Kernel functions and elliptic OP

Simon Ruijsenaars

School of Mathematics
University of Leeds

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1. Background

- **Physical perspective**: Systems of Calogero-Moser type are integrable one-dimensional $N$-particle systems that come in various versions: classical/quantum, nonrelativistic/relativistic, with special interactions given by rational/trigonometric/hyperbolic/elliptic functions.

- **Harmonic analysis perspective**: The quantum systems amount to commutative algebras of operators associated with root systems, with the differential/difference operator case corresponding to Lie groups/quantum groups; their symbols Poisson commute and amount to the classical versions.

The Hamiltonian for the quantum relativistic trigonometric $BC_1$ case, acting on $\mathcal{H} := L^2([0, \pi/2r], dx)$, is diagonalised by the ONB

$$ e_n(x) = w(x)^{1/2} P_n(x), \quad n \in \mathbb{N}, $$

where $P_n(x) = p_n(\cos 2rx)$ are the normalised Askey-Wilson polynomials and $w(x)$ their weight function.
Physically speaking, the AW polynomials depend on 4 couplings $a, b, c, d$, whereas the shifts over $\pm is$, $s > 0$, involve the Compton wave length $s = \hbar/mc$. The parameter $q$ amounts to $\exp(-sr)$.

The highest level in the CM hierarchy is the elliptic one. For the Lie algebra $BC_1$ it involves 8 coupling parameters $\gamma_0, \ldots, \gamma_7$, and two length parameters $a_+, a_-$, yielding two $q$-parameters $q_\delta = \exp(-a_\delta r)$, $\delta = +, -$.

Consider first a second order difference operator of the form

$$A_+ = \exp(-i a_- d/dx) + V_{a,+}(x) \exp(i a_- d/dx) + V_{b,+}(x),$$

where the coefficients are elliptic functions with periods $\pi/r$ and $ia_+$ (also, $r, a_+, a_- > 0$). Thus, $A_+$ leaves the space of meromorphic functions invariant. Clearly, for $A_+$ to be formally self-adjoint on $\mathcal{H}$, it suffices to require real-valuedness of $V_{b,+}(x)$ on the real line, and the conjugacy relation

$$V_{a,+}^*(x + ia_-) = V_{a,+}(x), \quad V_{a,+}^*(x) := \overline{V_{a,+}(\bar{x})}, \quad x \in \mathbb{C}.$$
Van Diejen’s relativistic Heun Hamiltonian is of the form $A_+$ (after a similarity transformation). Now consider the operator

$$A_- = \exp(-ia_+ d/dx) + V_{a,-}(x) \exp(ia_+ d/dx) + V_{b,-}(x),$$

obtained by swapping $a_+$ and $a_-$. It commutes with $A_+$, since the elliptic functions act as quasi-constants for the shifts.

It seems very unlikely that any joint meromorphic eigenfunctions of $A_+$ and $A_-$ exist in this generality. For the special coefficients at hand, however, the existence of meromorphic joint eigenfunctions can be proved for a discrete set of eigenvalues in $\mathbb{R}^2$. They are such that their restrictions to $[0, \pi/2r]$ yield an orthogonal base for $\mathcal{H}$.

This is shown in:
The key tool in this paper is a so-called kernel function, which is a product of 4 elliptic gamma functions. This building block is given by

\[ G(r, a_+, a_-; z) = \frac{E(z)}{E(-z)}, \quad E(z) \equiv \prod_{m,n=0}^{\infty} \left(1 - q_+^{2m+1} q_-^{2n+1} \exp(-2irz)\right). \]

Notice that

\[ \lim_{a_+ \to \infty} G(r, a_+, a_-; z - ia_+/2) = 1/(q_- \exp(2irz); q_-^2). \]

Another crucial ingredient for obtaining eigenvalue asymptotics is an auxiliary 8-parameter family of orthogonal polynomials, whose weight function only involves \( E(z) \). For \( a_+ \to \infty \) they turn into the \( q_-^2 \)-Hermite polynomials and the kernel function turns into the Poisson kernel known from Rogers’ work, cf. below.

The above SIGMA paper leaves many questions open, some of which are mentioned in Section 6.
2. Kernel functions

- Given a pair of operators $H_1(x)$ and $H_2(y)$, a kernel function is a function $K(x, y)$ satisfying

$$H_1(x)K(x, y) = H_2(y)K(x, y).$$

Here, $x$ and $y$ may vary over spaces of different dimension. The concept of a kernel function is still unfamiliar to many colleagues. A natural question is: What are kernel functions good for?

- Indeed, given an operator $H(x)$ with eigenfunctions

$$H(x)\psi_m(x) = E_m\psi_m(x), \quad m = 0, 1, 2, \ldots, M \leq \infty,$$

any function $K(x, y)$ of the form

$$K(x, y) = \sum_{m=0}^{M} \lambda_m \psi_m(x)\psi_m(y),$$

satisfies the kernel identity

$$H(x)K(x, y) = H(y)K(x, y).$$

Hence kernel functions exist in profusion.
Key point: Once one has found such a kernel identity for a given Hamiltonian \( H \), one can use \( K(x, y) \) in "favorable" cases as the kernel of an integral operator \( \mathcal{I} \) whose eigenfunctions are also \( H \)-eigenfunctions.

To explain why the qualifier "favorable" is needed, consider e. g. a finite-rank kernel of the form

\[
K(x, y) = \sum_{m=0}^{M} \lambda_m \psi_m(x) \psi_m(y), \quad \lambda_0 > \cdots > \lambda_M > 0,
\]

with \( \psi_m(x) \) real-valued smooth functions such that

\[
\int_{0}^{1} \psi_m(x) \psi_n(x) dx = \delta_{nm}.
\]

Thus \( \mathcal{I} \) is a self-adjoint operator on \( L^2((0, 1), dx) \) with eigenfunctions \( \psi_0, \ldots, \psi_M \) and infinite-dimensional null space.
Snag: A kernel identity \((H(x) - H(y))K(x, y) = 0\) does not imply that \(H(x)\) has eigenfunctions \(\psi_m(x)\). For instance, take \(M = 1\) and define \(H\) to be zero on \(\{\psi_0, \psi_1\}^\perp\), and

\[
(H\psi_0)(x) := E_0\psi_0(x) + c\lambda_1\psi_1(x), \quad (H\psi_1)(x) := E_1\psi_1(x) + c\lambda_0\psi_0(x),
\]

with \(E_0, E_1, c > 0\) (say). Then the kernel identity easily follows, yet it is plain that \(\psi_0\) and \(\psi_1\) are not eigenfunctions of \(H\).

Crux: It can be shown that the (very special) relativistic Heun kernel function does give rise to a "favorable" case, provided the couplings \(\gamma_0, \ldots, \gamma_7\) are restricted to suitable polytopes.

More specifically, the kernel function furnishes the only tool (to date) to solve the long-standing problem of promoting the commuting relativistic Heun \(A\Delta O\)s \(A_{\pm}\) to bona fide self-adjoint commuting Hilbert space operators. The point is that it yields an orthonormal base of eigenfunctions via the Hilbert-Schmidt integral operator associated to the kernel function.
3. Hilbert-Schmidt operators

- Assume $\mathcal{K}(x, y)$ is a function that is continuous on $[0, \pi/2r]^2$. Thus we have
  \[ \int_{[0,\pi/2r]^2} |\mathcal{K}(x, y)|^2 \, dx \, dy < \infty. \]

As a consequence the integral operator

\[ (I f)(x) := \int_0^{\pi/2r} \mathcal{K}(x, y) f(y) \, dy, \]

is a Hilbert-Schmidt operator on $\mathcal{H}$.

- **Crux:** This entails that there exist two sets of pairwise orthogonal unit vectors $f_0, f_1, \ldots$ and $g_0, g_1, \ldots$ such that

  \[ \mathcal{K}(x, y) = \sum_{m=0}^M s_m f_m(x) g_m(y), \quad s_0 \geq s_1 \geq s_2 \geq \cdots > 0, \quad M \leq \infty, \]

with the so-called **singular values** $s_m$ satisfying $\sum s_m^2 < \infty$. 

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The self-adjoint operators $I^*I$ and $II^*$ are then given by

\[
I^*I = \sum_{m=0}^{M} s_m^2 g_m \otimes \overline{g_m}, \quad II^* = \sum_{m=0}^{M} s_m^2 f_m \otimes \overline{f_m},
\]

so they are trace class and non-negative.

Let us call an HS operator complete when it has trivial null space and dense range. Equivalently, the vectors $f_0, f_1, \ldots$ and $g_0, g_1, \ldots$ are ONBs (orthonormal bases) for $\mathcal{H}$.

It seems there is no useful general way to recognize a complete HS operator when you meet one. (This is an open problem!)

For large classes of special HS operators, however, completeness can be shown. The proofs only involve elementary Fourier analysis (S. R., On positive Hilbert-Schmidt operators, Integr. Eq. Oper. Theory 75 (2013) 393–407). These results apply in particular to the elliptic kernel function below, provided the parameters are suitably restricted.
Specifically, the kernel function is given by

\[ S(t; x, y) \equiv \prod_{\delta_1, \delta_2 = +, -} G(\delta_1 x + \delta_2 y - ia + it), \quad a := (a_+ + a_-)/2. \]

Hence we have

\[ \lim_{a_+ \to \infty} S(t; x, y) = 1 / \prod_{\delta_1, \delta_2 = +, -} (e^{-2rt} \exp(2ir(\delta_1 x + \delta_2 y)); q^2)_{\infty}, \]

which amounts to the \( q \)-Mehler kernel from Rogers’ work with \( q = q^2_\). Taking \( x, y \) real and \( t \in (0, 2a) \), it can also be written

\[ S(t; x, y) = \exp \left( \sum_{n=1}^{\infty} \frac{2 \cos(2nrx) \cos(2nry) \sinh(2nr(a - t))}{n \sinh(nra_+) \sinh(nra_-)} \right). \]

Moreover, this \( t \)-choice ensures continuity on \([0, \pi/2r]^2\), hence the Hilbert-Schmidt property. But for \( t = a \) the associated HS integral operator has rank one!
For $t \in (0, a)$, however, $S(t; x, y)$ is of the form

$$\exp \left( \sum_{n=1}^{\infty} c_n \cos(2nrx) \cos(2nry) \right), \quad c_n > 0.$$ 

It is not obvious, but true that the integral operator on $\mathcal{H}$ with such a kernel is complete. Indeed, this is one of the results in (S. R., 2013).

The HS and completeness properties remain valid for integral kernels of the form

$$M_l(x)S(t; x, y)M_r(y),$$

where $M_l$ and $M_r$ are multiplication by functions that are continuous on $[0, \pi/2r]$ and nonzero on $(0, \pi/2r)$. 

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4. The commuting Hamiltonians

- Such multipliers arise via the generalized Harish-Chandra function

\[ c(\gamma; x) \equiv \frac{1}{G(2x + ia)} \prod_{\mu=0}^{7} G(x - i\gamma_\mu). \]

The point is that we need the kernel function

\[ K(\gamma; x, y) \equiv \frac{S(t; x, y)}{c(\gamma; x)c(\gamma'; -y)}, \]

to get the kernel identities

\[ A_\delta(\gamma; x)K(\gamma; x, y) = A_\delta(\gamma'; -y)K(\gamma; x, y), \quad \delta = +, - . \]

- Here we have \( t \equiv -\frac{1}{4} \sum_{\mu=0}^{7} \gamma_\mu \), and

\[ \gamma' \equiv -J\gamma, \quad J \equiv 1_8 - \frac{1}{4} \zeta \otimes \zeta, \]

where \( \zeta_0 = \cdots = \zeta_7 \equiv 1 \).
Also, the shift coefficients can be written

\[ V_{a,\delta}(x) = \frac{c(-x)}{c(x)} \cdot \frac{c(x + ia_{-\delta})}{c(-x - ia_{-\delta})}, \]

but for brevity we do not specify the additive parts \( V_{b,\pm}(x) \).

A more telling form for \( V_{a,\pm}(x) \) is obtained by using the right-hand side functions in the A\( \Delta \)Es satisfied by the elliptic gamma function, viz.,

\[ \frac{G(z + ia_\delta/2)}{G(z - ia_\delta/2)} = R_{-\delta}(z), \quad \delta = +, -. \]

Using these building blocks (basically renormalised theta functions with periods \( \pi/r, ia_{-\delta} \)), \( V_{a,\delta}(x) \) turns into

\[ \prod_{\mu=0}^{7} \prod_{s=+,-} R_{\delta}(x + ia_{-\delta}/2 + is\gamma_{\mu}) \prod_{s=+,-} R_{\delta}(2x + ia_{-\delta} + i\frac{s}{2}a_{\delta}) \]
From this second form it is clear by inspection that \( V_{a,\delta}(x) \) is an elliptic function that is invariant under the action of the Weyl group of \( B_8 \) on the coupling vector \( \gamma \) (permutations and sign changes). The additive coefficients \( V_{b,\pm}(x) \), however, are only invariant under \( \mathcal{W}(D_8) \) (even sign changes).

The ONB-vectors \( \{ f_n(\gamma) \}_{n=0}^{\infty} \) featuring in the singular value decomposition

\[
\mathcal{K}(\gamma; x, y) = \sum_{n=0}^{\infty} \lambda_n(\gamma) f_n(\gamma; x) \overline{f_n(\gamma'; y)}, \quad \lambda_0 > \lambda_1 \geq \lambda_2 \geq \cdots > 0,
\]

of the HS operator on \( \mathcal{H} \) corresponding to the kernel \( \mathcal{K}(\gamma; x, y) \) can be shown to have an analytic continuation to meromorphic joint eigenfunctions of the A\(\Delta\)Os \( \mathcal{A}_{\pm}(\gamma; x) \). The kernel identities play a pivotal role in this proof. Moreover, the eigenvalues \( E_{n,\pm}(\gamma) \) are real. Hence we can associate bona fide self-adjoint commuting operators with pure point spectrum to the A\(\Delta\)Os.

The above map \( J \) can be viewed as the reflection in the highest \( E_8 \)-root. This is how a \( \mathcal{W}(E_8) \) spectral invariance arises.
5. Eigenvalue asymptotics vs. elliptic OP

- Recall the elliptic gamma function is of the form $G(z) = E(z)/E(-z)$, with $E(z)$ an entire function. Define an auxiliary Harish-Chandra and weight function by

$$c_P(\gamma; x) \equiv \prod_{\mu=0}^{7} E(x \pm i\gamma_{\mu})/(1 - \exp(-4irx))E(2x \pm i(a_+ - a_-)/2),$$

$$w_P(\gamma; x) \equiv 1/c_P(\gamma; x)c_P(\gamma; -x).$$

- The point of this is that $c_P(\gamma; x)$ is related to $c(\gamma; x)$ via the identity

$$\frac{c(x)}{c(-x)} = \frac{c_P(x)}{c_P(-x)}.$$

This explains (to some extent!) that the $n \to \infty$ asymptotics of the orthonormal polynomials $P_n(\gamma; x), n \in \mathbb{N}$, associated with $w_P(\gamma; x)$, can be expected to be related to that of the ONB $f_n(\gamma; x)$. 
Here is what has been proved under certain restrictions in the above SIGMA paper. There is a sequence of signs \( s_n \) such that

\[
\| \psi_n(\gamma) - s_n f_n(\gamma) \|_{\mathcal{H}} \to 0, \quad n \to \infty,
\]

where

\[
\psi_n(\gamma; x) := \sqrt{\frac{r}{\pi}} P_n(\gamma; x) / c_P(\gamma; x).
\]

For \( \gamma \) fixed, the singular values are **distinct** for sufficiently large \( n \), and they satisfy

\[
\lambda_n(\gamma) \sim \kappa_n(t), \quad n \to \infty,
\]

(recall \( t = -\sum \gamma_{\mu} / 4 \)), where

\[
\kappa_n(t) = \frac{\pi G(2t - ia)}{r(q^2_+; q^2_+)\infty(q^2_-; q^2_-)\infty} \cdot \exp(-2nrt).
\]
6. Some open problems

- In the above SIGMA paper, complications arise from the eventuality of degeneracy of the singular values $\lambda_n(\gamma)$ and the eigenvalues $E_{n,\pm}(\gamma)$. There is, however, circumstantial evidence that no such degeneracy occurs. Any progress towards a proof of these nondegeneracy conjectures would be welcome.

- Other conjectures concern the eigenvalues $E_{n,\pm}(\gamma)$. In particular, we expect they are strictly increasing in $n$. Moreover, we expect large-$n$ asymptotics

$$E_{n,\pm}(\gamma) \sim \exp(2nra_{\mp}), \quad n \to \infty.$$  

- Quite likely, the signs $s_n$ in the above norm difference are all $\mp$. This boils down to a question about OP, cf. below.
As mentioned before, for \( a_+ \to \infty \) (\( q_+ \to 0 \)) the above elliptic OP reduce to the (normalised) \( q \)-Hermite polynomials:

\[
\lim_{a_+ \to \infty} p_n(\gamma; \cos 2rx) = \tilde{H}_n(\cos 2rx|q^2_-).
\]

Also, the \( A\Delta O \) \( A_+ \) reduces to (a similarity transform of) the usual \( AW \) \( A\Delta O \), specialised to the \( q \)-Hermite case,

\[
A = V(x)(\exp(ia_\Delta d/dx) - 1) + V(-x)(\exp(-ia_\Delta d/dx) - 1),
\]

\[
V(x) = 1/(1 - e^{4irx})(1 - q^2_- e^{4irx}),
\]

up to an additive constant. The \( A\Delta O \) \( A \) has eigenvalue \( q_-^{-2n} - 1 = \exp(2nra_-) - 1 \) on \( H_n(\cos 2rx|q^2_-) \), so the conjectured asymptotics yields the exact result for this limit!

This makes it very likely that we also have

\[
\lim_{a_+ \to \infty} c(\gamma; x)f_n(\gamma; x) = \tilde{H}_n(\cos 2rx|q^2_-) \quad (?)
\]

We can take \( s_n = + \) in this limit case, since the \( q \)-Hermite polynomials are positive for \( x \) near 0.
Is this positivity property more generally true for the orthonormal polynomials whose sign is fixed by

\[ P_n(x) = \nu_n \cos(2nr x) + \text{lower order Tchebyshevs}, \quad \nu_n > 0. \]

Do the functions \( f_n(\gamma; x) \) have \( n \) interlacing zeros in \((0, \pi/2r)\) (just as OP)?

Can one find the recurrence coefficients of the elliptic OP ‘explicitly’?

We have

\[
\lim_{a_+ \to \infty} \kappa_n(t) = \frac{2\pi}{(e^{-4rt}; q^2_\infty) (q^2_\infty; q^2_\infty) \exp(-2nrt)} =: \mu_n(t),
\]

and Rogers’ \( q \)-Mehler formula says

\[
\frac{1}{(e^{-2rt} e^{2ir(\pm x \pm y)}; q^2_\infty)} = \sum_{n=0}^{\infty} \mu_n(t) \tilde{H}_n(\cos 2rx | q^2_\infty) \tilde{H}_n(\cos 2ry | q^2_\infty).
\]

Hence our asymptotic formula yields the exact result for this limit!