A CLAIM OF RAMANUJAN INVOLVING EULERIAN SERIES

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Abstract. In this Research Problem Session I shall talk about a claim of Ramanujan involving Eulerian series, which appear in the last letter of Ramanujan to Hardy.

1. Introduction and statement of problems

Eulerian series are combinatorial formal power series which are constructed from basic hypergeometric series. In his last letter to Hardy, Ramanujan listed 17 examples of functions in Eulerian series that he called mock theta functions. To explain what he meant a "mock \( \vartheta \)-function", what is the differences between the usual Eulerian series, mock \( \vartheta \)-function and modular function, Ramanujan gives three pages of explanation. I will first give some detail Ramanujan was mentioned.

Let \( q = e^{-t} \) with \( t > 0 \), note that Dedekind eta function \( \eta(i t 2 \pi) = q^{1/24} (q; q)_{\infty} \) and using modular transform we obtain that

\[
(q; q)_{\infty} = \sqrt{\frac{2\pi}{t}} \exp\left( -\frac{\pi^2}{6t} + \frac{t}{24} \right) \prod_{n \geq 1} (1 - e^{-4\pi^2 n/t}) .
\]

Here we use the the \( q \)-Pochhammer symbol \((a; q)_m = \prod_{k=0}^{m-1} (1 - aq^k)\) for \( a, q \in \mathbb{C}, |q| < 1 \) and \( m \in \mathbb{N} \cup \{\infty\} \). The formula (1) actually gives the nice asymptotics of the following \( q \)-series in Eulerian form

\[
\frac{1}{(q; q)_{\infty}} = \sum_{n \geq 0} \frac{q^{n^2}}{(q; q)_n^2} = \sqrt{\frac{t}{2\pi}} \exp\left( \frac{\pi^2}{6t} - \frac{t}{24} \right) + O(1)
\]

as \( t \to 0^+ \). Ramanujan point out that the number of functions like above, which has beautifully asymptotic formula can be expressed in a very neat and closed exponential form, is very few; for other Eulerian series, approximations analogous to (2) may not exist. The following is a claim of Ramanujan.

Claim 1 (Ramanujan). Let \( q = e^{-t} \) with \( t \to 0^+ \). Then we have

\[
\sum_{m=0}^{\infty} \frac{q^{m(m+1)}}{2(q; q)_m^2} = \sqrt{\frac{t}{2\pi \sqrt{5}}} \exp\left( \frac{\pi^2}{5t} + c_1 t + \cdots + c_p t^p + o(t^p) \right)
\]

holds for each integer \( p \geq 1 \), all \( c_j \) are constants with infinitely many \( c_j \neq 0 \).

Although this claim have been discussed by Watson \[1\], McIntosh \[2\] and et al.. The asymptotic expansion have been proved by McIntosh \[2\], Zagier \[3\] and the author \[4\], we can’t prove this claim until today.

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2. A POSSIBLE APPROACH

Watson [1] proved that
\[
\sum_{m=0}^{\infty} q^{m(m+1)/2m} (q^2; q^2)_m = \frac{1}{(q; q)_\infty} \sum_{n \geq 0} \frac{(q^2)_n^2 q^{n+1}}{(q^2; q^2)_n} R(q^2)
\]
with
\[
R(q) = \sum_{n \geq 0} q^{n^2+n/2}(q^{n+1}; q)_\infty.
\]

From [1] we can find the asymptotics for \((q; q)_\infty(q^2; q^2)_\infty\), and hence we just need to consider the asymptotics for \(R(q)\), \(q = e^{-t}\) with \(t \to 0^+\). In [4], we have proved that
\[
R(e^{-t}) = (1 + o(t^p)) \int_{\mathbb{R}^+} q^{x^2+x/2}(q^{x+1}; q)_\infty dx
\]
holds for each \(p \geq 0\). We note the quantum dilogarithm function \(\text{Li}_2(x; q) = \sum_{k \geq 1} x^k / k(q^k - 1)\), and then it is easy to see that Claim 1 equivalent to the following:

Claim 2. We have as \(t \to 0^+\),
\[
\int_{\mathbb{R}^+} \exp \left( -(x^2 + x/2)t - \text{Li}_2 \left( e^{-(x+1)t}; e^{-t} \right) \right) dx = \frac{1}{\sqrt{t}} \exp \left( \sum_{k=1}^{p} d_k t^k + o(t^p) \right)
\]
holds for each integer \(p \geq 1\), all \(d_j\) are constants with infinitely many \(d_j \neq 0\).

Remark 3. Let \(A_q(z)\) be the Ramanujans entire function be defined as
\[
A_q(z) = \sum_{n \geq 0} \frac{q^{n^2}(z)^n}{(q; q)_n}.
\]
Clearly \((q^2; q^2)R(q^2) = A_q(-q)\). From the work of Ismail and Zhang [5], we have the follows integral representation:
\[
A_q(-q) = \sqrt{-\log q} \frac{1}{\pi} \int_{\mathbb{R}} \left( q^{2y^2}; q^2 \right)_\infty q^y dy.
\]
One may from above integral representation find the asymptotic expansion for \(R(q)\), and further prove Claim 1.

References