)_n} a^n,
\]

and the q-analogue of Gauss’ theorem

\[
2\phi_1(a, b; c; q, q) = \frac{(c/a, c/b; q)_\infty}{(c, c/ab; q)_\infty}, \; |c/ab| < 1.
\]

Proof. Let \(d = abq^{1-n}/c\) in (4.2) then let \(n \to \infty\). Taking the limit inside the sum is justified since \((a, b; q)_k/(q, c; q)_k\) is bounded, [9]. The result is (4.6). When \(b = q^{-n}\) then (4.6) becomes

\[
2\phi_1(q^{-n}, a; c; c/aq^n/a) = \frac{(c/a; q)_n}{(c; q)_n}.
\]

To prove (4.5) express the left-hand side of the above equation as a sum, over \(k\) say, replace \(k\) by \(n - k\), then apply (2.7) and arrive at (4.5) after some simplifications and substitutions. This completes the proof.

Replace \(a, b, c\) by \(q^a, q^b, q^c\), respectively, in (4.6), then let \(q \to 1^-\) to see that (4.6) reduces to Gauss’s theorem; [36], [4]:

\[
2F_1(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c - a - b)}{\Gamma(c - a)\Gamma(c - b)}, \; \Re(c - a - b) > 0.
\]
**Theorem 4.3.** If \(|z| < 1\) or \(q = q^{-n}\) then the \(q\)-binomial theorem

\[
\phi_0(a; -; q, z) = \left(\frac{az; q}{z; q}\right)_\infty,
\]

holds.

**Proof.** Let \(c = abz\) in (4.6) then let \(b \to 0\).

As \(q \to 1^-, \phi_0(q^a; -; q, z)\) tends to \(\sum_{n=0}^{\infty} (a)_n z^n / n! = (1 - z)^{-a}\), by the binomial theorem. Thus the right-hand side of (4.9) is a \(q\)-analogue of \((1 - z)^{-a}\).

**Theorem 4.4.** (Euler). We have

\[
e_q(z) := \sum_{n=0}^{\infty} \frac{z^n}{(q; q)_n} = \frac{1}{(z; q)_\infty}, \quad |z| < 1,
\]

\[
E_q(z) := \sum_{n=0}^{\infty} \frac{z^n q^{n(n-1)/2}}{(q; q)_n} = (-z; q)_\infty.
\]

**Proof.** Formula (4.10) is the special case \(a = 0\) of (4.9). To get (4.11), we replace \(z\) by \(-z/a\) in (4.9) and let \(a \to \infty\). This implies (4.11) and the proof is complete.

The left-hand sides of (4.10) and (4.11) are \(q\)-analogues of the exponential function.

The terminating version of the \(q\)-binomial theorem may be written as

\[
(z; q)_n = \sum_{k=0}^{n} \binom{n}{k}_q q^{k} (-z)^k.
\]

**Theorem 4.5.** The following summation theorem holds

\[
\frac{(ae^{i\theta}, ae^{-i\theta}; q^2)_n}{(aq^{-1/2}; q)_2 n} = 4\phi_3 \left( q^{n}, -q^{-n}, q^{1/2}e^{i\theta}, q^{1/2}e^{-i\theta}, -q, aq^{-1/2}, q^{2n+3/2}/a \right| q, q).
\]

**Proof.** Let \(f(x) = (ae^{i\theta}, ae^{-i\theta}; q)_n\) in (3.11). Using (3.5) we see that the coefficient of \(\phi_n(x)\) is

\[
\frac{(q - 1)^k q^{-k/2}}{2^k(q; q)_k} \frac{(2a)^k(q; q)_n}{(q - 1)^k(q; q)_{n-k}} q^{k(k-1)/4} (aq^{2k+1/4}, aq^{2k-1/4}; q)_{n-k} = \frac{a^k q^{-k/2}}{(q; q)_k(q; q)_{n-k}} q^{k(k-2)/4} (aq^{2k-1/4}; q^{1/2})_{2n-2k}.
\]

Replace \(q\) by \(q^2\) and after some simplification we establish (4.13).

Ismail and Stanton proved Theorem 4.5 in [24]. Schlosser and Yoo [40] observed that Theorem 4.5 follows from (1.4) in [14].

For completeness we mention without a proof a non-polynomial version of \(q\)-Taylor series. The maximum modulus of an entire functions \(f, M(r; f)\), is [8]

\[
M(r; f) = \sup \{|f(z)| : |z| \leq r\}.
\]
Theorem 4.6. ([25]) Let $f$ be analytic in a bounded domain $D$ and let $C$ be a contour within $D$ and $x$ belong to the interior of $C$. If the distance between $C$ and the set of zeros of $\phi_\infty(x; a)$ is positive then

$$f(x) = \frac{\phi_\infty(x; a)}{2\pi i} \oint_C \frac{f(y) \, dy}{y - x} \phi_\infty(y; a) - \frac{a}{\pi i} \sum_{n=0}^{\infty} q^n \phi_n(x; a) \oint_C \frac{f(y) \, dy}{\phi_{n+1}(y; a)},$$

where $\phi_\infty(\cos \theta; a) := (ae^{i\theta}, ae^{-i\theta}; q)_\infty$.

This is true for any entire function but if we let all points on $C$ tend to $\infty$, the integral may or may not converge to 0. We now restrict ourselves to entire function satisfying

$$\lim sup_{r \to +\infty} \frac{\ln M(r; f)}{\ln^2 r} = c,$$

for a particular $c$ which, of course, will depend upon $q$.

Theorem 4.7. ([25]) Any entire function $f$ satisfying (4.16) with $c < 1/(2 \ln q^{-1})$ has a convergent expansion

$$f(x) = \sum_{k=0}^{\infty} f_{k,\phi} \phi_k(x; a),$$

with $\{f_{k,\phi}\}$ as defined in Theorem 3.2. Moreover any such $f$ is uniquely determined by its values at the points $x_n : n \geq 0$ defined in (3.10).

The $q$-shifted factorial $(a; q)_n$ for $n < 0$ has been stated in (2.6). A bilateral basic hypergeometric function is

$$m_\psi m \left( \begin{array}{c} a_1, \ldots, a_m \\ b_1, \ldots, b_m \end{array} \bigg| q, z \right) = \sum_{n=-\infty}^{\infty} (a_1, \ldots, a_m; q)_n (b_1, \ldots, b_m; q)_n z^n.$$

It is easy to see that the series in (4.18) converges if

$$|b_1 b_2 \cdots b_m| \frac{a_1 a_2 \cdots a_m}{|a_1 a_2 \cdots a_m|} < |z| < 1.$$

Our next result is the Ramanujan $1_1\psi_1$ sum.

Theorem 4.8. The following summation theorem holds for $|b/a| < |z| < 1$

$$1_1\psi_1(a; b; q, z) = \frac{(b/a, q, q/a, az; q)_\infty}{(b, b/az, q/a, z; q)_\infty}.$$

Proof. ([17]). Observe that both sides of (4.20) are analytic function of $b$ for $|b| < |az|$ since, by (2.4), we have

$$1_1\psi_1(a; b; q, z) = \sum_{n=0}^{\infty} \frac{(a; q)_n}{(b; q)_n} z^n + \sum_{n=0}^{\infty} \frac{(q/b; q)_n}{(q/a; q)_n} \left( \frac{b}{az} \right)^n.$$
Moreover when \( b = q^{m+1}, m \) a positive integer, then \( 1/(b;q)_n = (q^n b;q)_{-n} = 0 \) for \( n < -m \), see (2.4). Therefore the \( q \)-binomial theorem (4.9) gives

\[
\psi_1(a; q^{m+1}; q, z) = \sum_{n=-m}^{\infty} (a;q)_n (q^{m+1};q)_n z^n = z^{-m} \frac{(a;q)_{-m}}{(q^{m+1};q)_m} \sum_{n=0}^{\infty} (a q^{-m};q)_n z^n
\]

Using (2.4)–(2.5) we simplify the above formula to

\[
\psi_1(a; q^{m+1}, q, z) = \frac{(q^{m+1}/a, q, q/a z, a z; q)_\infty}{(q^{m+1}, q^{m+1}/a z, q/a, z; q)_\infty},
\]

which is (4.20) with \( b = q^{m+1} \). The identity theorem for analytic functions then establishes the theorem.

For other proofs and references see [16] and [4].

**Theorem 4.9.** (Jacobi Triple Product Identity). We have

\[
\sum_{n=-\infty}^{\infty} q^{n^2} z^n = (q^2, -qz, -q/z; q^2)_\infty.
\]

**Proof.** It readily follows from (4.20) that

\[
\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \lim_{c \to 0} \psi_1(-1/c; 0; q^2, q z c) = \lim_{c \to 0} \frac{(q^2 z, -q z, -q/z; q^2)_\infty}{(-q^2 c, q e z; q^2)_\infty} = (q^2, -q z, -q/z; q^2)_\infty,
\]

which is (4.21).

## 5 Transformations

The Euler transformation for a hypergeometric function is, [36], [4],

\[
_2F_1(a, b; c; z) = (1 - z)^{c-a-b} 2F_1(c - a, c - b; c; z).
\]

The next theorem is a \( q \)-analogue of (5.1).

**Theorem 5.1.** We have the \( q \)-analogue of the Euler transformation

\[
_2\phi_1 \left( \begin{array}{c} A, B \\ C \end{array} \mid q, Z \right) = (ABZ/C; q)_\infty \frac{(C/A, C/B; q)_\infty}{(Z; q)_\infty} _2\phi_1 \left( \begin{array}{c} C/A, C/B \\ q, ABZ/C \end{array} \right)
\]

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and the symmetric form of the Sears transformation

\[(ab, b\beta, b\gamma; q)_m b^{-m} 4\phi_3 \left( \begin{array}{c|c} q^{-m}, q^{m-1} ab\beta\gamma, bz, b/z & q, q \\ \hline ab, b\beta, b\gamma & \end{array} \right) = (ab, a\beta, a\gamma; q)_m a^{-m} 4\phi_3 \left( \begin{array}{c|c} q^{-m}, q^{m-1} ab\beta\gamma, az, a/z & q, q \\ \hline ab, a\beta, a\gamma & \end{array} \right). \tag{5.3} \]

**Proof.** As usual we use \( z = e^{i\theta} \). Multiply (4.3) by \( t^n/(ab; q)_n \) and add for \( n \geq 0 \). The result is

\[2\phi_1 \left( \begin{array}{c|c} bz, b/z & q, t \\ \hline ab & \end{array} \right) = 1\phi_0 (b/a; -; q, t) 2\phi_1 \left( \begin{array}{c|c} az, a/z & q, b/a \\ \hline ab & \end{array} \right). \]

We sum the \( 1\phi_0 \) by the \( q \)-binomial theorem (4.9) and establish (5.2). The proof of (5.3) is similar. Multiply (4.3) by

\[\frac{(q^{-m}, q^{m-1} ab\beta\gamma; q)_n}{(ab, b\beta, b\gamma; q)_n} q^n \]

and sum for \( 0 \leq n \leq m \). After interchanging the \( k \) and \( n \) sums we get

\[4\phi_3 \left( \begin{array}{c|c} q^{-m}, q^{m-1} ab\beta\gamma, bz, b/z & q, q \\ \hline ab, b\beta, b\gamma & \end{array} \right) = \sum_{k=0}^{m} \frac{(q^{-m}, q^{m-1} ab\beta\gamma; q)_k}{(q; q)_k} (az, a/z; q)_k \left( \frac{b}{a} \right)^k \sum_{n=k}^{m} \frac{(q^{k-m}, q^{k+m-1} ab\beta\gamma, b/a; q)_n}{(ab; q)_k (b\beta, b\gamma; q)_n (q; q)_n} q^n \]

\[= \sum_{k=0}^{m} \frac{(q^{-m}, q^{m-1} ab\beta\gamma; q)_k}{(q, ab, b\beta, b\gamma; q)_k} (az, a/z; q)_k \left( \frac{ab}{a} \right)^k 3\phi_2 \left( \begin{array}{c|c} q^{-m}, q^{m-k} ab\beta\gamma, b/a & q, q \\ \hline b\beta q^k, b\gamma q^k & \end{array} \right). \]

By the \( q \)-Pfaff–Saalschütz theorem the \( 3\phi_2 \) is

\[\frac{(a\beta q^k, q^{-1-m}/a\gamma; q)_{m-k}}{(b\beta q^k, q^{-1-m}/b\gamma; q)_{m-k}} = \frac{(a\beta, a\gamma; q)_m (b\beta, b\gamma; q)_k}{(b\beta, b\gamma; q)_m (a\beta, a\gamma; q)_k} \left( \frac{b}{a} \right)^{m-k} q^n. \]

This establishes (5.3). \( \square \)

The transformation (5.2) is one of three transformations known as Heine transformations. The Sears transformation is usually written as, [16],

\[4\phi_3 \left( \begin{array}{c|c} q^{-n}, A, B, C & q, q \\ \hline D, E, F & \end{array} \right) = \frac{(E/A, F/A; q)_n}{(E, F; q)_n} A^n 4\phi_3 \left( \begin{array}{c|c} q^{-n}, A, D/B, D/C & q, q \\ \hline D, Aq^{1-n}/E, Aq^{1-n}/F & \end{array} \right), \tag{5.4} \]

where \( DEF = ABC q^{1-n} \). We leave as an exercise for the reader to show the equivalence of (5.4) and (5.3).
An iterate of the Sears transformation is

\[
\begin{aligned}
\phi_3^n \left( \begin{array}{c}
q^{-n}, A, B, C \\
D, E, F
\end{array} \middle| q, q \right) \\
\left( \begin{array}{c}
q^{-n}, E/A, F/A, EFD/ABC \\
EF/AB, EF/AC, q^{1-n}/A
\end{array} \middle| q, q \right)
\end{aligned}
\]

(5.5)

where \( DEF = ABCq^{1-n} \).

Note that the form which we proved of the \( q \)-analogue of the Euler transformation is the symmetric form

\[
\begin{aligned}
(t; q)_\infty \phi_1 \left( \begin{array}{c}
bz, b/z \\
ab
\end{array} \middle| q, t \right) = (bt/a; q)_\infty \phi_1 \left( \begin{array}{c}
az, a/z \\
ab
\end{array} \middle| q, bt/a \right).
\end{aligned}
\]

(5.6)

A limiting case of the Sears transformation is the useful transformation stated below.

**Theorem 5.2.** The following \( 3 \phi_2 \) transformation holds

\[
\begin{aligned}
3\phi_2 \left( \begin{array}{c}
q^{-n}, a, b \\
c, d
\end{array} \middle| q, q \right) = \frac{b^n(d/b; q)_n}{(d; q)_n} 3\phi_2 \left( \begin{array}{c}
q^{-n}, b/c, a \\
q^{-n}b/d
\end{array} \middle| q, aq/d \right).
\end{aligned}
\]

(5.7)

**Proof.** In (5.4) set \( F = ABCq^{1-n}/DE \) then let \( C = F \to 0 \) while all the other parameters remain constant. The result is

\[
\begin{aligned}
3\phi_2 \left( \begin{array}{c}
q^{-n}, A, B \\
D, E
\end{array} \middle| q, q \right) = \frac{A^n(E/A; q)_n}{(E; q)_n} 3\phi_2 \left( \begin{array}{c}
q^{-n}, A, D/B \\
q^{-n}A/E
\end{array} \middle| q, Bq/E \right).
\end{aligned}
\]

The result now follows with the parameter identification \( A = b, B = a, D = c, E = d \). \( \Box \)

An interesting application of (5.7) follows by taking \( b = \lambda d \) and letting \( d \to \infty \). The result is

\[
\begin{aligned}
2\phi_1 \left( \begin{array}{c}
q^{-n}, a, \gamma \\
c
\end{array} \middle| q, q \lambda \right) = (\lambda q^{1-n}; q)_n \sum_{j=0}^{\infty} \frac{(q^{-n}, c/a; q)_j q^{j(1/2)}(-\lambda a\gamma)^j}{(q, c, \lambda q^{1-n}; q)_j}.
\end{aligned}
\]

Now replace \( \lambda \) by \( \lambda q^{n-1} \) and observe that the above identity becomes the special case \( \gamma = q^n \) of

\[
\begin{aligned}
2\phi_1 \left( \begin{array}{c}
a, 1/\gamma \\
c
\end{array} \middle| q, \gamma \lambda \right) = (\lambda; q)_\infty \sum_{j=0}^{\infty} \frac{(1/\gamma, c/a; q)_j q^{j(1/2)}(-\lambda a\gamma)^j}{(q, c, \lambda; q)_j}.
\end{aligned}
\]

(5.8)

Since both sides of the relationship (5.8) are analytic functions of \( \gamma \) when \( |\gamma| < 1 \) and they are equal when \( \gamma = q^n \) then they must be identical for all \( \gamma \) if \( |\gamma| < 1 \).

It is more convenient to write the identity (5.8) in the form

\[
\begin{aligned}
\sum_{n=0}^{\infty} \frac{(A, C/B; q)_n q^{n(n-1)/2}(-Bz)^n}{(q, C, Az; q)_n} = \frac{(z; q)_\infty}{(Az; q)_\infty} 2\phi_1(A, B; C; q, z).
\end{aligned}
\]

(5.9)
In terms of basic hypergeometric functions (5.9) takes the form
\begin{equation}
2\phi_2(A, C/B; C, Az; q, Bz) = \frac{(z; q)_\infty}{(Az; q)_\infty} 2\phi_1(A, B; C; q, z).
\end{equation}

Observe that (5.9) or (5.10) is the q-analogue of the Pfaff-Kummer transformation, [36],
\begin{equation}
2F_1(a, b; c; z) = (1 - z)^{-a} 2F_1(a - b; c; z/(z - 1)),
\end{equation}
which holds when |z| < 1 and |z/(z - 1)| < 1.

Later on we will come across the following 3\phi_2 transformation
\begin{equation}
3\phi_2 \left( \frac{q^{-n}, a, b}{c, 0} \bigg| q, q \right) = \frac{(b; q)_n a^n}{(c; q)_n} 2\phi_1 \left( \frac{q^{-n}, c/b}{q^{1-n}/b} \bigg| q, q/a \right).
\end{equation}

**Proof of (5.12).** Let c → 0 in Theorem 5.2 to get
\begin{equation}
3\phi_2 \left( \frac{q^{-n}, a, b}{d, 0} \bigg| q, q \right) = \frac{(d/b; q)_n b^n}{(d; q)_n} 2\phi_1 \left( \frac{q^{-n}, b}{q^{1-n}b/d} \bigg| q, qa/d \right).
\end{equation}

On the 2\phi_1 side replace the summation index, say k, by n − k then apply (2.7) to obtain a result equivalent to (5.12).

The transformation (5.12) has an interesting consequence. Since the left-hand side is symmetric in a, b then
\begin{equation}
(b; q)_n a^n 2\phi_1 \left( \frac{q^{-n}, c/b}{q^{1-n}/b} \bigg| q, q/a \right) = (a; q)_n b^n 2\phi_1 \left( \frac{q^{-n}, c/a}{q^{1-n}/a} \bigg| q, q/b \right).
\end{equation}

**Theorem 5.3.** The following two Heine transformations hold when both sides are well-defined.
\begin{align}
2\phi_1 \left( \frac{a, b}{c} \bigg| q, z \right) &= \frac{(az, c/a; q)_\infty}{(c, z; q)_\infty} 2\phi_1 \left( \frac{a, abz/c}{az} \bigg| q, c/a \right), \\
2\phi_1 \left( \frac{a, b}{c} \bigg| q, z \right) &= \frac{(b, az; q)_\infty}{(c, z; q)_\infty} 2\phi_1 \left( \frac{c/b, z}{az} \bigg| q, b \right).
\end{align}

**Proof.** In (5.13) replace a and b by $q^{1-n}/a$ and $q^{1-n}/b$, respectively and conclude that
\begin{align*}
2\phi_1 \left( \frac{q^{-n}, q^{n-1}b}{b} \bigg| q, q^n a \right) &= \frac{(q^{1-n}/a; q)_n a^n}{(q^{1-n}/b; q)_n b^n} 2\phi_1 \left( \frac{q^{-n}, q^{n-1}ac}{a} \bigg| q, q^n b \right) \\
&= \frac{(a, bq^n; q)_\infty}{(b, aq^n; q)_\infty} 2\phi_1 \left( \frac{q^{-n}, q^{n-1}ac}{a} \bigg| q, q^n b \right).
\end{align*}

Now observe that the above equation, with c replaced by cq, is the case $\gamma = q^n$ of the transformation
\begin{align*}
2\phi_1 \left( \frac{1/\gamma, bc\gamma}{b} \bigg| q, \gamma a \right) &= \frac{(a, b\gamma; q)_\infty}{(b, a\gamma; q)_\infty} 2\phi_1 \left( \frac{1/\gamma, ac\gamma}{a} \bigg| q, \gamma b \right).
\end{align*}
With $|a| \leq 1, |b| \leq 1$, both sides of the above identity are analytic functions of $\gamma$ in the open unit disc, hence its validity for the sequence $\gamma = q^n$ implies its validity for $|\gamma| < 1$. This is an equivalent form of (5.14). Apply (5.2) to (5.14) to derive (5.15).

## 6 $q$-Hermite Polynomials

Our approach to the theory of $q$-series is based on the theory of Askey–Wilson operators and polynomials. The first step in this program is to develop the continuous $q$-Hermite polynomials $\{H_n(x \mid q)\}$. They are generated by the recursion relation

$$2xH_n(x \mid q) = H_{n+1}(x \mid q) + (1 - q^n)H_{n-1}(x \mid q), \tag{6.1}$$

and the initial conditions

$$H_0(x \mid q) = 1, \quad H_1(x \mid q) = 2x. \tag{6.2}$$

We solve this recurrence relation using the generating function technique. Set

$$H(x, t) := \sum_{n=0}^{\infty} H_n(x \mid q) \frac{t^n}{(q; q)_n}. \tag{6.3}$$

Multiply (6.1) by $t^n/(q; q)_n$, and sum over $n = 1, 2, \ldots$. After the use of the initial conditions (6.2) we obtain the $q$-difference equation

$$H(x, t) = \frac{H(x, qt)}{1 - 2xt + t^2} = \frac{H(x, qt)}{(te^{i\theta}; q)_1(te^{-i\theta}; q)_1}, \quad x = \cos \theta. \tag{6.6}$$

After iterating the above functional equation $n$ times we get

$$H(\cos \theta, t) = \frac{H(\cos \theta, q^n t)}{(te^{i\theta}, te^{-i\theta}; q)_n}. \tag{6.6}$$

As $n \to \infty$, $H(x, q^n t) \to H(x, 0) = 1$. This establishes the generating function

$$\sum_{n=0}^{\infty} H_n(\cos \theta \mid q) \frac{t^n}{(q; q)_n} = \frac{1}{(te^{i\theta}, te^{-i\theta}; q)_\infty}. \tag{6.4}$$

To obtain an explicit formula for the $H_n$’s we expand $1/(te^{\pm i\theta}; q)_\infty$ by (4.10), then multiply the resulting series. This gives

$$H_n(\cos \theta \mid q) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q e^{i(n-2k)\theta} = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_q \cos(n - 2k)\theta. \tag{6.5}$$

The representation (6.5) reflects the polynomial character of $H_n(x \mid q)$. The generating function (6.4) implies the symmetry property

$$H_n(-x \mid q) = (-1)^n H_n(x \mid q). \tag{6.6}$$
It is worth noting that
\begin{equation}
H_n(x \mid q) = (2x)^n + \text{lower order terms},
\end{equation}
which follows from (6.1) and (6.2).

Our next goal is to prove the orthogonality relation
\begin{equation}
\int_{-1}^{1} H_m(x \mid q)H_n(x \mid q)w(x \mid q)\,dx = \frac{2\pi(q; q)_\infty}{(q; q)_\infty} \delta_{m,n},
\end{equation}
where
\begin{equation}
w(x \mid q) = \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\sqrt{1-x^2}}, \quad x = \cos \theta, \quad 0 \leq \theta \leq \pi.
\end{equation}

The proof of (6.8) is based on the following lemma.

Lemma 6.1. We have the following evaluation
\begin{equation}
\int_0^{\pi} e^{2ij\theta} (e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta = \frac{\pi(-1)^j}{(q; q)_\infty} (1 + q^j)q^{j-1/2}. \tag{6.10}
\end{equation}

Proof. Let $I_j$ denote the left side of (6.10). The Jacobi triple product identity (4.21) gives
\begin{align*}
I_j &= \int_0^{\pi} e^{2ij\theta}(1 - e^{2i\theta})(qe^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta \\
&= \int_0^{\pi} e^{2ij\theta}(1 - e^{2i\theta})\sum_{n=-\infty}^{\infty} (-1)^n q^{n(n+1)/2} e^{2in\theta} d\theta \\
&= \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{2(q; q)_\infty} \int_{-\pi}^{\pi} (1 - e^{i\theta})e^{i(j+n)\theta} d\theta.
\end{align*}
The result now follows from the orthogonality of the trigonometric functions on $[-\pi, \pi]$. \hfill \Box

Proof of (6.8). Since the weight function $w(x \mid q)$ is an even function of $x$, it follows that (6.8) trivially holds if $|m-n|$ is odd. Thus there is no loss of generality in assuming $m \leq n$ and $n-m$ is even. It is clear that we can replace $n-2k$ by $|n-2k|$ in (6.5). Therefore we evaluate the following integrals only for $0 \leq j \leq n/2$. We now have
\begin{align*}
\int_0^{\pi} e^{i(n-2j)\theta} H_n(\cos \theta \mid q)(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta &= \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}} \int_0^{\pi} e^{2i(n-j-k)\theta} (e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta \\
&= \frac{\pi}{(q; q)_\infty} \sum_{k=0}^{n} \frac{(-1)^{j+k+n}(q; q)_n}{(q; q)_k(q; q)_{n-k}} \frac{1}{1 + q^{n-j-k}q^{(n-j-k)(n-j-k-1)/2}} \\
&= \frac{(-1)^{n+j}\pi}{(q; q)_\infty} q^{(n-j)(n-j-1)/2}[q_0(q^{-n}; -; q, q^{+1}) + q^{n-j}q_0(q^{-n}; -; q, q^{+j})].
\end{align*}
We evaluate the $1q_0$ functions by the $q$-binomial theorem and after some simplification we arrive at
\begin{align*}
\int_0^{\pi} e^{i(n-2j)\theta} H_n(\cos \theta \mid q)(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta \\
&= \frac{(-1)^{n+j}\pi}{(q; q)_\infty} q^{(n-j)(n-j-1)/2}[(q^{-n+j+1}; q)_n + q^{n-j}(q^{-n+j}; q)_n]. \tag{6.11}
\end{align*}
For $0 < j < n$ it is clear that the right-hand side of (6.11) vanishes. When $j = 0$, the right-hand side of (6.11) is
\[ \frac{\pi}{(q; q)_\infty} q^{n(n-1)/2} (-1)^n q^n (q^{-n}; q)_n. \]
Thus
\[ (6.12) \quad \int_0^\pi e^{i(n-2j)\theta} H_n(\cos \theta | q)(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta = \frac{\pi(q; q)_n}{(q; q)_\infty} \delta_{j,0}, \quad 0 \leq j < n. \]
This calculation establishes (6.8) when $m < n$. When $m = n$ we use (6.11) and (6.7) to obtain
\[ \int_0^\pi H_m(\cos \theta | q) H_n(\cos \theta | q)(e^{2i\theta}, e^{-2i\theta}; q)_\infty d\theta = \frac{2\pi(q; q)_n}{(q; q)_\infty}. \]

It is a fact that
\[ (6.13) \quad \lim_{q \to 1^-} \left( \frac{2}{1-q} \right)^{n/2} H_n(x \sqrt{(1-q)/2} | q) = H_n(x). \]
This can be verified using (6.1) and (6.2) and compare them with the recursive definition of the Hermite polynomials, namely
\[ (6.14) \quad H_0(x) = 1, \quad H_1(x) = 2x, \quad H_{n+1}(x) = 2xH_n(x) - H_{n-1}(x), \]
[19], [36], [41]. It is an interesting exercise to see how the orthogonality relation for \( \{H_n(x | q)\} \) tends to the orthogonality relation for \( \{H_n(x)\} \).

**Theorem 6.2.** The linearization of products of continuous $q$-Hermite polynomials is given by
\[ (6.15) \quad H_m(x | q) H_n(x | q) = \sum_{k=0}^{m \wedge n} \frac{(q; q)_m(q; q)_n}{(q; q)_k(q; q)_{m-k}(q; q)_{n-k}} H_{m+n-2k}(x | q), \]
where $m \wedge n := \min\{m, n\}$.

**Proof.** It is clear from (6.6) that $H_m(x | q) H_n(x | q)$ has the same parity as $H_{m+n}(x | q)$. Therefore there exists a sequence \( \{a_{m,n,k} : 0 \leq k \leq m \wedge n\} \) such that
\[ (6.16) \quad H_m(x | q) H_n(x | q) = \sum_{k=0}^{m \wedge n} a_{m,n,k} H_{m+n-2k}(x | q) \]
and $a_{m,n,k}$ is symmetric in $m$ and $n$. Furthermore
\[ (6.17) \quad a_{m,0,k} = a_{0,n,k} = \delta_{k,0} \]
holds and (6.7) implies

\[(6.18)\]

\[a_{m,0} = 1.\]

Multiply (6.16) by 2x and use the three-term recurrence relation (6.1) to obtain

\[
\sum_{k=0}^{(m+1)\wedge n} a_{m+1,n,k} H_{m+n+1-2k}(x | q) + (1 - q^m) \sum_{k=0}^{(m-1)\wedge n} a_{m-1,n,k} H_{m+n-2k}(x | q)
\]

which leads to

\[
\sum_{k=0}^{m\wedge n} a_{m,n,k} [H_{m+n+1-2k}(x | q) + (1 - q^{m+n-2k}) H_{m+n-1-2k}(x | q)],
\]

with \(H_{-1}(x | q) := 0\). This leads us to the system of difference equations

\[(6.19)\]

\[a_{m+1,n,k+1} - a_{m,n,k+1} = (1 - q^{m+n-2k}) a_{m,n,k} - (1 - q^m) a_{m-1,n,k},\]

subject to the initial conditions (6.17) and (6.18). When \(k = 0\) equations (6.19) and (6.18) imply

\[a_{m+1,n,1} = a_{m,n,1} + q^m(1 - q^n),\]

which leads to

\[(6.20)\]

\[a_{m,n,1} = (1 - q^n) \sum_{k=0}^{m-1} q^k = \frac{(1 - q^m)(1 - q^n)}{1 - q}.
\]

Setting \(k = 1\) in (6.19) and applying (6.20) we find

\[a_{m+1,n,2} = a_{m,n,2} + q^{m-1}(1 - q^m)(1 - q^n)(1 - q^{n-1})/(1 - q),\]

whose solution is

\[a_{m,n,2} = \frac{(1 - q^n)(1 - q^{n-1})(1 - q^m)(1 - q^{m-1})}{(1 - q)(1 - q^2)}.
\]

From here we suspect the pattern

\[a_{m,n,k} = \frac{(q; q)_m(q; q)_n}{(q; q)_{m-k}(q; q)_{n-k}(q; q)_{k}},\]

which can be proved from (6.19) by a straightforward induction. \(\square\)

The next result is the Poisson kernel

\[(6.21)\]

\[\sum_{n=0}^{\infty} \frac{H_n(\cos \theta | q) H_n(\cos \phi | q)}{(q; q)_n} t^n = \frac{(t^2; q)_\infty}{(te^{i(\theta + \phi)}, te^{i(\theta - \phi)}, te^{-i(\theta + \phi)}, te^{-i(\theta - \phi)}; q)_\infty},\]

To prove (6.21) multiply (6.15) by \(t^m s^n/(q; q)_m(q; q)_n\) and add for \(m, n \geq 0\). The result is

\[
1/(te^{i\theta}, te^{-i\theta}, se^{i\theta}, se^{-i\theta}; q)_\infty = \sum_{k=0}^{\infty} \sum_{m\geq k, n\geq k} \frac{t^m s^n H_{m+n-2k}(\cos \theta | q)}{(q; q)_{m-k}(q; q)_{n-k}(q; q)_{m+n-2k}}
\]

\[= \sum_{k=0}^{\infty} \left(\frac{ts}{q; q}_k \right) \sum_{m,n=0}^{\infty} \frac{t^m s^n H_{m+n}(\cos \theta | q)}{(q; q)_m(q; q)_n} = \frac{1}{(st; q)_\infty} \sum_{p=0}^{\infty} \frac{H_p(\cos \theta | q)}{(q; q)_p} \sum_{n=0}^{\infty} \left[ \frac{p}{n} \right] t^{p-n} s^n.
\]
Now let \( t = ue^{i\phi}, s = ue^{-i\phi} \) and use (6.5) to write the last equation as
\[
\frac{(u^2; q)_\infty}{(ue^{i(\phi+\theta)}, ue^{i(\phi-\theta)}, ue^{i(\phi-\phi)}, ue^{-i(\phi+\theta)}; q)_\infty} = \sum_{p=0}^{\infty} \frac{w^p}{(q; q)_p} H_p(\cos \theta | q) H_p(\cos \phi | q).
\]
This proves (6.21).

The Askey–Wilson operator acts on \( H_n(x | q) \) in a natural way.

**Theorem 6.3.** The polynomials \( \{ H_n(x | q) \} \) have the ladder operators
\[
(6.22) \quad D_q H_n(x | q) = \frac{2(1 - q^n)}{1 - q} q^{(1-n)/2} H_{n-1}(x | q)
\]
and
\[
(6.23) \quad \frac{1}{w(x | q)} D_q \{ w(x | q) H_n(x | q) \} = -\frac{2q^{-n/2}}{1 - q} H_{n+1}(x | q),
\]
where \( w(x | q) \) is as defined in (6.9).

**Proof.** Apply \( D_q \) to (6.4) and get
\[
\sum_{n=0}^{\infty} \frac{t^n}{(q; q)_n} D_q H_n(x | q) = \frac{2t(1 - q)}{(tq^{-1/2}e^{i\theta}, tq^{-1/2}e^{-i\theta}; q)_\infty},
\]
The above and (6.4) imply (6.22). Similarly one can prove (6.23).

Since \( (q^{-1}; q^{-1})_n = (-1)^n(q; q)_nq^{-n(n+1)/2} \), we derive
\[
(6.24) \quad H_n(\cos \theta | q^{-1}) = \sum_{k=0}^{n} \frac{(q; q)_n}{(q; q)_{n-k} (q; q)_{n-k}} q^{k(k-n)} e^{i(2k-2n)\theta}
\]
from (6.4).

**Theorem 6.4.** The polynomials \( \{ H_n(x | q^{-1}) \} \) have the generating function
\[
(6.25) \quad \sum_{n=0}^{\infty} \frac{H_n(\cos \theta | q^{-1})}{(q; q)_n} (-1)^n t^n q(n)_2 = (te^{i\theta}, te^{-i\theta}; q)_\infty.
\]

**Proof.** Insert \( H_n(\cos \theta | q^{-1}) \) from (6.24) into the left-hand side of (6.25) to see that
\[
\sum_{n=0}^{\infty} \frac{H_n(\cos \theta | q^{-1})}{(q; q)_n} (-1)^n t^n q(n)_2
\]
\[
= \sum_{n\geq k\geq 0} \frac{q^{k(k-n)+n(n-1)/2}}{(q; q)_k (q; q)_{n-k}} (-t)^n e^{i(2k-2n)\theta}
\]
\[
= \sum_{k=0}^{\infty} \frac{(-t)^k}{(q; q)_k} q^{k(k-1)/2} e^{-ik\theta} \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2}(-t)^n}{(q; q)_n} e^{in\theta}
\]
and the result follows from Euler’s theorem (4.11).
Exercises

1. Prove that the Poisson kernel (6.21) is equivalent to the linearization formula (6.15).

2. Prove that the linearization formula (6.15) has the inverse relation

\[
H_n(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(k-1)/2}}{(q; q)_k} \frac{H_{n-k}(x|q)}{(q; q)_{n-k}} \frac{H_{m-k}(x|q)}{(q; q)_{m-k}}.
\]

3. Show that if \(g_n(z) = \sum_{k=0}^{\infty} \frac{[n]}{k} q^k z^k\) then

\[
\sum_{n=0}^{\infty} t^n g_n(z)/(q; q)_n = 1/(t, tz;q)_\infty.
\]

4. Prove that

\[
H_{2n+1}(0|q) = 0, \quad \text{and} \quad H_{2n}(0|q) = (-1)^n (q; q^2)_n.
\]

5. Show that

\[
H_n((q^{1/4} + q^{-1/4})/2 | q) = (-1)^n H_n(-(q^{1/4} + q^{-1/4})/2 | q)
\]

\[
= q^{-n/4}(-q^{1/2}; q^{1/2})_n.
\]

6. Prove (6.23) using the orthogonality relation, (6.22), Theorem 2.1, and the completeness of \(\{H_n(x|q)\}\) in \(L_2[-1, 1, w(x|q)]\).

7. Prove that

\[
H_n(x|q) = \sum_{k=0}^{\infty} \frac{(-1)^k q^{k(3k-2n-1)/2}}{(q; q)_k (q; q)_{n-2k}} H_{n-2k}(x|q^{-1}).
\]

8. Prove the inverse relation

\[
H_n(x|q^{-1}) = \sum_{s=0}^{\infty} \frac{q^{-s(s-n)}(q; q)_s}{(q; q)_s (q; q)_{n-2s}} H_{n-2s}(x|q),
\]

7 The Askey–Wilson Polynomials

**Theorem 7.1.** For max \(\{|t_j| : 1 \leq j \leq 4\} < 1\), we have the Askey–Wilson q-beta integral evaluation,

\[
\int_0^\pi \frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^{4} (t_j e^{i\theta}, t_j e^{-i\theta}; q)_\infty} d\theta = \frac{2\pi (t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq j < k \leq 4} (t_j t_k; q)_\infty}.
\]
Proof. Note that each factor $1/(t_je^{i\theta}, t_{j+1}e^{-i\theta}; q)_{\infty}$ is a generating function for $q$-Hermite polynomials. Expanding these factors and using the linearization formula (6.15) we find that the left-hand side of (7.1) is

$$\frac{2\pi}{(q; q)_{\infty}} \int_0^\pi \sum_{k_1,k_2,n_1,n_2,n_3,n_4=0}^{\infty} \frac{\langle e^{2i\theta}, e^{-2i\theta}; q \rangle_{\infty} \prod_{j=1}^4 t_j^{n_j}}{H_{n_1+n_2-2k_1}(\cos \theta)} H_{n_3+n_4-2k_2}(\cos \theta) \text{d}\theta.$$ 

We now put $n_1 = k_1 + m_1$, $n_2 = k_1 + m_2$, $n_3 = k_2 + m_3$, $n_4 = k_2 + m_4$. Thus $m_1 + m_2 = m_3 + m_4 = m$, say, since the other terms when $m_1 + m_2 \neq m_3 + m_4$ vanish due to the orthogonality relation (6.8). With this notation the above expression becomes

$$\frac{2\pi}{(q; q)_{\infty}} \sum_{m_1,m_2,m_3,m_4} \frac{\langle q; q \rangle_{\infty} (q; q)_{m_1+k_1+m_1} \langle q; q \rangle_{k_2+m_2+m_1} \langle q; q \rangle_{k_3+m_3+m_2} \langle q; q \rangle_{k_4+m_4+m_3}}{(q, t_1 t_2, t_3 t_4; q)_{\infty} \prod_{m=0}^{\infty} \frac{1}{(q, q)_m} \sum_{m_1=0}^{m} \left[ \frac{m}{m_1} \right] \left[ \frac{m}{m_2} \right] \left[ \frac{m}{m_3} \right] \left[ \frac{m}{m_4} \right]}.$$

We set $t_1 = re^{-i\theta}$, $t_2 = re^{i\theta}$, $t_3 = pe^{-i\phi}$, $t_4 = pe^{i\phi}$ then use (6.5) and the Poisson kernel (6.21) to see that the above expression equals

$$\frac{2\pi(r^2p^2; q)_{\infty}}{(q, t_1 t_2, t_3 t_4; q)_{\infty} (rpe^{i(\theta+\phi)}, rpe^{i(\theta-\phi)}, rpe^{-i(\theta-\phi)}, rpe^{i(\theta-\phi)}; q)_{\infty}},$$

which is the desired result. \qed

The above evaluation is due to Ismail and Stanton [23]. Other proofs are in [16] and [4]. The original proof by R. Askey and J. Wilson in [7] is rather lengthy.

The Askey–Wilson polynomials are orthogonal with respect to the weight function whose total mass is given by (7.1). To save space we shall use the vector notation $\mathbf{t}$ to denote the ordered tuple $(t_1, t_2, t_3, t_4)$. Their weight function is

$$(7.2) \quad w(x; \mathbf{t}) = w(x; t_1, t_2, t_3, t_4; q) = \frac{\langle e^{2i\theta}, e^{-2i\theta}; q \rangle_{\infty}}{\prod_{j=1}^4 \langle t_j e^{i\theta}, t_j e^{-i\theta}; q \rangle_{\infty}} \frac{1}{\sqrt{1-x^2}},$$

$x = \cos \theta.$

**Theorem 7.2.** The Askey–Wilson polynomials are given by

$$(7.3) \quad p_n(x; \mathbf{t} | q) = t_1^{-n}(t_1 t_2, t_1 t_3, t_1 t_4; q)_{\infty} \phi_3 \left( q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_1 e^{i\theta}, t_1 e^{-i\theta}; q \right).$$

They satisfy the orthogonality relation

$$(7.4) \quad \int_{-1}^1 p_n(x; \mathbf{t} | q) p_m(x; \mathbf{t} | q) w(x; \mathbf{t} | q) \text{d}x = 2\pi \frac{(t_1 t_2 t_3 t_4 q^{2n}; q)_{\infty} \prod_{j=k} q_{\infty}}{(q^{n+1}; q)_{\infty}} \delta_{m,n}.$$
Proof. We evaluate the integral of \( p_n \) as given by (7.3) times any polynomial of degree \( \leq n \) against the weight function in (7.2). From (7.1) it is clear that for \( j \leq n \) we have

\[
\frac{t_1^n}{\prod_{s=2}^{4}(t_1 t_s; q)_n} \int_0^\pi (t_2 e^{i\theta}, t_2 e^{-i\theta}; q)_j p_n(\cos \theta; t | q) w(\cos \theta; t | q) \sin \theta \, d\theta
\]

\[
= \sum_{k=0}^{n} \frac{(q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}; q)_k}{(q, t_1 t_2, t_1 t_3, t_1 t_4; q)_k} q^{k} \int_0^1 w(x; t_1 q^k, t_2 q^k; t_3, t_4) \, dx
\]

\[
= \sum_{k=0}^{n} \frac{(q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}; q)_k}{(q, t_1 t_2, t_1 t_3, t_1 t_4; q)_k} q^{k} \frac{2\pi (q^{j+k} t_1 t_2 t_3 t_4; q)_\infty / (q; q)_\infty}{(q^{j+k} t_1 t_2, q^k t_1 t_3, q^k t_1 t_4, q^k t_2 t_3, q^k t_2 t_4, t_3 t_4; q)_\infty}
\]

\[
= \frac{2\pi (q^{j} t_1 t_2 t_3 t_4; q)_\infty / (q; q)_\infty}{(q^j t_1 t_2, q^j t_2 t_3, q^j t_2 t_4, t_1 t_3, t_1 t_4, t_3 t_4; q)_\infty} 3\phi_2 \left( \begin{array}{c}
q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_1 t_2 q^j \\
1, 1, 1, 1, 1, 1
\end{array} \right| q, q \right).
\]

We apply (4.2) to see that the sum of the \( 3\phi_2 \) is

\[
\frac{(t_3 t_4, q^{j-n+1}; q)_n}{(t_1 t_2 t_3 t_4 q^j, q^{1-n}/t_1 t_2; q)_n},
\]

which vanishes if \( j < n \). Therefore

\[
\frac{t_1^n}{\prod_{s=2}^{4}(t_1 t_s; q)_n} \int_0^\pi (t_2 e^{i\theta}, t_2 e^{-i\theta}; q)_j p_n(\cos \theta; t | q) w(\cos \theta; t | q) \sin \theta \, d\theta
\]

\[
= \frac{2\pi (q^n t_1 t_2 t_3 t_4; q)_\infty / (q; q)_\infty}{(q^n t_1 t_2, q^n t_2 t_3, q^n t_2 t_4, t_1 t_3, t_1 t_4, t_3 t_4; q)_\infty} \frac{(-t_1 t_2)^n q^{(n)}(t_3 t_4, q; q)_n}{(t_1 t_2 t_3 t_4 q^n, t_1 t_2; q)_n} \delta_{j,n}.
\]

Using

\[
(a e^{i\theta}, a e^{-i\theta}; q)_n = (-2a)^n q^{(n)} x^n + \text{lower order terms}
\]

and the explicit representation (7.3) we conclude that

\[
\int_0^\pi (t_1 e^{i\theta}, t_1 e^{-i\theta}; q)_n p_n(\cos \theta; t | q) w(\cos \theta; t | q) \sin \theta \, d\theta
\]

\[
= \frac{2\pi (q^n t_1 t_2 t_3 t_4; q)_\infty / (q; q)_\infty}{(q^n t_1 t_2, q^n t_2 t_3, q^n t_2 t_4, t_1 t_3, t_1 t_4, t_3 t_4; q)_\infty} \frac{(-t_1)^n q^{(n)}(t_3 t_4, q; q)_n}{(t_1 t_2 t_3 t_4 q^n, t_1 t_2; q)_n}.
\]

We then use the explicit form (7.3) and the above evaluation and establish the desired orthogonality relation. \( \square \)

Observe that the weight function in (7.2) and the right-hand side of (7.4) are symmetric functions of \( t_1, t_2, t_3, t_4 \). The uniqueness of the polynomials orthogonal with respect to a positive measure shows that the Askey–Wilson polynomials are symmetric in the four parameters \( t_1, t_2, t_3, t_4 \). This symmetry is the Sears transformation in the form

\[
t_1^n(t_1 t_2, t_1 t_3, t_1 t_4; q)_n 4\phi_3 \left( \begin{array}{c}
q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_1 e^{i\theta}, t_1 e^{-i\theta} \\
t_1 t_2, t_1 t_3, t_1 t_4
\end{array} \right| q, q \right)
\]

\[
t_2^n(t_2 t_1, t_2 t_3, t_2 t_4; q)_n 4\phi_3 \left( \begin{array}{c}
q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_2 e^{i\theta}, t_2 e^{-i\theta} \\
t_2 t_1, t_2 t_3, t_2 t_4
\end{array} \right| q, q \right),
\]

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Theorem 7.3. The Askey–Wilson polynomials have the generating function

\[ (7.5) \sum_{n=0}^{\infty} \frac{p_n(\cos \theta; t | q)}{(q, t_1 t_2, t_3 t_4; q)_n} t^n = 2 \phi_1 \left( \begin{array}{c} t_1 e^{i\theta}, t_2 e^{i\theta} \\ t_1 t_2 \end{array} \right) | q, t e^{i\theta} \right) t_3 e^{-i\theta}, t_4 e^{-i\theta} | q, t e^{i\theta} \right). \]

Proof. Apply (5.4) with

\[ A = t_1 e^{i\theta}, \ B = t_1 e^{-i\theta}, \ C = t_1 t_2 t_3 t_4 q^{n-1}, \ D = t_1 t_2, \ E = t_1 t_3, \ F = t_1 t_4, \]

to obtain

\[ (7.6) p_n(x; t | q) = (t_1 t_2, q^{1-n} e^{i\theta}/t_3, q^{1-n} e^{i\theta}/t_4; q)_n (t_3 t_4 q^{n-1} e^{-i\theta})^n \]

Using (2.6) we write

\[ (q^{1-n} e^{i\theta}/t_3, q^{1-n} e^{i\theta}/t_4; q)_n = e^{2n\theta} q^{n(n-1)} (t_3 t_4)^{-n} (t_3 e^{-i\theta}, t_4 e^{-i\theta}; q)_n. \]

Furthermore, if the summation index of the \(4 \phi_3\) in (7.6) is \(k\) then we may use (2.5) to get

\[ \frac{(q^{-n}, q^{1-n}/t_3 t_4; q)_k}{(q^{1-n} e^{i\theta}/t_3, q^{1-n} e^{i\theta}/t_4; q)_k} = \frac{(q, t_3 t_4; q)_n}{(q, t_3 t_4; q)_{n-k}} \frac{(t_3 e^{-i\theta}, t_4 e^{-i\theta}; q)_{n-k}}{(t_3 e^{-i\theta}, t_4 e^{-i\theta}; q)_n} (q e^{2i\theta})^{-k}. \]

Therefore

\[ \frac{p_n(x; t_1, t_2, t_3, t_4 | q)}{(q, t_1 t_2, t_3 t_4; q)_n} = \sum_{k=0}^{n} \frac{(t_1 e^{i\theta}, t_2 e^{i\theta}; q)_k}{(q, t_1 t_2; q)_k} e^{-i\theta k} \frac{(t_3 e^{-i\theta}, t_4 e^{-i\theta}; q)_{n-k}}{(q, t_3 t_4; q)_{n-k}} e^{i(n-k)\theta}. \]

It is clear that (7.7) implies (7.5) and the proof is complete. \(\square\)

The Askey–Wilson polynomials are orthogonal polynomials, hence they satisfy a three-term recurrence relation of the form

\[ (7.8) 2 x p_n(x; t | q) = A_n p_{n+1}(x; t | q) + B_n p_n(x; t | q) + C_n p_{n-1}(x; t | q). \]

The coefficient of \(x^n\) in \(p_n\) is \(2^n (t_1 t_2 t_3 t_4 q^{n-1}; q)_n\). Equating the coefficients of \(x^{n+1}\) on both sides of (7.8) we get

\[ A_n = \frac{1 - t_1 t_2 t_3 t_4 q^{n-1}}{(1 - t_1 t_2 t_3 t_4 q^{2n-1})(1 - t_1 t_2 t_3 t_4 q^{2n})}. \]
We next choose the special values \( e^{-i\theta} = t_1, t_2 \) in (7.3) and obtain

\[
p_n((t_1 + 1/t_1)/2; t | q) = (t_1 t_2, t_1 t_3, t_1 t_4; q)_n t_1^{-n},
\]

\[
p_n((t_2 + 1/t_2)/2; t | q) = (t_2 t_1, t_2 t_3, t_2 t_4; q)_n t_2^{-n}.
\]

With \( A_n \) given by (7.9), we substitute \( x = (t_j + 1/t_j)/2, j = 1, 2 \) in (7.8) and solve for \( B_n \) and \( C_n \). The result is

\[
C_n = \frac{(1 - q^n)\prod_{1 \leq j < k \leq 4}(1 - t_j t_k q^{n-1})}{(1 - t_1 t_2 t_3 t_4 q^{2n-2})(1 - t_1 t_2 t_3 t_4 q^{2n-1})},
\]

\[\text{Theorem 7.4.} \quad \text{With } z = e^{i\theta} \text{ and } x = \cos \theta \text{ we have}
\]

\[
p_n(\cos \theta; t | q) = z^n (t_1/z, t_3/z; q)_\infty \sum_{m=0}^{\infty} \frac{(q z/t_1, q z/t_3; q)_m}{(q, q z^2; q)_m} (t_1 t_3 q^n)_m
\]

\[
\times \genfrac{\langle}{\rangle}{0pt}{}{2}{2} \left( \begin{array}{c} t_1 z, t_1/z \\ t_1 t_2, t_1 q^{-m}/z \end{array} \right) _2 \genfrac{\langle}{\rangle}{0pt}{}{2}{2} \left( \begin{array}{c} t_3 z, t_3/z \\ t_3 t_4, t_3 q^{-m}/z \end{array} \right) _2 \frac{t_4, t_4 q^{-m}/z}{q}
\]

+a similar term with \( z \) and \( 1/z \) interchanged.

\text{Proof.} \quad \text{Let } F(x, t) \text{ denote the right-hand side of (7.5). Apply the } q\text{-analogue of the Pfaff–Kummer transformation, (5.9) to the } 2\phi_1 \text{’s in } F(x, t). \text{ Thus}

\[
F(x, t) = \frac{(t t_1, t t_3; q)_\infty}{(t z, t/z; q)_\infty} \sum_{k,j=0}^{\infty} (t_1 z, t_1/z; q)_k (t_3 z, t_3/z; q)_j
\]

\[
\times q^{(j)_2 + (k)_2} (-t t_4)^j (-t t_2)^k.
\]

Cauchy’s theorem shows that

\[\text{(7.12)} \]

\[
\frac{p_n(\cos \theta; t | q)}{(q, t_1 t_2, t_3 t_4; q)_n} = \frac{1}{2\pi i} \int_C F(x, t) t^{-n-1} dt,
\]

where \( C \) is the contour \( \{ t : |t| = r \} \), with \( r < |e^{-i\theta}| = 1/|z| \). Now think of the contour \( C \) as a contour around the point \( t = \infty \) with the wrong orientation, so it encloses all the poles of \( F(x, t) \). Therefore the right-hand side of (7.12) is \(- \sum \text{ Residues. Now } t = 0 \text{ is outside the contour and the singularities of } F \text{ inside the contour are } t = q^{-m} z^\pm, m = 0, 1, \ldots \). It is straightforward to
see that

\[ \text{Res}\{F(x,t) : t = zq^{-m}\} = -\frac{(q^{-m}zt_1,q^{-m}z^3;q)_\infty}{(q^{-m};q)_m(q,q^{-m}z^2;q)_\infty}(zq^{-m})^{-n} \]

\[ \times \sum_{k,j=0}^{\infty} \frac{(t_1z,t_1/z;q)_j(t_2z,t_2/z;q)_j}{(q,t_1t_2,t_1zq^{-m};q)_k(q,t_3t_4,t_2zq^{-m};q)_k} q^{(j)(j)}(-t_4)^j(-t_2)^k(q^{-m}z)^{j+k} \]

\[ = -z^{-n}(zt_1,zt_3;q)_\infty(q/zt_1,q/zt_3;q)_m(t_1t_3q^n)_m \]

\[ \times \phi_2 \left( t_1z,t_1/z \mid q,t_2zq^{-m} \right) 2\phi_2 \left( t_3z,t_3/z \mid q,t_4zq^{-m} \right). \]

For the residue at \( t = q^{-m}/z \) replace \( z \) by \( 1/z \), and we establish the theorem. \( \square \)

Observe that the series (7.11) is both an explicit formula and an asymptotic series for large \( n \).

8 Ladder Operators and Rodrigues Formulas

**Theorem 8.1.** The Askey–Wilson polynomials satisfy the lowering and raising relations

\[ (8.1) \quad \mathcal{D}_q p_n(x; t \mid q) = 2 \frac{(1 - q^n)(1 - t_1t_2t_3t_4q^{n-1})}{(1 - q)(q^{n-1}q)} p_{n-1}(x; q^{1/2}t \mid q) \]

\[ (8.2) \quad \frac{2q^{(1-n)/2}}{q-1} p_n(x; t \mid q) = \frac{1}{w(x; t \mid q)} \mathcal{D}_q \left[ w(x; q^{1/2}t \mid q)p_{n-1}(x; q^{1/2}t \mid q) \right], \]

respectively, where \( w(x; t \mid q) \) is defined in (7.2).

**Proof.** The lowering relation (8.1) follows from (3.5) and the representation (7.3). To prove (8.2) we use (8.1) and Theorem 2.1 to write the orthogonality relation (7.4) in the form

\[ 2 \int_0^1 p_m(x; q^{1/2}t \mid q) w(x; q^{1/2}t \mid q) \mathcal{D}_q p_{n+1}(x; t \mid q) dx \]

\[ = -\int_0^1 p_{n+1}(x; t \mid q) \mathcal{D}_q \left[ w(x; q^{1/2}t \mid q)p_m(x; q^{1/2}t \mid q) \right] dx. \]

In the last step we used integration by parts for the Askey–Wilson operator (Theorem 2.1) and the fact that the boundary term vanishes since the weight function, as a function of \( z \), vanishes when \( z = q^{\pm 1/2} \). Therefore

\[ \frac{1}{w(x; tq)} \mathcal{D}_q \left[ w(x; q^{1/2}t \mid q)p_m(x; q^{1/2}t \mid q) \right] \]

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must be a constant multiple of $p_{m+1}(x; t|q)$ because the Askey–Wilson polynomials are complete in $L_2[-1, 1, w(x; t|q)]$. From the orthogonality relation we find that the sought multiple is $2q^{-m/2}/(q-1)$ and (8.2) follows.

By iterating (8.2) one derive the Rodrigues type formula

\[(8.3)\quad w(x; t|q) p_n(x; t|q) = \left(\frac{q-1}{2}\right)^n q^{n(n-1)/4} D^n_q \left[w(x; q^{n/2} t|q)\right].\]

Combining (8.1) and (8.2) we conclude that the Askey–Wilson polynomials solve the second order Sturm–Liouville equation

\[(8.4)\quad \frac{1}{w(x; t|q)} D_q \left[w(x; q^{1/2} t|q) D_q y\right] = \frac{4q}{(1-q)^2} (1-q^{-n})(1-t_1 t_2 t_3 t_4 q^{n-1}) y.\]

**Theorem 8.2.** The Askey–Wilson polynomials have the $8W_7$ representation

\[(8.5)\quad p_n(\cos \theta; t|q) = \frac{1}{(1/z^2; q)_n} 8W_7(q^{-n} z^2; q^{-n}, t_1 z, t_2 z, t_3 z, t_4 z; q, q^{2-n}/t_1 t_2 t_3 t_4).\]

**Proof.** From the Rodrigues formula (8.3) and Cooper’s formula (2.29) and with $z = e^{i \theta}$ we see that

\[
\begin{align*}
p_n(x; t|q) &= \sum_{k=0}^{n} \binom{n}{k} q^k (1-z^2)^{n-k} q^{n/2} (t_j ; q)_k / (q^2; q)_n, \\
&= \sum_{k=0}^{n} \binom{n}{k} q^k (1-z^2)^{n-k} q^{n/2} (t_j ; q)_k / (q^2; q)_n.
\end{align*}
\]

After routine manipulations we arrive at the representation (8.5). \qed

Note that the representation (8.5) exhibits the symmetry of the Askey–Wilson polynomials under permutations of $(t_1, t_2, t_3, t_4)$ but does not show the polynomial nature of $p_n(x; t|q)$.

**Theorem 8.3.** We have the Watson transformation

\[(8.6)\quad 8\phi_7 \begin{pmatrix} a, qa^{1/2}, -qa^{1/2}, b, c, d, e, q^{-n} \\ a^{1/2}, -a^{1/2}, qa/b, qa/c, qa/d, qa/e, qa^{n+1} \end{pmatrix} = 8W_7(q, a^2 q^{n+2} bcde) \]

\[
\begin{pmatrix} q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_1 e^{i \theta}, t_1 e^{-i \theta} \\ t_1 t_2, t_1 t_3, t_1 t_4 \end{pmatrix}.
\]

**Proof.** We equate the right-hand sides of the representations of $p_n(x; t|q)$ in (7.3) and (8.5) and we arrive at the representation

\[(8.7)\quad p_n(\cos \theta; t|q) = t_1^{-n}(t_1 t_2, t_1 t_3, t_1 t_4; q)_n 4\phi_3 \begin{pmatrix} q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}, t_1 e^{i \theta}, t_1 e^{-i \theta} \\ t_1 t_2, t_1 t_3, t_1 t_4 \end{pmatrix}.
\]

To reach the form (8.6) we apply the Sears transformation (5.4) with $A = t_1 z, D = t_1 t_4$. \qed
We now evaluate certain integrals involving $q$-functions. Our first result is an integral representation for a $\phi_5$ function.

**Theorem 8.4.** If $|t_j| < 1$ for $1 \leq j \leq 6$ then

$$
\int_0^\pi \prod_{j=5}^6 2\phi_1(t_{1j}e^{i\theta}, t_{2j}e^{i\theta} \mid q, t_{j}e^{-i\theta}) 2\phi_1(t_{3j}e^{-i\theta}, t_{4j}e^{-i\theta} \mid q, t_{j}e^{i\theta}) \\
\frac{(e^{2i\theta}, e^{-2i\theta}; q)_\infty}{\prod_{j=1}^6 (t_{j}e^{i\theta}, t_{j}e^{-i\theta}; q)_\infty} d\theta = \frac{2\pi (t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq j < k \leq 4} (t_{j}t_{k}; q)_\infty} 6\phi_5(q, t_{1j} t_{2j} t_{3j} t_{4j}; q)_\infty
$$

Proof. Multiply the orthogonality relation (7.4) by

$$
\frac{t_{m}^n t_{n}^m}{(q; t_{1j} t_{2j} t_{3j} t_{4j}; q)_m (q, t_{1j} t_{2j} t_{3j} t_{4j}; q)_n}
$$

and add for all $m, n \geq 0$ and then make use of (7.5). \qed

The next integral to consider is the Nassrallah–Rahman integral in the following theorem.

**Theorem 8.5.** We have the Nassrallah–Rahman integral evaluation

$$
\int_0^\pi \prod_{j=1}^5 (e^{2i\theta}, e^{-2i\theta}; q)_\infty (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty d\theta = \frac{2\pi (t_1 t_2 t_3 t_4; q)_\infty \prod_{j=2}^4 (at_j; q)_\infty}{(q, \alpha t_2 t_3 t_4; q)_\infty \prod_{1 \leq j < k \leq 5} (t_{j}t_{k}; q)_\infty} \times \sum_{k=0}^{n} \frac{(q; q)_n}{(\alpha t_4, t_1 t_2 t_3 t_4; q)_k (q; q)_n} \frac{(\alpha/t_4)_k}{\phi_3(q; q)_n (q; q)_n (q; q)_n (q; q)_n (q; q)_n} (t_{j}t_{k}; q)_\infty
$$

when $|t_j| < 1; 1 \leq j \leq 5$.

The proof is based on the following lemma.

**Lemma 8.6.** We have the evaluation

$$
\int_0^\pi \prod_{j=1}^4 (e^{2i\theta}, e^{-2i\theta}; q)_\infty (\alpha e^{i\theta}, \alpha e^{-i\theta}; q)_\infty d\theta = \frac{2\pi \alpha/t_4, (at_4; q)_n (t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq j < k \leq 4} (t_{j}t_{k}; q)_\infty} 4\phi_3(q; q)_n \\
\frac{(q; q)_n}{(\alpha t_4, t_1 t_2 t_3 t_4; q)_k (q; q)_n} \frac{(\alpha/t_4)_k}{\phi_3(q; q)_n (q; q)_n (q; q)_n (q; q)_n (q; q)_n} (t_{j}t_{k}; q)_\infty
$$

where $|t_j| < 1, 1 \leq j \leq 4$. 29
Proof. Denote the Askey–Wilson integral in (7.1) by \( I(t_1, t_2, t_3, t_4) \). Apply (4.3) with \( b = \alpha \) and \( a = t_4 \) to see that the extreme left-hand side of (8.10) is

\[
\sum_{k=0}^{n} \frac{(q, \alpha t_4; q)_{n} (\alpha/t_4)_k}{(\alpha t_4, q; q)_{k} (q; q)_{n-k}} (\alpha/t_4; q)_{n-k} I(t_1, t_2, t_3, q^k t_4) = 2\pi (q, \alpha t_4; q)_{n} (t_1 t_2 t_3 t_4; q)_{\infty} \prod_{1 \leq j < k \leq 4} (t_j t_k; q)_{\infty} \sum_{k=0}^{n} \frac{(t_1 t_4, t_2 t_4, t_3 t_4; q)_{k} (\alpha/t_4; q)_{n-k}}{(q, \alpha t_4, t_1 t_2 t_3 t_4; q)_{k} (q; q)_{n-k}} (\alpha/t_4)_k,
\]

and the lemma follows.

Proof of (8.9). We first take \( t_5 = \alpha q^n \) and apply Lemma 8.6. Apply the Watson transformation (8.6) to the \( 4\phi_3 \) in Lemma 8.6 with the choices:

\[
aq = \alpha t_2 t_3 t_4, \quad b = t_2 t_3, \quad c = \alpha/t_1, \quad d = t_2 t_4, \quad e = t_3 t_4.
\]

This establishes the theorem when \( \alpha = t_5 q^n \). Since both sides of (8.9) are analytic functions of \( \alpha \) the identity theorem for analytic functions establishes the result.

Note that if we did not know the right-hand side of (8.9) we would have discovered it by replacing \( q^n \) in our calculations by \( t_5/\alpha \), then continue the argument given in the proof. It is also important to note that the left-hand side of (8.9) is symmetric under interchanging \( t_j \) and \( t_k \) for any \( 1 \leq j, k \leq 5 \). The right-hand side is obviously symmetric under \( t_1 \leftrightarrow t_5 \). The symmetry under \( t_i \leftrightarrow t_j \) gives transformation formulas for the \( sW_7 \) functions.

We next solve the connection coefficient problem for the Askey–Wilson polynomials.

**Theorem 8.7.** The Askey–Wilson polynomials have the connection relation

\[
p_n(x; b|q) = \sum_{k=0}^{n} c_{n,k}(a, b) p_k(x; a|q),
\]

where

\[
c_{n,k}(a, b) = \frac{(q; q)_{n} (b_{4}^{n-k} b_1 b_2 b_3 b_4 q^{n-1}; q)_k (b_1 b_4, b_2 b_4, b_3 b_4; q)_{n-k} (t_1 t_2 t_3 t_4 q^{k-1}; q)_k (b_1 b_4, b_2 b_4, b_3 b_4; q)_{k}}{(q; q)_{n-k} (q, t_1 t_2 t_3 t_4 q^{k-1}; q)_k (b_1 b_4, b_2 b_4, b_3 b_4; q)_{k}} \times q^{k(k-n)} \sum_{j,l \geq 0} \frac{(q^{k-n}, b_1 b_2 b_3 b_4 q^{n+k-1}, t_3 b_4 q^{k}; q)_{j+l} q^{j+l}}{(b_1 b_4 q^k, b_2 b_4 q^k, b_3 b_4 q^k; q)_{j+l}(q; q)_{j+l}(q; q)_{j+l}} \times \frac{(a_1 a_4 q^k, a_2 a_4 q^k, a_3 a_4 q^k; q)_j (b_4/a_4 q^k; q)_l}{(a_1 b_4 q^k, a_1 a_2 a_3 a_4 q^k; q)_l} \left( \frac{b_4}{t_4} \right)^l.
\]

Proof. Denote the coefficient of \( \delta_{m,n} \) in (7.4) by \( h_n(t) \). The coefficients \( c_{n,k} \) are given by

\[
h_k(a)c_{n,k}(a, b) = \left( \sqrt{1 - x^2} p_n(x; b|q), w(x; a|q)p_k(x; a|q) \right).
\]
where \((f, g)\) is the inner product \(\int_{-1}^{1} f(x)g(x)(1-x^2)^{-1/2} \, dx\), see (2.23). We use the Rodrigues formula (8.3) and the integration by parts formula (2.24) to find

\[
h_k(a)c_{n,k} = \left[ \frac{q-1}{2} \right]^k q^{k(k-1)/4} \left\langle \sqrt{1-x^2} p_n(x; b|q), D_q^k w(x; q^{k/2}a|q) \right\rangle
\]

\[
= \left[ \frac{1-q}{2} \right]^k q^{k(k-1)/4} \left\langle D_q^k p_n(x; b|q), \sqrt{1-x^2} w(x; q^{k/2}a|q) \right\rangle
\]

\[
= q^{k(k-n)/2} (b_1 b_2 b_3 b_4 q^{n-1}; q)_k (q|q)_n
\]

\[
\times \left\langle p_{n-k}(x; q^{k/2}|b|q), \sqrt{1-x^2} w(x; q^{k/2}a|q) \right\rangle
\]

\[
= b_1^{-k-n} (b_1 b_2 b_3 b_4 q^{n-1}; q)_k (b_1 b_4 q^k, b_2 b_4 q^k, b_3 b_4 q^k; q)_{n-k}
\]

\[
x q^{k(k-n)} (q|q)_n \sum_{j=0}^{n-k} \frac{(q^{k-n}, b_1 b_2 b_3 b_4 q^{n+k-1}; q)_j}{(q, b_1 b_4 q^k, b_2 b_4 q^k, b_3 b_4 q^k; q)_j} q^j
\]

\[
\times \left\langle \phi_j(x; b_4 q^{k/2}), \sqrt{1-x^2} w(x; q^{k/2}a|q) \right\rangle.
\]

In the above steps we applied Lemma 2.1 repeatedly and used the fact that the boundary terms vanish since the weight function vanished when \(z = q z_j/2\) for any nonnegative integer \(j\). Using Lemma 8.6 we see that the \(j\)-sum is

\[
\frac{2\pi (t_1 t_2 t_3 t_4 q^{2k}; q)_{\infty}}{(q|q)_1} \prod_{1 \leq r < s \leq 4} (t_r t_s q^k; q)_{\infty} \sum_{j=0}^{n-k} \frac{(q^{k-n}, b_1 b_2 b_3 b_4 q^{n+k-1}, t_4 b_4 q^k; q)_j}{(b_1 b_4 q^k, b_2 b_4 q^k, b_3 b_4 q^k; q)_j}
\]

\[
\times \sum_{l=0}^{j} \frac{(a_1 a_4 q^k, a_2 a_4 q^k, a_3 a_4 q^k; q)_l}{(q, a_4 b_4 q^k, a_1 a_2 a_3 a_4 q^{2k}; q)_l} \frac{(b_4/a_4 q)_{j-l}}{(q|q)_{j-l}} \left( \frac{b_1}{t_4} \right)^l,
\]

and after some manipulations one completes the proof. \(\square\)

The special case \(a_4 = b_4\) is worth noting.

**Corollary 8.8.** (Askey and Wilson [7]) We have the connection relation

\[
p_n(x; b_1, b_2, b_3, a_4|q) = \sum_{k=0}^{n} d_{n,k} p_k(x; a_1, a_2, a_3, a_4|q),
\]

where

\[
d_{n,k} = \frac{a_1^{k-n}(b_1 b_2 b_3 a_4 q^{n-1}; q)_k (q, b_1 a_4, b_2 a_4, b_3 a_4; q)_n q^{k(n-k)}}{(q|q)_{n-k}(q, a_1 a_2 a_3 a_4 q^{k-1}; q)_k (b_1 b_4, b_2 b_4, b_3 b_4; q)_k}
\]

\[
\times 5 \phi_4 \left( \begin{array}{c} q^{k-n}, b_1 b_2 b_3 a_4 q^{n+k-1}, a_1 a_4 q^k, a_2 a_4 q^k, a_3 a_4 q^k \\ b_1 a_4 q^k, b_2 a_4 q^k, b_3 a_4 q^k, a_1 a_2 a_3 a_4 q^{2k} \end{array} \right| q, q \right).
\]

The proof follows because the terms in the double series in (8.11) vanish unless \(j = 0\) so the double sum reduces to a \(5 \phi_4\).
The Askey–Wilson polynomials contain many special and limiting cases. For details see the Askey scheme in [31]. We will only mention the Al-Salam–Chihara polynomials introduced by W. Al-Salam and T. S. Chihara in [1]. Their weight function was first found in [6]. Al-Salam–Chihara polynomials correspond to the case \( t_3 = t_4 = 0 \) in the Askey–Wilson polynomials. They are defined by

\[
(9.1) \quad p_n(x; t_1, t_2) = p_n(x; t_1, t_2, 0, 0|q) = t_1^{-n}(t_1 t_2; q)_n 3\phi_2 \left( \begin{array}{c} q^{-n}, t_1 e^{i\theta}, t_1 e^{-i\theta} \\ t_1 t_2, 0 \end{array} \bigg| q, q \right).
\]

Their generating function is

\[
(8.17) \quad \sum_{n=0}^\infty p_n(\cos \theta; t_1, t_2 \bigg| q) = (t_1 t_2; q)_\infty\frac{t^n}{(q; q)_n}.
\]

9 Identities and Summation Theorems

We next derive the sum of a very well-poised \( 6\phi_5 \) series.

**Theorem 9.1.** (Rogers) The sum of a very well-poised \( 6\phi_5 \) series is

\[
(9.1) \quad 6\phi_5 \left( \begin{array}{c} A, qA^{1/2}, -qA^{1/2}, B, C, D \\ A^{1/2}, -A^{1/2}, qA/B, qA/C, qA/D \end{array} \bigg| q, qA \bigg/ BCD \right) = \frac{(q A, q A/BC, q A/BD, q A/CD; q)\infty}{(q A, q A/B, q A/C, q A/D, q A/BCD; q)\infty}.
\]

and when \( D = q^{-n} \) is

\[
(9.2) \quad 6\phi_5 \left( \begin{array}{c} A, qA^{1/2}, -qA^{1/2}, B, C, q^{-n} \\ A^{1/2}, -A^{1/2}, qA/B, qA/C, q^{n+1} A \end{array} \bigg| q, q^{n+1} A \bigg/ BC \right) = \frac{(q A, q A/BC; q)\infty}{(q A, q A/B, q A/C; q)\infty}.
\]

Proof. With \( z = e^{i\theta} \) is straightforward to check that

\[
D_q(a z, a/z; q)\infty = \frac{2b(1-a/b)}{[q^{1/2} - q^{-1/2}]} \frac{(aq^{1/2}z, aq^{1/2}/z; q)\infty}{(bq^{-1/2}z, bq^{-1/2}/z; q)\infty}.
\]

Therefore

\[
D_q^n(a z, a/z; q)\infty = \frac{2^n b^n (a/b; q)_n q^{-n(n-1)/4}}{(q^{1/2} - q^{-1/2})^n} \frac{(aq^{n/2}z, aq^{n/2}/z; q)\infty}{(bq^{-n/2}z, bq^{-n/2}/z; q)\infty}.
\]

Using (2.29) we see that the left-hand side of the above equation is

\[
\frac{2^n q^{n(1-n)/4}}{(q^{1/2} - q^{-1/2})^n} \sum_{k=0}^n \binom{n}{k} q^{k(n-k) - 2k - n} (aq^{-k+n/2}z, aq^{k-n/2}/z; q)\infty.
\]
We note that
\[
(bq^{-n/2}z, bq^{-n/2}/z; q)_{\infty} (aq^{-k+n/2}z, aq^{k-n/2}/z; q)_{\infty}
\]
\[
(aqn^2z, aqn^2/z; q)_{\infty} (bq^{-k+n/2}z, bq^{k-n/2}/z; q)_{\infty}
\]
\[
= (bq^{-n/2}z, bq^{-n/2}/z; q)^n (aq^{-k+n/2}z; q)_{k} (aq^{k-n/2}/z; q)_{n-k}
\]
\[
(bq^{-k+n/2}z; q)_{k} (bq^{k-n/2}/z; q)_{n-k}.
\]
After some simplifications we find that
\[
\frac{b^n(a/b, q^{1-n}/z^2; q)_n}{(bq^{-n/2}z, bq^{-n/2}/z; q)_n} = 6\phi_5 \left( z^{n}\frac{q^n a}{b} \right)
\]
which is equivalent to (9.2). Now both sides of (9.1) are analytic in $1/D$ for $|1/D| < 1$ and are equal when $1/D = q^n$ by (9.2), hence are equal for all $D$, $|1/D| < 1$. We then analytically continue the result for all $D$ for which both sides are well-defined.

Another proof of (9.2) follows by choosing $d = \lambda/c$ in Theorem 8.3 and then letting $c \to 0$.

**Theorem 9.2.** (Bailey) We have the $6\psi_6$ sum
\[
6\psi_6 \left( qa^{1/2}, -qa^{1/2}, b, c, d, e \atop a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e \right) \frac{q^n a}{bcde} = (qa, qa/bc, qa/bd, qa/be, qa/cd, qa/ce, qa/de, q, q/a; q)_{\infty}
\]
\[
(qa/b, qa/c, qa/d, qa/e, q/b, q/c, q/d, q/e, qa^2/bcde; q)_{\infty}
\]

**Proof.** Note that both sides of (9.3) are analytic in $z := qa/e$. If $z = q^{m+1}$, then $e = qa^{-m}$. Using (2.4) we see that the sum in the $6\psi_6$ is now over all $n, n \geq -m$. Thus the $6\psi_6$ becomes
\[
\sum_{n=-m}^{\infty} \frac{(qa^{1/2}, -qa^{1/2}, b, c, d, e; q)_{n}}{(a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, aq/e; q)_{n}} \left( \frac{q^{n+1} a}{bcde} \right)^n
\]
\[
= \sum_{n=0}^{\infty} \frac{(qa^{1/2}, -qa^{1/2}, b, c, d, q^{-m}; q)_{n-m}}{(a^{1/2}, -a^{1/2}, aq/b, aq/c, aq/d, q^{m+1}; q)_{n-m}} \left( \frac{q^{n+1} a}{bcde} \right)^{n-m}
\]
\[
\times 6\phi_5 \left( qa^{-2m}, q^{-m}a^{1/2}, -q^{-m}a^{1/2}, bx^{-m}, cq^{-m}, dq^{-m} \atop qa^{1/2}, -qa^{1/2}, aq/b, aq/c, aq/d, q^{m+1}; q \right)_{-m}
\]
\[
= \left( \frac{bcde}{q^{m+1} a} \right)^m \left( \frac{q^{1-2m} a, qa/bc, qa/bd, qa/ce; q)_{\infty}}{(qa/b, qa/c, qa/d, q^{m+1} a/bcde; q)_{\infty}} \right)_{-m}
\]
\[
= \left( \frac{q^{1-2m} a, qa^{1/2}, -qa^{1/2}; q)_{-m}}{(a^{1/2}, -a^{1/2}, q^{m+1}; q)_{-m}} \right) = \left( \frac{q^{1-2m} a; q)_{\infty}}{(a^{1/2}, -a^{1/2}, q^{m+1}; q)_{-m}} \right) = \left( \frac{q^{1-2m} a; q)_{\infty}}{(a^{1/2}, -a^{1/2}, q^{m+1}; q)_{-m}} \right).
This shows that the $q$-$\psi_0$ is

$$\frac{(qa/bc, qa/bd, qa/cd; q)_\infty}{(qa/b, qa/c, qa/d, q^{m+1}a/bcd; q)_\infty} \frac{(qa/a; q)_\infty}{(q/b, q/c, q/d; q)_m},$$

which, with $e = aq^{-m}$, takes the form

$$\frac{(q, qa/bc, qa/bd, qa/cd; q)_\infty}{(qa/b, qa/c, qa/d, qa/e, qa^2/bcd; q)_\infty} \frac{(q, qa/a, qa/be, qa/cd; q)_\infty}{(q/b, q/c, q/d, q/e; q)_\infty}.$$

This completes the proof. \hfill $\Box$

10 Expansions

This section is mostly based on [20] and [26]. We first expand an Askey-Wilson basis in terms of Askey-Wilson polynomials.

**Proposition 10.1.** For any non-negative $n$,

$$(be^{i\theta}, be^{-i\theta}; q)_n = \sum_{k=0}^{n} f_{n,k}(b, t)p_k(x; t|q),$$

where

$$f_{n,k}(b, t) = \frac{(-b)^k q^{k(1/2)}(q; q)_n(b/t^4, bt^4 q^k; q)_{n-k}}{(q, t^4 t^3 t^4 q^{k-1}; q)_{k} (q; q)_{n-k}} \times_{4}\phi_3 \left( q^{k-n}, t^4 q^k, t^4 q^k, t^4 q^k; \frac{bt^4 q^k}{q}, \frac{1 + k^{1-k} t^4/b}{q}, q, q \right),$$

**Proof.** Denote the coefficient of $\delta_{m,n}$ in (7.4) by $h_k(t)$. It is clear that

$$f_{n,k} h_k(t) = \langle p_k(x; t|q)w(x; t|q), \sqrt{1 - x^2(\text{be}^{i\theta}, \text{be}^{-i\theta}; q)_n} \rangle$$

$$= \left( \frac{q - 1}{2} \right)^k q^{k(k-1)/4} C_q^{k} w(x; q^{k/2} t|q), \sqrt{1 - x^2(\text{be}^{i\theta}, \text{be}^{-i\theta}; q)_n}$$

$$= \left( \frac{1 - q}{2} \right)^k q^{k(k-1)/4} \int_{-1}^{1} w(x; q^{k/2} t|q) D^{k}_q(\text{be}^{i\theta}, \text{be}^{-i\theta}; q)_n dx$$

$$= \frac{(-b)^k (q; q)_n q^{k(1/2)}}{(q; q)_{n-k}} \int_{-1}^{1} (b q^{k/2} e^{i\theta}, b q^{k/2} e^{-i\theta}; q)_{n-k} w(x; q^{k/2} t|q) dx.$$

In the above steps we used the Rodrigues formula (8.3), Theorem 2.1, and (3.5). The result follows from Lemma 8.6. \hfill $\Box$

The special case $b = t_1$ of Proposition 10.1 is interesting.

$$(t_1 e^{i\theta}, t_1 e^{-i\theta}; q)_n = \sum_{k=0}^{n} \left[ \frac{n}{k} \right] (-1)^k q^{k(1/2)} \left( \frac{t_1 t_2, t_1 t_3, t_1 t_4; q)_n}{t_1 t_2, t_1 t_3, t_1 t_4; q)_k \right) \frac{1 - t_1 t_2 t_3 t_4^{2k-1}}{1 - t_1 t_2 t_3 t_4/q} \times \frac{(t_1 t_2 t_3 t_4; q)_k}{(t_1 t_2 t_3 t_4; q)_{n+k}} p_k(x; t|q),$$

where $n \in \mathbb{N}, x \in \mathbb{R}, t \in \mathbb{R}$, and $q \in (0, 1)$. The proof follows from the general result. \hfill $\Box$
We have the following expansion

\[ \sum_{n=0}^{\infty} c_n(u, t, a)p_n(x; a|q) = \frac{(ue^{i\theta}, ue^{-i\theta}; q)_{\infty}}{(te^{i\theta}, te^{-i\theta}; q)_{\infty}}, \]

when \( \max\{|a_1|, \ldots, |a_4|, |t|\} < 1 \), where

\[ c_n(u, t, a) = \frac{t^n(a_1 a_2 a_3 t; q)_{\infty}}{(q, q^{n-1} a_1 a_2 a_3 q; q)_{\infty} (a_1 t, a_2 t, a_3 t, q b_1; q)_{\infty} \times_{\phi_7} (b_1, q b_1^{1/2}, -q b_1^{1/2}, q^n a_1 a_2, q^n a_1 a_3, q^n a_2 a_3, u/a_4, q^n u/t t^{1/2}, -q^{n-1} a_1 u, q^n a_2 u, q^n a_3 u, q^n a_2 a_3 a_4, q^n a_1 a_2 a_3 t \mid q, a_4 t)} \]

with \( b_1 = q^{2n-1} a_1 a_2 a_3 u \). In particular,

\[ c_n(0, t, a) = \frac{t^n(a_1 a_2 a_3 t; q)_{\infty}}{(q, q^{n-1} a_1 a_2 a_3 q; q)_{\infty} (a_1 t, a_2 t, a_3 t; q)_{\infty} \times_{\phi_2} (q^n a_1 a_2, q^n a_1 a_3, q^n a_2 a_3 | q, a_4 t)} \]

**Theorem 10.3.** We have the expansion

\[ \sum_{n=0}^{\infty} \frac{(t_4 z, t_4/z; q)_n}{(q; q)_n} \Lambda_n \zeta^n = \sum_{k=0}^{\infty} p_k(x; t|q) \frac{(-t_4 \zeta q^{k} z^{k})}{(q, t_1 t_2 t_3 t_4 q^{k-1}; q)_k} \sum_{n=0}^{\infty} \frac{\Lambda_{n+k}}{(q, t_1 t_2 t_3 t_4 q^{2k}; q)_n} \zeta^n. \]

**Proof.** Interchange \( t_1 \) and \( t_4 \) in (10.3) then multiply the result by \( \Lambda_n \zeta^n/(q; q)_n \) and sum over \( n \). \( \Box \)

When the \( \Lambda_n \) is a quotient of products of \( q \)-shifted factorials we establish the following corollary.

**Corollary 10.4.** We have the following expansion

\[ p+1 \phi_p \left( \frac{a_1, \ldots, a_p-1, t_4 z, t_4/z}{t_1 t_4, t_2 t_4, t_3 t_4, b_1, \ldots, b_{p-3}} \mid q, \zeta \right) = \sum_{k=0}^{\infty} p_k(x; t|q) \frac{(a_1, \ldots, a_p-1; q)_k}{(t_1 t_4, t_2 t_4, t_3 t_4, b_1, \ldots, b_{p-3}; q)_k} \times \frac{(-t_4 \zeta q^{k} z^{k})}{(q, t_1 t_2 t_3 t_4 q^{k-1}; q)_k} p^{-1} \phi_{p-2} \left( \frac{q^k a_1, \ldots, q^k a_{p-1}}{q^k b_1, \ldots, q^k b_{p-3}, t_1 t_2 t_3 t_4 q^{2k}} \mid q, \zeta \right). \]

The special case \( p = 2 \) and a special choice of the parameters lead to the following theorem.
Theorem 10.5. We have the expansion formula

\[ _{3}\Phi_{2}\left(\begin{array}{c} A, b z, b / z \\ b t_{4}, B \\
\end{array}q, B t_{4} \bigg| \frac{b A}{b A} \right) \]

\[ = \frac{(B / A, B t_{4} / b q)}{(B, B t_{4} / b A ; q)_{\infty}} \sum_{k=0}^{\infty} \frac{(A ; q)_{k}}{(b t_{4}, B t_{4} / b q)_{k}} \left(\frac{B t_{4}}{b A}\right)^{k} \frac{(-b)^{k} q^{(k)}_{2}}{(q, t_{1} t_{2} t_{3} t_{4} q^{k-1} ; q)_{k}} \times_{4} \phi_{3}\left(\begin{array}{c} A q^{k}, t_{1} t_{4} q^{k}, t_{2} t_{4} q^{k}, t_{3} t_{4} q^{k} \\ b t_{4} q^{k}, t_{1} t_{2} t_{3} t_{4} q^{2 k} ; q, B t_{4} / b \end{array} q, B / A \right) p_{k}(x ; t | q). \]

Theorem 10.5 appeared as Theorem 2.9 in [26]. Another result from [26] follows directly from (10.3).

Proposition 10.6. We have the expansion

\[ \sum_{n=0}^{\infty} \frac{\Lambda_{n}}{(q, b t_{4} ; q)_{n}} (b z, b / z ; q)_{n} = \sum_{k=0}^{\infty} p_{k}(x ; t | q) \frac{(-b)^{k} q^{(k)}_{2}}{(q, b t_{4}, t_{1} t_{2} t_{3} t_{4} q^{k-1} ; q)_{k}} \times \sum_{s=0}^{N} \frac{(t_{1} t_{4} q^{k}, t_{2} t_{4} q^{k}, t_{3} t_{4} q^{k} ; q)_{s}}{(q, b t_{4} q^{k}, t_{1} t_{2} t_{3} t_{4} q^{2 k} ; q)_{s}} \left(\frac{b}{t_{4}}\right)^{s} \sum_{n=0}^{\infty} \frac{(b / t_{4} ; q)_{n}}{(q, q)_{n}} \Lambda_{n+k+s}. \]

Another choice for \( \Lambda_{n} \) in Proposition 10.6 is

\[ \Lambda_{n} = \frac{(q^{-N}, A ; q)_{n}}{(q, t_{4}^{-N} A b / B t_{4} ; q)_{n}} q^{n}. \]

This time the \( n \)-sum is evaluable by the \( q \)-Pfaff-Saalschütz theorem. The result is the following.

Theorem 10.7. The expansion of a general terminating \( _{4}\Phi_{3} \) in the Askey-Wilson polynomials is given by

\[ _{4}\Phi_{3}\left(\begin{array}{c} q^{-N}, A, b z, b / z \\ b t_{4}, B, b A q^{1-N} / B t_{4} \\
\end{array}q, q \right) \]

\[ = \frac{(B / A, B t_{4} / b q)}{(B, B t_{4} / b A ; q)_{N}} \sum_{k=0}^{N} \frac{(-t_{4})^{k} q^{(k+1)}_{2}}{(q, b t_{4}, t_{1} t_{2} t_{4} q^{k-1} ; B t_{4} / b, q^{1-N} A / B ; q)_{k}} p_{k}(x ; t | q) \times_{5} \phi_{4}\left(\begin{array}{c} q^{-N+k}, A q^{k}, t_{1} t_{4} q^{k}, t_{2} t_{4} q^{k}, t_{3} t_{4} q^{k} \\ b t_{4} q^{k}, B t_{4} q^{k} / b, A q^{k+1-N} / B, t_{1} t_{2} t_{3} t_{4} q^{2 k} ; q, q \end{array} q, q \right). \]

In Theorem 10.7 if we replace \( A \) by \( A q^{N-1} \), we can then identify parameters \( a_{2}, a_{3} \) such that the \( _{4}\Phi_{3} \) in Theorem 10.7 is a multiple of \( p_{N}(x ; b, a_{2}, a_{3}, t_{4} | q) \). As such Theorem 10.7 is equivalent to a connection coefficient problem solved in [7]. We also note that although Theorem 10.5 is the limiting case \( N \rightarrow \infty \) of Theorem 10.7, Theorem 10.5 is not available in the literature.

Remark 10.8. If we specialize Theorem 10.7 to

\[ b = t_{2}, \quad B = t_{1} t_{2}, \quad z = t_{3}. \]
the $3\phi_4$ in Theorem 10.7 reduces to a balanced $3\phi_2$ which is again evaluable by the $q$-Pfaff–Saalschütz theorem [16, (II.12)]. The resulting identity is the terminating case of the Watson transformation [16, (III.18)]. The nonterminating case Watson transformation [16, (III.18)] follows by analytic continuation in the variable $d = q^N$.

We now go back to (10.7) and observe that \{$(t_4z, t_4/z; q)_n$\} is a basis for the space of polynomials, hence we can replace $(t_4z, t_4/z; q)_n$ by $A_n(t_4z, t_4/z; q)_n$ and (10.7) will remain valid as long as the series on both sides converge. This establishes the following expansion theorem.

**Proposition 10.9.** We have the general expansion

$$
\sum_{n=0}^{\infty} \frac{(az, a/z; q)_n}{(q; q)_n} A_n B_n \zeta^n
$$

(10.8)

$$
= \sum_{k=0}^{\infty} \frac{(-\zeta)^k q^{\frac{k}{2}}}{(q, Cq^{k-1}; q)_k} \left[ \sum_{j=0}^{k} \frac{(q^{-k}, Cq^{k-1}; q)_j}{(q; q)_j} A_j (az, a/z; q)_j q^j \right]
\times \left[ \sum_{n=0}^{\infty} \frac{B_{n+k} \zeta^n}{(q, Cq^{k+1}; q)_n} \right].
$$

Proposition 10.9 writes a triple sum as a single sum. Fields and Ismail [12] pointed out that identities like (10.8) follow from a matrix inversion for upper triangular matrices. For the definition of inverse relations see [37]. Indeed if $A = (a_{i,j}), B = (b_{i,j})$ are two upper infinite triangular matrices and $B = A^{-1}$, and \{$(u_n(x))$\} is a sequence of polynomials with $u_n$ of degree $n$ then

$$
P_n(x) = \sum_{j=0}^{n} a_{n,j} A_j u_j(x) \iff A_m u_m(x) = \sum_{n=0}^{m} b_{m,n} P_n(x).
$$

We now find that for any basis \{$(v_n(x))$\} of the space of polynomials we have

$$
\sum_{m=0}^{\infty} A_m B_m u_m(x) v_m(y) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} b_{n,j} A_j u_j(x) \right) \sum_{m=0}^{\infty} B_{m+n} v_{m+n}(y) b_{m+n,n}.
$$

Since we can interchange $A$ and $B$ we also have the dual expansion

$$
\sum_{m=0}^{\infty} A_m B_m u_m(x) v_m(y) = \sum_{n=0}^{\infty} \left( \sum_{j=0}^{n} a_{n,j} A_j u_j(x) \right) \sum_{m=0}^{\infty} B_{m+n} v_{m+n}(y) a_{m+n,n}.
$$

Formulas of the type in (10.8) have a long history. Fields and Wimp [13] expanded hypergeometric functions into Jacobi type polynomials. In [42] Verma generalized their expansion to expansion with arbitrary coefficients in Jacobi type polynomials. His formula is

$$
\sum_{m=0}^{\infty} a_m b_m \frac{(zu)^m}{m!} = \sum_{n=0}^{\infty} \frac{(-z)^n}{n!(\gamma+n)n} \left( \sum_{r=0}^{\infty} \frac{b_{n+r} z^r}{r!(\gamma+2n+1)r} \right) \left( \sum_{s=0}^{\infty} \frac{(-n)_s (\gamma+s)_s}{s!} a_s w^s \right).
$$

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Fields and Wimp as well as Verma noted a Laguerre type expansion where \( w \) is replaced by \( \frac{w}{\gamma} \), \( b_n \) is replaced by \( \gamma b_n \) and \( \gamma \rightarrow \infty \). The expansion (10.8) extends all those expansions to expansions in Askey–Wilson type polynomials.

**Remark 10.10.** One may take \( \Lambda_n \) to be 0 unless \( n \equiv a (\text{mod } b) \) for fixed integers \( a, b, a > 0, b \geq 0 \). This leads to hypergeometric expansions where the differences of consecutive parameters in a certain group is \( 1/b \).

### 11 Askey-Wilson expansions

Andrews [2] proved the terminating basic hypergeometric identity

\[
\begin{align*}
&\genfrac{[}{]}{0pt}{}{5\phi_4}{q^{-N}, \rho_1, \rho_2, b, c} \left( \frac{aq}{\rho_1, \rho_2 q^{-N}/a, e, f, g}{q, q} \right) = \frac{(aq/\rho_1, \rho_2 q^{-N}/a, e, f, g; q)_N}{(aq/\rho_1 \rho_2 q^{-N}/a, e, f, g; q)_N} \\
&\quad \times \sum_{n=0}^{N} \left( \frac{aq/\rho_1, \rho_2 q^{-N}/a, e, f, g}{q, \rho_1 \rho_2 q^{-N}/a, e, f, g} \right)_n (1 - a q^{2n}) \left( \frac{aq/\rho_1 \rho_2 q^{-N}/a, e, f, g}{q, \rho_1 \rho_2 q^{-N}/a, e, f, g} \right)_n w_n,
\end{align*}
\]

where \( N \) is a non-negative integer, \( qabc = efg \), and

\[
u_n = 4\phi_3 \left( \frac{q^{-n} aq^n, b, c}{e, f, g, q, q} \right).
\]

**Remark 11.1.** The Andrews formula (11.1) is the case \( p = 4 \) in Corollary 10.4 with the parameter identification

\[
a_1 = q^{-N}, \quad a_2 = \rho_1, \quad a_3 = \rho_2, \quad b_1 = \rho_1 \rho_2 q^{-N}/a, \quad \zeta = q.
\]

In this case the \( 3\phi_2 \) can be summed by the \( q \)-Pfaff–Saalschütz theorem, [16, (II.12)].

**Remark 11.2.** Another application of Corollary 10.4 is the case \( p = 6 \) with

\[
a_1 = q^{-N}, \quad a_2 = c_1 c_2 c_3 t_4 q^{N-1}, \quad a_j = t_{j-2} t_4 \quad \text{for } 3 \leq j \leq 5,
\]

\[
b_k = t_4 c_k \quad \text{for } 1 \leq j \leq 3.
\]

The reader is encouraged to write down the resulting formula.

Upon setting \( z = t_1 \) in Corollary 10.4, we have the next corollary.

**Corollary 11.3.** We have the following identity

\[
\begin{align*}
&\genfrac{[}{]}{0pt}{}{p\phi_{p-1}}{a_1, \ldots, a_{p-1}, t_4/t_1} \left( t_2 t_4, t_3 t_4, b_1, \ldots, b_{p-3} \right) q, \zeta \right) \\
&= \sum_{k=0}^{\infty} \frac{(a_1, \ldots, a_{p-1}, t_1 t_2, t_1 t_2; q)_k}{(t_2 t_4, t_3 t_4, b_1, \ldots, b_{p-3}; q)_k} \\
&\quad \times \left( \frac{-t_4 \zeta/t_1 q^{k/2}}{q, t_1 t_2 t_3 t_4 q^{k-1}; q}_k \right) p-1\phi_{p-2} \left( q^k a_1, \ldots, q^k a_{p-1} \right) \left( q^k b_1, \ldots, q^k b_{p-3}, t_1 t_2 t_3 t_4 q^{2k} \right) q, \zeta.
\end{align*}
\]
We note that by equating coefficients of $ζ^n$ on both sides of the equation in Corollary 11.3 is equivalent to the sum of a terminating very well-poised $₆φ₅$, [16, (II.21)]

We proceed to derive other generating functions for the Askey–Wilson polynomials. In this section we give two generating functions for Askey-Wilson polynomials: Theorem 11.4, which follows from Proposition 10.1, and Theorem 11.5, for which we provide an independent proof.

**Theorem 11.4.** The Askey–Wilson polynomials have the generating function

$$
\frac{(be^{iθ}, be^{-iθ}; q)_∞}{(bt_4, b/t_4; q)_∞} = \sum_{k=0}^{∞} \frac{(-b)^k q^{(k)}_2}{(q, bt_4, t_1t_2t_3t_4q^{k-1}; q)_k} p_k(x; t|q)
$$

(11.3)

and satisfy the relationship

$$
\frac{(t_1z, t_1/z, t_1t_2t_3t_4z; q)_∞}{(t_1t_2, t_1t_3, t_1t_4; q)_∞} = \sum_{k=0}^{∞} \frac{(-t_1)_k (t_1t_2t_3t_4z/q; q)_k q^{(k)}_2}{(q, t_1t_2, t_1t_3, t_1t_4; q)_k} \frac{1 - t_1t_2t_3t_4q^{2k-1}}{1 - t_1t_2t_3t_4/q} p_k(x; t|q)
$$

(11.4)

**Proof.** To prove (11.3) we let $n → ∞$ in Proposition 10.1. Taking the limit inside the sum is justified by Tannery’s theorem, [9], the discrete analogue of the Lebesgue dominated convergence theorem. We omit the details. The identity (11.4) is the case $b = t_1$ of (11.3), because the $3φ₂$ becomes a $2φ₁$ and is summed by the $q$-Gauss theorem [16, (II.8)].

One may ask for a version of Theorem 11.4 in which the infinite products in $z$ are in the denominator.

**Theorem 11.5.** The Askey–Wilson polynomials have the generating function

$$
\frac{1}{(be^{dθ}, be^{-dθ}; q)_∞} = \sum_{n=0}^{∞} c_n(t, b)p_n(x; t|q),
$$

(11.5)

where

$$
c_n(t, b) = \frac{b^n (t_2t_3t_4bq^{n}; q)_∞}{(q, t_1t_2t_3t_4q^{n-1}; q)_n \prod_{j=2}^{4} (t_jb; q)_∞} \times ₃φ₂\left(\frac{q^n t_2t_3, q^n t_2t_4, q^n t_3t_4}{q^{2n} t_1t_2t_3t_4, q^n t_2t_3t_4b} \left| q, t_1b\right)\right).
$$

(11.6)

Of course Theorem 11.5 is a special case of Theorem 10.2 but we will give a proof of the special case since we did not provide a prove of the more general result.
Proof of Theorem 11.5. We use two facts to prove Theorem 11.5. The first fact is the orthogonality relation for Askey-Wilson polynomials (7.4), where the coefficient of $\delta_{m,n}$ is $h_n(t)A(t)$,

$$h_n(t) = \frac{(q; q)_n \prod_{1 \leq j < k \leq 4} (t_j t_k; q)_n (t_1 t_2 t_3 t_4 q^{n-1}; q)_n}{(t_1 t_2 t_3 t_4; q)_2n},$$

(11.7) 

$$A(t) = \frac{2\pi (t_1 t_2 t_3 t_4; q)_\infty}{(q; q)_\infty \prod_{1 \leq j < k \leq 4} (t_j t_k; q)_\infty},$$

and we have assumed that $\max \{|t_1|, |t_2|, |t_3|, |t_4|\} < 1$. The second fact is

$$\frac{(q; q)_\infty}{2\pi} \int_0^\pi \frac{w(\cos \theta; x_1, x_2, x_3, x_4)}{(x_5 e^{i\theta}, x_5 e^{-i\theta}; q)_\infty} \sin \theta \, d\theta$$

(11.8) 

$$= \frac{(x_1 x_2 x_4 x_1, x_2 x_3 x_4 x_5, x_4 x_5; q)_\infty}{\prod_{1 \leq r < s \leq 5} (x_r x_s; q)_\infty} \Phi(3)\left(\begin{array}{c} x_2 x_3, x_2 x_4, x_3 x_4 \\ x_1 x_2 x_3 x_4, x_2 x_3 x_4 x_5 \end{array}\right)_{q, x_1 x_5}.$$ 

The integral (11.8) is a special case of the Nassrallah–Rahman integral (8.9).

For symmetry we replace $t$ by $t_5$. We shall find the coefficient $c_n(t, t_5)$ of $p_n(x; t|q)$ using orthogonality, setting

$$\sum_{n=0}^\infty c_n(t, t_5) p_n(x; t|q) = \frac{1}{(t_5 e^{i\theta}, t_5 e^{-i\theta}; q)_\infty}.$$ 

Such a formula exists because the right-hand side is $\in L^2[-1, 1, w(x; t|q)]$. Moreover

$$c_n(t, t_5) h_n(t) A(t) = \int_{-1}^1 \frac{w(x; t|q)}{(t_5 e^{i\theta}, t_5 e^{-i\theta}; q)_\infty} p_n(x; t|q) dx$$

Therefore, using (7.3) we see that

$$c_n(t, t_5) h_n(t) A(t) = \frac{(t_1 t_2, t_1 t_3, t_1 t_4; q)_n}{t_1^n} \sum_{k=0}^n \frac{(q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}; q)_k}{(q, t_1 t_2, t_1 t_3, t_1 t_4; q)_k} k^k$$

$$\times \int_0^\pi \frac{w(\cos \theta; t_1 q^k, t_2, t_3, t_4|q)}{(t_5 e^{i\theta}, t_5 e^{-i\theta}; q)_\infty} \sin \theta \, d\theta.$$ 

The integral is now evaluated by (11.8) and we obtain

$$\frac{(q; q)_\infty}{2\pi} c_n(t, t_5) h_n(t) A(t) = \frac{(t_1 t_2, t_1 t_3, t_1 t_4; q)_n}{t_1^n} \sum_{k=0}^n \frac{(q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}; q)_k}{(q, t_1 t_2, t_1 t_3, t_1 t_4; q)_k} k^k$$

$$\times \frac{(q; q)_\infty}{\prod_{j=2}^5 (q^k t_1 t_j; q)_\infty} \prod_{2 \leq r < s \leq 5} (t_r t_s; q)_\infty \Phi(3)\left(\begin{array}{c} t_2 t_3, t_2 t_4, t_3 t_4 \\ q^k t_1 t_2 t_3 t_4, t_2 t_3 t_4 t_5 \end{array}\right)_{q, q^k t_1 t_5}.$$ 

Write the $\Phi(3)$ as a sum over $s$ and interchange the $k$ and $s$ sums to see that

$$\frac{(q; q)_\infty}{2\pi} c_n(t, t_5) h_n(t) A(t) = \frac{(t_1 t_2, t_1 t_3, t_1 t_4; q)_n (t_1 t_2 t_3 t_4 t_5; q)_\infty}{t_1^n} \prod_{j=2}^5 (t_1 t_j; q)_\infty \prod_{2 \leq r < s \leq 5} (t_r t_s; q)_\infty$$

$$\times \sum_{s=0}^\infty \frac{(t_2 t_3, t_2 t_4, t_3 t_4; q)_s (t_1 t_5)^s}{(q, t_1 t_2 t_3 t_4; q)_s} \sum_{k=0}^n \frac{(q^{-n}, t_1 t_2 t_3 t_4 q^{n-1}; q)_k}{(q, q^k t_1 t_2 t_3 t_4; q)_k} q^{k(s+1)}.$$
The $k$ sum is an evaluable terminating $2\phi_1$, and we obtain
\[
\frac{(q; q)_{\infty}}{2\pi} c_n(t, t_3) h_n(t) A(t) = \frac{(t_1 t_2, t_1 t_3, t_1 t_4; q)_n (t_1 t_2 t_3 t_4, t_2 t_3 t_4 t_5; q)_\infty}{t_1^n \prod_{j=2}^n (t_1 t_j; q)_{\infty} \prod_{1 \leq r < s \leq 5} (t_r t_s; q)_{\infty}} \times \sum_{s=0}^{\infty} \frac{(t_2 t_3, t_2 t_4, t_3 t_4; q)_s}{(q, t_1 t_2 t_3 t_4; q)_s} (t_1 t_5)^s \frac{(q^{s+1-n}; q)_n}{(q^{s+1}; q^{s+1}; q)_{s+n}}.
\]
Thus $s \geq n$, so shift $s$ by $n$. Therefore the left-hand side in the above equation is the statement of the theorem.

An attractive special case of Theorem 11.5 is a corollary due to Kim and Stanton [29].

**Corollary 11.6.** The continuous dual $q$-Hahn polynomials $p_n(x; t_1, t_2, t_3, 0|q)$ have the generating function
\[
\sum_{k=0}^{\infty} p_k(x; t_1, t_2, t_3, 0|q) t^k = \frac{(bt_1, bt_2, bt_3; q)_{\infty}}{(be^{i\theta}, be^{-i\theta}, bt_1 t_2 t_3; q)_{\infty}}
\]

*Proof.* Take $t_2 = 0$ in Theorem 11.5 and relabel the $t_j$’s. The $3\phi_2$ becomes a $1\phi_0$ which we sum by the $q$-binomial theorem.

We note that the case $t_3 = 0$ is the generating function (8.17) of the Al-Salam–Chihara polynomials, while the case $t_1 = t_2 = t_3 = 0$ is the generating function for the $q$-Hermite polynomials.

We now recast some of the results proved so far in integral form using the orthogonality relation (7.4).

Proposition 10.1 becomes
\[
\int_0^\pi \frac{(be^{i\theta}, be^{-i\theta}, q)_{n} (e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_{\infty}} p_k(\cos \theta; t) d\theta = \frac{(-b)^k q^{(k)}(t; q)_n (b/t_4, bt_4 q^k; q)_{n-k}}{(q; q)_{n-k} \prod_{1 \leq r < s \leq 4} (t_r t_s q^k; q)_{\infty}} \times 3\phi_2 \left( \begin{array}{c} q^{k-n}, t_1 t_4 q^k, t_2 t_4 q^k, t_3 t_4 q^k \\ bt_4 q^k, t_1 t_2 t_3 t_4 q^{-2k}, q^{1+k-n} t_4/b \end{array} \bigg| q, q \right).
\]
In view of (11.3), the limiting case $n \to \infty$ of (11.9) is
\[
\int_0^\pi \frac{(be^{i\theta}, be^{-i\theta}, e^{2i\theta}, e^{-2i\theta}; q)_{\infty}}{\prod_{j=1}^4 (t_j e^{i\theta}, t_j e^{-i\theta}; q)_{\infty}} p_k(\cos \theta; t) d\theta = \frac{(-b)^k q^{(k)}(b/t_4, bt_4 q^k; q)_{\infty}}{(q; q)_{\infty} \prod_{1 \leq r < s \leq 4} (t_r t_s q^k; q)_{\infty}} \times 3\phi_2 \left( \begin{array}{c} t_1 t_4 q^k, t_2 t_4 q^k, t_3 t_4 q^k \\ bt_4 q^k, t_1 t_2 t_3 t_4 q^{-2k} \end{array} \bigg| q, t_4 \right).
\]

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When \( b = t_1 \), for example, the \( 3\phi_2 \) in (11.10) sums. The result is known because it is the constant term in the expansion of \( p_k(x; t_1, t_2, t_3, t_4|q) \) in \( p_k(x; 0, t_2, t_3, t_4|q) \), see Theorem 8.7.

It is important to note that the left-hand side of (11.10) is symmetric under permutations of \( \{t_1, t_2, t_3, t_4\} \). But the symmetry of the right-hand side under interchanging \( t_1 \) and \( t_4 \) gives the transformation

\[
(b/t_4, bt_4q^k; q)_\infty 3\phi_2 \left( \begin{array}{c} t_1t_4q^k, t_2t_4q^k, t_3t_4q^k \\ bt_4q^k, t_1t_2t_3q^{2k} \end{array} \right| q, \frac{t_4}{t_1} )
\]

(11.11)

\[
= (b/t_1, bt_1q^k; q)_\infty 3\phi_2 \left( \begin{array}{c} t_1t_4q^k, t_1t_2q^k, t_1t_3q^k \\ bt_1q^k, t_1t_2t_3q^{2k} \end{array} \right| q, \frac{b}{t_1} )
\]

Remark 11.7. By multiplying (11.9) by \( \Lambda_n \zeta^n \) and adding for \( n \geq 0 \) we obtain a general expansion formula with arbitrary coefficients. This can be specialized to derive a general formula involving integrals of power series with arbitrary coefficients times a product of an Askey–Wilson polynomial and its weight function. This can further specialized by taking \( \Lambda \) to be of hypergeometric or basic hypergeometric form.

We mention without proof the following theorem which, among other things, evaluates the moments of the Askey–Wilson weight function.

Theorem 11.8. We have the integral evaluation

\[
\int_0^{\pi} \left( e^{i\theta}, e^{-2i\theta}; q \right)_\infty (be^{i\theta}, be^{-i\theta}; q)_\infty \prod_{j=1}^4 (a_j e^{i\theta}, a_j e^{-i\theta}; q)_\infty \frac{d\theta}{q,a} (a_1bq^k, bq^{-k}/a_1; p)_n
\]

\[
\times q^{k(n+1)} \left( \frac{1 - a_1}{1 - a_1q^k} \right) (1 - a_1a_4) (1 - a_1a_3) (1 - a_1a_2a_3a_4) (q^{-1}, q, a_1a_4, a_1q^{k+1}/a_2, q^{3/4} a_1a_2 q^{-3/4})
\]

(11.12)

Theorem 11.8 is due to Ismail and Rahman in [20]. Note that if \( p = 1 \) then, with \( x = \cos \theta, y = (b + 1/b)/2 \), we have

\[
\left( b e^{i\theta}, b e^{-i\theta}; q \right)_n \bigg|_{p = 1} = (1 - 2bx + b^2)^n = (-2b)^n (x - y)^n.
\]

In particular when \( y = 0 \), Theorem 11.8 gives the moments of the Askey–Wilson weight function.

Exercises:

1. Show that for any positive integer \( n \), \( p_n(z, t|q) = 0 \) if

\[
z = -\gamma, \quad t_1 = \gamma, \quad t_2 = \gamma^3, \quad t_3 = \gamma^5, \quad t_4 = 0, \quad \gamma = e^{2\pi i/6}.
\]

2. Let \( \omega \) be a primitive cubic root of unity. Prove that

\[
\sum_{k=0}^n \left[ \begin{array}{c} n \\ k \end{array} \right] q^{k/3} p_k(-1/2; c, \omega c, \omega^2 c, 0|q) = \begin{cases} 0 & \text{if } 3 \nmid n, \\ \left( q, q^2, q^3 \right)_{n/3} & \text{if } 3 | n. 
\end{cases}
\]

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12 A $q$-Exponential Function

Ismail and Zhang [28] introduced the $q$-exponential function

$$\mathcal{E}_q(\cos \theta; t) := (t^2; q^2)_{\infty} \sum_{n=0}^{\infty} (-i e^{i \theta} q^{(1-n)/2}, -i e^{-i \theta} q^{(1-n)/2}; q)_n \frac{(-it)^n}{(q; q^2)_n} q^{n^2/4}.$$  

(12.1)

Define $u_n(x, y)$ by

$$u_n(\cos \theta, \cos \phi) = e^{-i n \theta} (-e^{i (\phi + \theta)} q^{(1-n)/2}, -e^{i (\phi - \theta)} q^{(1-n)/2}; q)_n.$$  

(12.2)

It is easy to see that $u_n(x, y) \to 2^n (x + y)^n$ as $q \to 1$. Hence

$$\lim_{q \to 1} \mathcal{E}_q(x; (1 - q)t/2) = \exp(tx).$$

This shows that $\mathcal{E}_q(x; t)$ is a $q$-analogue of $e^{tx}$. The factor in front of the sum in (12.1) is to normalize $\mathcal{E}_q$ by $\mathcal{E}_q(0; t) = 1$.

Following Rainville [36] we say that a polynomial sequence $\{p_n(x)\}$ belongs to a linear operator $T$ which reduces the degree of a polynomial by 1 if $T p_n(x) = p_{n-1}(x)$. One can repeat the same arguments used by Rainville and prove the following theorem.

**Theorem 12.1.** Two polynomial sequences $\{p_n(x)\}$ and $\{q_n(x)\}$ belong to the same operator $T$ if and only if there is a sequence of constants $\{a_n\}$ with $a_0 \neq 0$ such that

$$p_n(x) = \sum_{k=0}^{n} a_k q_{n-k}(x).$$  

(12.3)

This is equivalent to

$$\sum_{n=0}^{\infty} p_n(x) t^n = \left[ \sum_{n=0}^{\infty} a_n t^n \right] \left[ \sum_{k=0}^{\infty} q_k(x) t^k \right] = h(t) \sum_{k=0}^{\infty} q_k(x) t^k,$$  

(12.4)

and $h(0) \neq 0$.

We now apply Theorem 12.1 to the operator $T = D_q$ with different $p_n$'s. Let

$$s_n(x; a) = (a q^{-n/2} e^{i \theta}, a q^{-n/2} e^{-i \theta}; q)_n.$$  

(12.5)

It is easy to see that

$$D_q s_n(x; a) = \frac{-2a q^{-n/2}}{1 - q} (1 - q^n) s_{n-1}(x; a).$$  

(12.6)

Therefore $\{s_n(x; a)(1 - q)^n (-2a)^{-n} q^{u(n+1)/4}/(q; q)_n\}$ belongs to $D_q$. Formulas (12.6), (3.6), and (3.7) show that $[(q-1)/2]^n \phi_n(x) q^{-n/4}/(q; q)_n,$ and $[(1-q)/2]^n \rho_n(x) q^{u(n-1)/4}/(q; q)_n,$ also belong to $D_q$. This proves the following theorem.
Theorem 12.2. We have the expansions

\[
(12.7) \quad \mathcal{E}_q(\cos \theta; \alpha) = \frac{(-t; q^{1/2})_\infty}{(q^{1/2}; q^2)_\infty} 2\phi_1 \left( \begin{array}{c} q^{1/4}e^{i\theta}, q^{1/4}e^{-i\theta} \\ q^{1/2} \end{array} \middle| q^{1/2}, -t \right),
\]

\[
\mathcal{E}_q(x; t) = g(t) \sum_{n=0}^{\infty} \frac{(aq^{-n/2}e^{i\theta},aq^{-n/2}e^{-i\theta}; q)_n}{(q; q)_n} \left( \frac{-t}{a} \right)^n,
\]

where \( g(t) \) is given by

\[
(12.8) \quad g(t) = \sum_{n=0}^{\infty} \frac{(aq^{-n/2}c,aq^{-n/2}/c; q)_n}{(q; q)_n} \left( -\frac{t}{a} \right)^n,
\]

and \( x_0 = (c + 1/c)/2, \) for \( c \neq 0. \)

We next define a function \( \mathcal{E}_q(x, y; t) \) by

\[
(12.9) \quad \mathcal{E}_q(x, y; t) = \frac{t^2; q^2}_\infty \sum_{n=0}^{\infty} (te^{-i\theta})^n q^{n^2/4} \times (-e^{i(\phi+\theta)}, q^{1-n/2}, -e^{i(\phi-\theta)}, q^{1-n/2}; q)_n.
\]

This function was introduced in [28].

We note that the symmetry \( \mathcal{E}_q(x, y; t) = \mathcal{E}_q(y, x; t) \) follows from the definition (12.9). It follows from (12.1) and (12.6) that

\[
(12.10) \quad \mathcal{D}_q\mathcal{E}_q(x; t) = \frac{2tq^{1/4}}{1-q} \mathcal{E}_q(x; t), \quad \text{and} \quad \mathcal{D}_q\mathcal{E}_q(x, y; t) = \frac{2tq^{1/4}}{1-q} \mathcal{E}_q(x, y; t),
\]

and \( \mathcal{D}_q \) acts on \( x. \) Therefore \( \mathcal{E}_q(x, y; t) = g(y, t)\mathcal{E}_q(x; t). \) On the other hand the definition (12.9) shows that \( \mathcal{E}_q(0, y; t) = \mathcal{E}_q(y; t). \) This establishes the addition theorem

\[
(12.11) \quad \mathcal{E}_q(x, y; t) = \mathcal{E}_q(x; t)\mathcal{E}_q(y; t).
\]

Further properties of \( \mathcal{E}_q(x; t) \) were developed in [28], [21] and many are recorded in [19].

There are several formulas expanding \( \mathcal{E}_q(x; t) \) in a series of orthogonal polynomials. One sample from [26] is

\[
\frac{(q^2t^4; q^4)_\infty}{(-t; q)_\infty} \mathcal{E}_q^2(x; t) = \sum_{k=0}^{\infty} \frac{(q, -q, t_2t_3t_4q^{k-1/2}; q)_k}{(q; q^{1/2}, t_2, t_3, t_4)_k} p_k(x; q^{1/2}, t_2, t_3, t_4|q)
\times 3\phi_2 \left( \begin{array}{c} q^{k+1/2}t_2, q^{k+1/2}t_3, q^{k+1/2}t_4 \\ -q^{k+1}, t_2t_3t_4q^{2k+1/2} \end{array} \middle| q, -t \right).
\]

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