INTEGRAL AND SERIES REPRESENTATIONS OF $q$-POLYNOMIALS AND FUNCTIONS: PART II
SCHUR POLYNOMIALS AND THE ROGERS-RAMANUJAN IDENTITIES

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Abstract. We give several expansion and identities involving the Ramanujan function $A_q$ and the Stieltjes–Wigert polynomials. Special values of our identities give $m$-versions of some of the items on the Slater list of Rogers-Ramanujan type identities. We also study some bilateral extensions of certain transformations in the theory of basic hypergeometric functions.

1. Introduction

In part I of this series of papers we derived several integral representations for many $q$-functions and polynomials including the $q$-exponential functions $e_q, E_q,$ and our $E_q,$ [9]. We also derived several series identities and transformations. The present work is part II, where we continue our studies and establish quite a few series identities and transformation formulas. Some of our formulas give new identities for the Schur polynomials introduced by I. Schur in [14]. We also treat a generalization of the Schur polynomials we introduced in [11].

For $0 < q < 1$ the $q$-Bessel function $J_{\nu}^{(2)}(z;q)$ and the Stieltjes-Wigert polynomials $S_n(x;q)$ are defined by [8]

\begin{equation}
J_{\nu}^{(2)}(z;q) = \frac{(q^{\nu+1};q)_{\infty}}{(q;q)_{\infty}} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{\nu+2n}}{(q,q^{\nu+1};q)_n q^{n(\nu+n)}}
\end{equation}

and

\begin{equation}
S_n(x;q) = \sum_{k=0}^{n} \frac{q^{k^2} (-x)^k}{(q;q)_k (q;q)_{n-k}}.
\end{equation}

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From them we get Ramanujan’s entire function,

\[ A_q(z) = \sum_{n=0}^{\infty} q^{n^2} (-z)^n (q; q)_n \]

by taking properly scaled limits.

Garrett, Ismail, and Stanton [4] generalized the Rogers-Ramanujan identities to

\[ \sum_{n=0}^{\infty} q^{n^2+mn} (q; q)_n = \frac{(-1)^m q^{-\binom{m}{2}} a_m(q)}{(q, q^4; q^5)_\infty} - \frac{(-1)^m q^{-\binom{m}{2}} b_m(q)}{(q^2, q^3; q^5)_\infty}, \]

where \( a_m(q) \) and \( b_m(q) \) are defined by

\[ a_m(q) = \sum_{j \geq 0} q^{j^2} \left[ \frac{m-j-2}{j} \right]_q, \quad b_m(q) = \sum_{j \geq 0} q^{j^2} \left[ \frac{m-j-1}{j} \right]_q. \]

They satisfy the initial conditions \( a_0(q) = 1 = b_1(q), a_1(q) = b_0(q) = 0. \) The Garrett–Ismail–Stanton result became known as the \( m \)-version of the Rogers-Ramanujan identities. Clearly, it gives a “closed” form evaluation of \( A_q(-q^m) \).

The polynomials \( a_m(q) \) and \( b_m(q) \) were considered by Schur in conjunction with his proof of the Rogers-Ramanujan identities; see [1] and [4] for details. We shall refer to \( a_m(q) \) and \( b_m(q) \) as the Schur polynomials. The closed form expressions for \( a_m \) and \( b_m \) in [12] were given by Andrews in [2], where he also gave a polynomial generalization of the Rogers-Ramanujan identities. We must note that \( a_{m+1}(q) \) and \( b_{m+1}(q) \) are the partial numerators and denominators of the Ramanujan continued fraction.

In this work we will use the following confluent limit of the \( q \)-Gauss sum:

\[ _1\phi_1(a; c; q, c/a) = \frac{(c/a; q)_\infty}{(c; q)_\infty}. \]

We will also use the integral representations

\[ (bq^\alpha; q)_\infty q^{\alpha^2/2} _1\phi_1(a; bq^\alpha; q, q^{\alpha+1/2}) \]

\[ = \frac{1}{\sqrt{\pi \log q^{-2}}} \int_{-\infty}^{\infty} \left( \frac{-ae^{ix}}{q} \right)_\infty \exp \left( \frac{x^2}{\log q^2} + i\alpha x \right) dx, \]

\[ q^{\alpha^2/2} A_q(q^{\alpha} z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left( z^{1/2} e^{ix} \right)_\infty \exp \left( \frac{x^2}{\log q^{-2}} + i\alpha x \right) dx, \]

\[ q^{\alpha^2/2} \left( -zq^{\alpha+1/2}; q \right)_\infty = \int_{-\infty}^{\infty} \exp \left( \frac{x^2}{\log q^2} + i\alpha x \right) dx, \]

and

\[ q^{\alpha^2/2} S_n \left( xq^{\alpha-1/2}; q \right) = \int_{-\infty}^{\infty} \left( xe^{iy}; q \right)_n \exp \left( \frac{y^2}{\log q^{-2}} + i\alpha y \right) dy, \]

which we proved in our forthcoming paper [12].

In Section 2 we prove the following theorem and discuss some of its implications.
Theorem 1.1. We have the identities

\[(q; q)_\infty \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} q^{2n} a_{2n}(q) - \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} q^{2n} b_{2n}(q),\]

\[(q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n} b_{2n+1}(q)}{(q^2; q^2)_n} - \frac{1}{(q^2; q^2)_\infty} \sum_{n=0}^{\infty} q^{2n} b_{2n+1}(q).\]

This theorem will follow from Theorem 1.2, which we now state.

Theorem 1.2. The following identities hold for all \(a, z \in \mathbb{C}\) but \(z \neq q^n, n = 0, -1, -2, \cdots:\)

\[(z; q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2 - n} z^{2n}}{(q^2; q^2)_n} A_q (q^{2n-1} a z),\]

\[(b; q)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2 - n} (ab)^n}{(q; q)_n} A_q (q^{2n-1} b^2),\]

\[\sum_{n=0}^{\infty} \frac{q^{2n^2 - n} b_{2n}}{(q^2; q^2)_n} A_q (q^{2n-1} ab) = \sum_{n=0}^{\infty} \frac{q^{2n^2 - n} (ab)^n}{(q; q)_n} A_q (q^{2n-1} b^2).\]

Formula (1.13) is (7.32) in our paper [11], while (1.14) is the result of writing \(A_q (q^{2n-1} a z)\) as a sum, then interchanging the two sums, which is then stated formally as (1.15).

Corollary 1.3. We have

\[(-z; q)_\infty = \sum_{n=0}^{\infty} \frac{q^{2n^2 - n} z^{2n}}{(q^2; q^2)_n} A_q (q^{2n-1} z).\]

This is just the case \(a = 1\) of (1.13). Recall the definition of the Jackson modified \(q\)-Bessel functions, using the notation from [6]:

\[(q^\nu; q)_\infty = \sum_{n=0}^{\infty} \frac{q^{n(q+1)} z^{2n}}{(q^2; q^2)_n} A_q (q^{2n} z),\]

\[\sum_{n=0}^{\infty} \frac{q^{n(q+1)} z^{2n}}{(q^2; q^2)_n} A_q (q^{2n} z),\]

\[\sum_{n=0}^{\infty} \frac{q^{n(q+1)} z^{2n}}{(q^2; q^2)_n} A_q (q^{2n} z).\]

In our work [12] we proved Theorem 1.4 below.

Theorem 1.4. We have the inverse pair

\[J^{(2)}_\nu (2z; q) = \frac{z^\nu}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-q^\nu)^k}{(q^2; q^2)_k} A_q (q^{\nu+k+1} z),\]

\[\frac{z^\nu A_q (q^\nu z^2)}{(q; q)_\infty} = \sum_{k=0}^{\infty} \frac{q^k}{(q^2; q^2)_k} \left( \frac{q^\nu}{z} \right)^k J^{(2)}_{k+\nu} (2z; q).\]
In the same work we also proved that

\[
A_q(z) \equiv \sum_{k=0}^{\infty} \frac{(-z)^k q^{k^2}}{(q^2; q^2)_k},
\]

(1.22)

\[
S_n(x; q) = \frac{1}{(q; q)_n} \sum_{k=0}^{\infty} \frac{(xq^n)^k}{(q; q)_k} q^{(k+1)/2} A_q(q^k x),
\]

(1.23)

\[
A_q(ab) = \sum_{k=0}^{\infty} \frac{(b; q)_k}{(q; q)_k} q^{(k+1)/2} a^k A_q(aq^k);
\]

(1.24)

see equations (7.6), (7.8), and (7.2), respectively, in [12].

In Section 2 we also discuss some consequences of the results stated so far. This leads, among other things, to new generating functions for the Schur polynomials. Section 3 contains a master identity for bilateral \( q \)-series and some noteworthy special and limiting cases of it. Section 4 has an extensive list of new identities involving the Ramanujan function, and special cases of them lead to \( m \)-versions of formulas on the Slater list [15]. It also contains several new results on the Stieltjes–Wigert polynomials.

2. Theorems 1.1, 1.2, and 1.4, and Corollary 1.3

It is obvious that (1.4) is nothing but

\[
A_q(-q^m) = \sum_{n=0}^{\infty} \frac{q^{2n^2+mn}}{(q; q)_n} \frac{(-1)^m q^{-\binom{m}{2}} a_m(q)}{(q, q^4; q^5)_{\infty}} - \frac{(-1)^m q^{-\binom{m}{2}} b_m(q)}{(q^2, q^3; q^5)_{\infty}}.
\]

(2.1)

This formula will be used repeatedly in this work.

Corollary 1.3 is the case \( a = 1 \) of Theorem 1.2

**Proof of Theorem 1.1** We take \( z = q \) in Corollary 1.3 and apply (1.4) to obtain

\[
(-q; q)_{\infty} = \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q^2; q^2)_n} A_q(-q^{2n})
\]

\[
= \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(q^2; q^2)_n} \left( q^{-\binom{2}{2}} a_{2n}(q) \frac{(q, q^4; q^5)_{\infty}}{(q^2, q^3; q^5)_{\infty}} - q^{-\binom{2}{2}} b_{2n}(q) \right),
\]

which simplifies to (1.11). Similarly \( z = q^2 \) in (1.16) leads to (1.12).

\[\square\]

Let \( a = 0 \) in (1.13) to get

\[
(z; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{2}{2}} z^n}{(q, z; q)_n} = \sum_{n=0}^{\infty} \frac{q^{2n^2-n} z^{2n}}{(q^2; q^2)_n}.
\]

Obviously the above equation is

\[
A_q^2(-z^2/q) = (z; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{\binom{2}{2}} z^n}{(q, z; q)_n},
\]

(2.2)
which relates the Ramanujan function to a modified $q$-Bessel function $I^{(3)}$. The special case $z = q^{m+1/2}$ is of interest and gives

$$
(q^{m+1/2}; q)_\infty \sum_{n=0}^{\infty} \frac{q^{mn+n^2/2}}{(q, q^{m+1/2}; q)_n} = (-1)^m q^{2m} \left[ \frac{a_m(q^2)}{(q^2, q^8; q^{10})_\infty} - \frac{b_m(q^2)}{(q^4, q^6; q^{10})_\infty} \right].
$$

(2.3)

It is clear that the left-hand side is a $q$-analogue of the fact that $I_{m+1/2}$ is a linear combination of $I_{\pm 1/2}$ with coefficients related to the Lommel polynomials. The cases $m = 0, 1$ are

$$
\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q; q)_{2n}} = \frac{1}{(q; q^2)_{\infty} (q^4, q^{10}; q^{20})_\infty},
$$

(2.4)

$$
\sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q; q)_{2n+1}} = \frac{1}{(q; q^2)_{\infty} (q^8, q^{12}; q^{20})_\infty}.
$$

(2.5)

The identities (2.4) and (2.5) are (79) and (96) on Lucy Slater’s list [15]. They go back to Rogers.

For $\ell \in \mathbb{N}_0$, we let $a = -q^{\ell+1}/z$ in (1.13) and establish

$$
(z; q)_\infty \phi_1 \left( -q^{\ell+1}/z; z; q, -z \right) = \sum_{n=0}^{\infty} \frac{q^{2n^2-n^2} z^{2n}}{(q^2; q^2)_n} A_q ( -q^{2n+\ell} )
$$

$$
= \frac{(-1)^\ell q^{-\ell}(z)}{(q, q^4; q^5)_\infty} \sum_{n=0}^{\infty} \frac{a_{2n+\ell}(q)}{(q^2; q^2)_n} \left( \frac{z}{q^2} \right)^{2n} - \frac{(-1)^\ell q^{-\ell}(z)}{(q^2, q^3; q^5)_\infty} \sum_{n=0}^{\infty} \frac{b_{2n+\ell}(q)}{(q^2, q^3; q^2)_n} \left( \frac{z}{q^2} \right)^{2n}.
$$

(2.6)

This proves the following theorem.

**Theorem 2.1.** If $\ell \in \mathbb{N}_0$, $z \notin \{1, q^{-1}, q^{-2}, \ldots \}$, then

$$
(z; q)_\infty \phi_1 \left( -q^{\ell+1}/z; z; q, -z \right) = \frac{(-1)^\ell q^{-\ell}(z)}{(q, q^4; q^5)_\infty} \sum_{n=0}^{\infty} \frac{a_{2n+\ell}(q)}{(q^2; q^2)_n} \left( \frac{z}{q^2} \right)^{2n} - \frac{(-1)^\ell q^{-\ell}(z)}{(q^2, q^3; q^5)_\infty} \sum_{n=0}^{\infty} \frac{b_{2n+\ell}(q)}{(q^2, q^3; q^2)_n} \left( \frac{z}{q^2} \right)^{2n}.
$$

(2.7)

Let $z = -q^{\ell+1}$ in (1.13) to get

$$
(-q^{\ell+1}; q)_\infty = (-1)^\ell q^{-\ell}(z) \sum_{n=0}^{\infty} \left[ \frac{a_{2n+\ell}(q) q^{2n}}{(q, q^4; q^5)_\infty (q^2; q^2)_n} - \frac{b_{2n+\ell}(q) q^{2n}}{(q^2, q^3; q^5)_\infty (q^2; q^2)_n} \right].
$$

(2.8)

From (1.24) we obtain

$$
A_q (-b) = \frac{1}{(q, q^4; q^5)_\infty} \sum_{k=0}^{\infty} \frac{b(q)_k q^k a_k (q)}{(q; q)_k} - \frac{1}{(q^2, q^3; q^5)_\infty} \sum_{k=0}^{\infty} \frac{b(q)_k q^k b_k (q)}{(q; q)_k}.
$$

(2.9)

The special case $b = q^m$ of (2.8) is the interesting identity

$$
(-1)^m q^{m/2} \left[ \frac{a_m(q)}{(q^1, q^4, q^5; q^8)_\infty} - \frac{b_m(q)}{(q^2, q^3, q^5; q^8)_\infty} \right] = \frac{1}{(q^1, q^4, q^5; q^5)_\infty} \sum_{k=0}^{\infty} \frac{(q^m)_k q^k a_k (q)}{(q; q)_k} - \frac{1}{(q^2, q^4, q^5; q^5)_\infty} \sum_{k=0}^{\infty} \frac{(q^m)_k q^k b_k (q)}{(q; q)_k}.
$$

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The further specialization, \( m = 1 \), is also interesting:

\[
1 + \sum_{k=0}^{\infty} b_k(q) q^k = \frac{(q^2, q^3; q^5)_\infty}{(q, q^4; q^5)_\infty} \sum_{k=0}^{\infty} a_k(q) q^k.
\]  

(2.10)

The identities (2.9) and (2.10) should have very interesting partition theoretic interpretations.

**Theorem 2.2.** For any \( m, n \in \mathbb{N} \) we have

\[
(-1)^n q^{-\binom{n}{2}} \left[ \frac{a_m(q)}{(q, q^4; q^5)_\infty} - \frac{b_m(q)}{(q^2, q^3; q^5)_\infty} \right] = (-1)^{m+n} \sum_{k=0}^{\infty} \frac{q^{k(k+m)}}{(q; q)_k} q^{-\binom{k+m}{2}} \left[ \frac{a_{m+n+2k}(q)}{(q, q^4, q^5; q^5)_\infty} - \frac{b_{m+n+2k}(q)}{(q^2, q^3, q^5; q^5)_\infty} \right].
\]

(2.11)

Proof. In [12] we proved that

\[
A_q(z) = \sum_{k=0}^{\infty} \frac{q^{k^2}}{(q; q)_k} (-z)^k (w; q)_k A_q(wzq^{2k}).
\]

(2.12)

The theorem is the special case \( z = -q^m, w = q^n \) in (2.12). \( \square \)

It must be noted that the series on the right-hand side of (2.11) converges because (1.4) shows that

\[
\lim_{m \to \infty} \left[ (-1)^n q^{-\binom{n}{2}} a_m(q) - (-1)^m q^{-\binom{m}{2}} b_m(q) \right] = 1.
\]

Using (1.22) and (2.1) we find that for \( m = 0, 1, \ldots \), we have

\[
(-1)^m q^{-\binom{m}{2}} \left[ \frac{a_m(q)}{(q, q^4; q^5)_\infty} - \frac{b_m(q)}{(q^2, q^3; q^5)_\infty} \right] = (-q^{m+2}; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+mn}}{(q^2, -q^{m+2}; q^2)_n}.
\]

(2.13)

Here again the cases \( m = 0 \) and \( m = 1 \) are

\[
\frac{1}{(q, q^4; q^5)_\infty} = (-q^2; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^2, -q^2; q^2)_n}.
\]

(2.14)

\[
\frac{1}{(q^2, q^3; q^5)_\infty} = (-q^3; q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^2, -q^3; q^2)_n}.
\]

(2.15)

Formulas (2.14) and (2.15) are equations (20) and (17) on the Slater list [15]. They were first proved by L. J. Rogers.
3. Bilateral sums

In this section we derive bilateral sum identities.

**Theorem 3.1.** Let \( r \geq s, \alpha_j \in \mathbb{C}, 1 \leq j \leq r, \beta_k \in \mathbb{C}, 1 \leq k \leq s, \frac{|r_1|}{|r_2|} < |w| < 1 \) and \(|z| < |w| \) if \( r = s + 1 \). Then

\[
\sum_{m=-\infty}^{\infty} \left( \frac{\alpha_1 q^m}{\beta_1 q^m} \right)_m r \phi_s \left( \begin{array}{c}
\alpha_1 q^m, \ldots, \alpha_r \\
\beta_1 q^m, \ldots, \beta_s \\
q, z
\end{array} \right) w^m \\
= \frac{(q/(\alpha_1 w), \alpha_1 w, q, \beta_1/\alpha_1; q)_\infty}{(\beta_1, q/\alpha_1, w, \beta_1/(\alpha_1 w); q)_\infty} r^{-1} \phi_{s-1} \left( \begin{array}{c}
\alpha_2, \ldots, \alpha_r \\
\beta_2, \ldots, \beta_s \\
q, \frac{z}{w}
\end{array} \right).
\]

**Proof.** The left-hand side is

\[
= \sum_{m=-\infty}^{\infty} w^m \sum_{n=0}^{\infty} \left( \frac{\alpha_1 q^n}{\beta_1 q^n} \right)_n \frac{z^n}{(q, \beta_2, \ldots, \beta_s; q)_n} \left( -\frac{q^{n-1/2}}{q^{n+1/2}} \right)^{n(s+1-r)}
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{\alpha_1, \ldots, \alpha_r; q)_n}{(q, \beta_1, \ldots, \beta_s; q)_n} \left( -\frac{q^{n-1/2}}{q^{n+1/2}} \right)^{n(s+1-r)} \sum_{m=-\infty}^{\infty} \frac{(\alpha_1 q^n; q)_m}{(\beta_1 q^n; q)_m} w^m
\]

\[
= \sum_{n=0}^{\infty} \left( \frac{\alpha_1, \ldots, \alpha_r; q)_n}{(q, \beta_1, \ldots, \beta_s; q)_n} \left( -\frac{q^{n-1/2}}{q^{n+1/2}} \right)^{n(s+1-r)} \frac{(\beta_1/q^n, \beta_1/(\alpha_1 w), q^{n-1}/(\alpha_1 w), q^n/\alpha_1, w, q)_{\infty}}{(q^{1-n}/\alpha_1; q)_{\infty}}
\]

\[
= \frac{(q/(\alpha_1 w), \alpha_1 w, q, \beta_1/\alpha_1; q)_{\infty}}{(\beta_1, q/\alpha_1, w, \beta_1/(\alpha_1 w); q)_{\infty}} \sum_{n=0}^{\infty} \frac{(\alpha_1, \ldots, \alpha_r; q)_n}{(q, \beta_2, \ldots, \beta_s; q)_n} \left( -\frac{q^{n-1/2}}{q^{n+1/2}} \right)^{n(s+1-r)} (\beta_1/q^n, \beta_1/(\alpha_1 w), q^{n-1}/(\alpha_1 w), q^n/(\alpha_2 \alpha_3 z), q/\alpha_1, \alpha_2 \alpha_3 z, \beta_2/\beta_1, \beta_2/(\alpha_1 \alpha_2 \alpha_3 z); q)_{\infty},
\]

and the theorem follows. \(\square\)

**Corollary 3.2.** For \( |\beta_1/\alpha_1| < |\alpha_2 \alpha_3 z/\beta_2| < 1 \) and \( |\alpha_2 \alpha_3| < |\beta_2| \) we have

\[
\sum_{m=-\infty}^{\infty} \left( \frac{\alpha_1 q^m}{\beta_1 q^m} \right)_m \frac{z^n}{(\beta_1, q/\alpha_1, w, \beta_1/(\alpha_1 w); q)_{\infty}} \left( -\frac{q^{n-1/2}}{q^{n+1/2}} \right)^{n(s+1-r)} \frac{(\alpha_2 \alpha_3 z; q^n/(\alpha_1 \alpha_2 \alpha_3 z); q)_{\infty}}{(\beta_2/\beta_1, \beta_2/(\alpha_1 \alpha_2 \alpha_3 z); q)_{\infty}}
\]

Proof. Apply Theorem 3.1 with \( z = w \beta_2/\alpha_1 \alpha_2 \) and use the \( q \)-Gauss sum. \(\square\)

**Corollary 3.3.**

(i) For \( |\beta_1/\alpha_1| < |w| < 1 \) and \( |z| < |w| \) we have

\[
\sum_{m=-\infty}^{\infty} \left( \frac{\alpha_1 q^m}{\beta_1 q^m} \right)_m \frac{z^n}{(\beta_1, q/\alpha_1, w, \beta_1/(\alpha_1 w), z/w; q)_{\infty}} \left( -\frac{q^{n-1/2}}{q^{n+1/2}} \right)^{n(s+1-r)} \frac{(\alpha_2 \alpha_3 z; q^n/(\alpha_1 \alpha_2 \alpha_3 z); q)_{\infty}}{(\beta_2/\beta_1, \beta_2/(\alpha_1 \alpha_2 \alpha_3 z); q)_{\infty}}
\]

(ii) For \( |\beta_1/\alpha_1| < |\alpha_2 z/\beta_2| < 1 \) we have

\[
\sum_{m=-\infty}^{\infty} \left( \frac{\alpha_1 q^m}{\beta_1 q^m} \right)_m \frac{z^n}{(\beta_1, q/\alpha_1, w, \beta_1/(\alpha_1 w), z/w; q)_{\infty}} \left( -\frac{q^{n-1/2}}{q^{n+1/2}} \right)^{n(s+1-r)} \frac{(\alpha_2 \alpha_3 z; q^n/(\alpha_1 \alpha_2 \alpha_3 z); q)_{\infty}}{(\beta_2/\beta_1, \beta_2/(\alpha_1 \alpha_2 \alpha_3 z); q)_{\infty}}
\]
Proof. The results follow from Theorem 3.1 and the $q$-binomial theorem and a confluent limit of the $q$-Gauss theorem.

It is important that we use the following definition of $q$-shifted factorial:

\[(a; q)_\infty = \prod_{k=0}^{\infty} (1 - aq^k),\]

where $a, q, n \in \mathbb{C}$, $|q| < 1$, and $aq^n \neq q^{-k}$, $k \in \mathbb{N}_0$. Then it is clear that

\[(a; q)_{n+m} = (a; q)_n (aq^n; q)_m, \quad a, n, m \in \mathbb{C}.\]

In the rest of this section we derive identities involving $I^{(2)}_n(z; q)$. The next theorem uses Ramanujan’s $1 \psi_1$ sum [5] (II.28)

\[\sum_{n=-\infty}^{\infty} (a; q)_n z^n = \frac{(q, b/a, az, q/az; q)_\infty}{(b, q/a, z, b/az; q)_\infty}, \quad |b/a| < |z| < 1.\]

**Theorem 3.4.** If $|q^{\nu+1}/a| < |w| < 1$, then

\[
\sum_{m=-\infty}^{\infty} (a; q)_m \left(\frac{w}{z}\right)^m I^{(2)}_n(z; q) = \frac{z^\nu (q^{\nu+1}/a, aw, q/aw; q)_\infty}{(q/a, q^{\nu+1}/aw, w; q)_\infty} \sum_{n=0}^{\infty} \left(\frac{z^2 q^{\nu+1}}{aw}\right)^n \frac{n^{(2)}(z)}{(q; q)_n} \frac{(w; q)_n}{(q, q^{\nu+1}/a; q)_n}.
\]

proof. The definition (1.18) implies that

\[I^{(2)}_n(z; q) = \frac{(z/2)^\nu}{(q; q)_\infty} \sum_{n=0}^{\infty} q^{n^2+n\nu} \left(\frac{z^2}{4}\right)^n (q^{n+\nu+1}; q)_\infty \frac{(z q^n/2)^m}{(q^{n+\nu+1}; q)_m}.\]

Assuming $|q^{\nu+1}/a| < |wz/2| < 1$ we use the above expansion and establish the generating function

\[\sum_{m=-\infty}^{\infty} (a; q)_m w^m I^{(2)}_n(z; q) = \frac{(z/2)^\nu}{(q; q)_\infty} \sum_{n=0}^{\infty} q^{n^2+n\nu} \left(\frac{z^2}{4}\right)^n (q^{n+\nu+1}; q)_\infty \times \frac{(q, q^{n+\nu+1}/a, awz q^n/2, 2q/(awz q^n); q)_\infty}{(q^{n+\nu+1}/a, q/a, wzq^n/2, 2q^{n+\nu+1}/(awz); q)_\infty} \frac{(z/2)^\nu (q^{\nu+1}/a, awz/2, q/(awz); q)_\infty}{(q/a, 2q^{\nu+1}/(awz), wz/2; q)_\infty} \sum_{n=0}^{\infty} \left(\frac{z^2 q^{\nu+1}}{2aw}\right)^n \frac{q^{(2)}(z)}{(q, q^{\nu+1}/a; q)_n},\]

where the Ramanujan $1 \psi_1$ was used in the second to last step. This establishes our theorem. \[\square\]
4. Series involving $A_q$ and $S_n$

In this section we prove the identities contained in Theorems 4.1 and 4.2 and consider some of their corollaries. The proofs use the identities

\begin{align*}
(4.1) \quad S_n(ab;q) &= b^n \sum_{k=0}^{n} \frac{(b^{-1};q)_k}{(q;q)_k} (-1)^k q^{-nk} q^{(k+1)/2} S_{n-k}(aq^k;q), \\
(4.2) \quad S_{2n+1}(q^{-2n-1};q) &= 0, \quad S_{2n}(q^{-2n};q) = \frac{(-1)^n q^{-n^2}}{(q^2;q^2)_n}, \\
(4.3) \quad S_n(-q^{-n+1/2};q) &= \frac{q^{-(n^2-n)/4}}{(q^{1/2};q^{1/2})_n}, \quad S_n(-q^{-n-1/2};q) = \frac{q^{-(n^2+n)/4}}{(q^{1/2};q^{1/2})_n}.
\end{align*}

These identities were proved in Section 6 of our paper [11].

**Theorem 4.1.** For $n = 0, 1, \ldots$ we have

\begin{align*}
(4.4) \quad \frac{q^{n/2} S_n(bq^{-n};q)}{(-b)^n} &= \sum_{j=0}^{[n/2]} \frac{q^{2j-n} (-1)^j (b^{-1};q)_{n-2j}}{(q^2;q^2)_j (q;q)_{n-2j}}, \\
(4.5) \quad \frac{S_n(-q^{-n+1/2};q) q^{n/2}}{(-b)^n} &= \sum_{k=0}^{n} \frac{q^{(k^2-k)/4}}{(q^{1/2};q^{1/2})_k} \frac{(b^{-1};q)_{n-k}}{(q;q)_{n-k}} (-1)^k, \\
(4.6) \quad \frac{S_n(-q^{-n-1/2};q) q^{n/2}}{(-b)^n} &= \sum_{k=0}^{n} \frac{q^{(k^2-3k)/4}}{(q^{1/2};q^{1/2})_k} \frac{(b^{-1};q)_{n-k}}{(q;q)_{n-k}} (-1)^k.
\end{align*}

**Proof.** The proof of (4.4) follows from (4.1) and (4.2). Formulas (4.5) and (4.6) follow from (4.1) and both parts of (4.3). \hfill \Box

The proof of the next theorem makes use of

\begin{align*}
(4.7) \quad I^{(2)}_\nu \left(2q^{-n/2};q\right) &= \frac{q^{\nu n/2} S_n(-q^{-\nu-n};q)}{(q^{n+1};q)_\infty} = \frac{q^{-\nu n/2} S_n(-q^{\nu-n};q)}{(q^{n+1};q)_\infty}.
\end{align*}

from our work [11].

**Theorem 4.2.** The Ramanujan function $A_q$ satisfies the following summation theorems:

\begin{align*}
(4.8) \quad \sum_{k=0}^\infty \frac{q^{k+1}}{(q;q)_k} A_q(aq^{k-2n}) &= (-q;q)_\infty (aq^{1-2n};q^2)_\infty, \\
(4.9) \quad \sum_{k=0}^\infty \frac{q^{k}}{(q;q)_k} A_q(-q^{k+n+1/2}) &= \frac{(q;q)_n}{q^{(n^2-n)/4}(q^{1/2};q^{1/2})_n}, \\
(4.10) \quad \sum_{k=0}^\infty \frac{q^{k^2/2}}{(q;q)_k} A_q(-q^{-k-n-1/2}) &= \frac{(-1)^n (q;q)_n}{q^{(n^2+n)/4}(q^{1/2};q^{1/2})_n}.
\end{align*}
Moreover we have the expansions

\begin{align}
(4.11) \quad A_q(w) &= (wq; q)_\infty \sum_{n=0}^\infty q^{3n^2} \frac{(-w^2)^n}{(q^2, wq, wq^2; q)_n}, \\
(4.12) \quad A_q(w) &= (-wq^{1/2}; q)_\infty \sum_{n=0}^\infty q^{n(3n-1)/4} \frac{(-w)^n}{(q^{1/2}; q^{1/2})_n (wq^{1/2}; q)_n}, \\
(4.13) \quad A_q(w) &= (-wq^{3/2}; q)_\infty \sum_{n=0}^\infty q^{n(3n+1)/4} \frac{(-w)^n}{(q^{1/2}; q^{1/2})_n (-wq^{3/2}; q)_n}.
\end{align}

The special cases \( a = 1 \) and \( a = q \) of (4.8) are worth recording. For \( n > 0 \) they are

\begin{align}
(4.14) \quad \sum_{k=0}^\infty q^{k+1} \frac{(k+1)}{(q; q)_k} A_q(q^{k-2n-1}) &\equiv 0, \\
(4.15) \quad \sum_{k=0}^\infty q^{k+1} \frac{(k+1)}{(q; q)_k} A_q(q^{k-2n}) &\equiv (-1)^n q^{-n^2} (q; q^2)_n.
\end{align}

Proof of Theorem of 4.2

The left-hand side of (4.8) is

\[
\sum_{k,m=0}^\infty \frac{(-a)^m q^{m^2+(k+1)/2} q^{km-2nm}}{(q; q)_k (q; q)_m} = \sum_{m=0}^\infty \frac{(-a)^m q^{m^2-2mn}}{(q; q)_m} (-q^{m+1}; q)_\infty
\]

\[
= (-q; q)_\infty \sum_{m=0}^\infty \frac{(-a)^m q^{m^2-2mn}}{(q^2; q^2)_m},
\]

and we have proved (4.8). Let us first rewrite (1.20) in the equivalent form

\begin{align}
(4.16) \quad I^{(2)}_\nu(2z; q) &= \frac{z^\nu}{(q; q)_\infty} \sum_{k=0}^\infty (-q^\nu)^k q^{k+1} A_q(-q^{\nu+k} z^2).
\end{align}

Formulas (4.9) and (4.10) are the special cases \( \nu = 1/2, -1/2 \), respectively, combined with (4.7). We now come to (4.11). Write \( (wq; q)_\infty (qw; q)_2 \) as \( (wq^{2n+1}; q)_\infty \); then expand it into powers of \( w \) to see that the right-hand side of (4.11) is

\[
\sum_{k,n=0}^\infty (-1)^{n+k} w^{2n+k} (q; q)_k (q^2; q^2)_n q^{3n^2+2nk+(k+1)/2} = \sum_{m=0}^\infty (-w)^m q^{(m+1)/2} \sum_{n=0}^{\lfloor n/2 \rfloor} (-1)^n q^{n(n-1)/4} \frac{w^{n+k}}{(q^2; q^2)_n (q; q)_m^{n-2n}}.
\]

Denote the \( n \)-sum by \( c_m \). It is easy to see that

\[
\sum_{m=0}^\infty c_m t^m = \frac{(t^2; q^2)_\infty}{(t; q)_\infty} = (-t; q)_\infty.
\]

Thus \( c_m = q^{(2)}(2t; q)_m \). This establishes (4.11). The proof of (4.12) is similar. Its right-hand side is

\[
\sum_{n=0}^\infty \frac{q^{n(3n-1)/4} (-w)^n}{(q^{1/2}; q^{1/2})_n} (-wq^{n+1/2}; q)_\infty = \sum_{n,k=0}^\infty \frac{q^{n(3n-1)/4} w^{n+k}}{(q^{1/2}; q^{1/2})_n (q; q)_k} (-1)^n q^{nk+k^2/2}
\]

\[
= \sum_{m=0}^\infty w^m q^{m^2/2} \sum_{n+k=m} (-1)^n q^{n(n-1)/4} \frac{1}{(q^{1/2}; q^{1/2})_n (q; q)_k}.
\]
Denoting the inner sum by $d_m$ we find that
\[
\sum_{m=0}^{\infty} d_m t^m = \frac{(t; q^{1/2})_{\infty}}{(t; q)_{\infty}} = (t q^{1/2}; q)_{\infty},
\]
and we conclude that $d_m = q^{m^2/2} (-1)^m / (q; q)_m$ and the proof of (4.12) is complete. The proof of (4.13) is parallel to the proof of (4.12) and will be omitted. \(\square\)

The cases $w = -q^m$ of (4.11)–(4.13) are interesting. They are
\[
\frac{(-1)^m q^{-\binom{m}{2}} a_m(q)}{(q, q^4; q^5)_{\infty}} = \frac{(-1)^m q^{-\binom{m}{2}} b_m(q)}{(q^2, q^3; q^5)_{\infty}}
\]
(4.17)
\[
= (-q^{m+1}; q)_{\infty} \sum_{n=0}^{\infty} \frac{\alpha q^{3n^2 + 2mn}}{(q^2; q^2)^n (-q^{m+1}; q)_n},
\]
(4.18)

The case $m = 0$ of (4.18) is on the Slater list; see (44)–(45) in [15].

The structure of the identities (4.12) and (4.13) suggests that we consider the series
\[
(-w q^{\alpha + 1/2}; q)_{\infty} \sum_{n=0}^{\infty} \frac{q^{n(3n+4\alpha)/4} (-w)^n}{(q^{1/2}; q^{1/2})_n (-w q^{\alpha + 1/2}; q)_n}
\]
\[
= \sum_{n,k=0}^{\infty} \frac{(-1)^n q^{n(3n+4\alpha)/4} w^{n+k} q^{k(\alpha+n)+k^2/2}}{(q^{1/2}; q^{1/2})_n (q; q)_k}
\]
\[
= \sum_{m=0}^{\infty} w^m q^{\alpha m + m^2/2} \sum_{n=0}^{m} \frac{(-1)^n q^{n(c-\alpha)+n^2/4}}{(q^{1/2}; q^{1/2})_n (q; q)_{m-n}}.
\]

Denote the $n$ sum by $c_m$. Thus
\[
\sum_{m=0}^{\infty} c_m t^m = \frac{(t q^{-\alpha + 1/4}; q^{1/2})_{\infty}}{(t; q)_{\infty}}.
\]
At this stage we take $c = \alpha - 1/4$ or $c = \alpha - 3/4$ and find that $c_m = (-1)^m q^{m^2/2} / (q; q)_m$ or $c_m = (-1)^m q^{m(m-1)/2} / (q; q)_m$, respectively. This essentially leads to (4.12) and (4.13). On the other hand we have other choices for $c$.

We let $c = \alpha - 1/4 - j/2$. In this case it turns out that there is no loss of generality in assuming $\alpha = 0$. We also assume $j > 0$, since $j = 0$ is already covered.
Thus
\[ (-wq^{1/2}; q) \infty \sum_{n=0}^{\infty} \frac{q^{n(3n-2j-1)/4} (-w)^n}{(q^{1/2}; q^{1/2})_n (wq^{1/2}; q)_n} = \sum_{m=0}^{\infty} w^m q^{m^2/2} \sum_{n=0}^{\infty} \frac{(-1)^n q^{-jn/2+n(n-1)/4}}{(q; q)k(q^{1/2}; q^{1/2})_n}. \]

Again we denote the $n$-sum by $c_m$ and find that
\[ \sum_{m=0}^{\infty} c_m t^m = \frac{(tq^{-j/2}; q^{1/2})_{\infty}}{(t; q)_{\infty}} = (tq^{-j/2}; q^{1/2})_{j-1}(tq^{-1/2}; q)_{\infty}. \]

Therefore
\[ c_m = (-1)^m \sum_{s+k=m} \left[ \frac{j-1}{s} \right] q^{s(s-1)/4} q^{-js/2} t^{k(k-2)/2} (q; q)_k. \]

This proves that
\[ (-wq^{1/2}; q) \infty \sum_{n=0}^{\infty} \frac{q^{n(3n-2j-1)/4} (-w)^n}{(q^{1/2}; q^{1/2})_n (wq^{1/2}; q)_n} = \sum_{m=0}^{\infty} c_m t^m = \frac{(tq^{-j/2}; q^{1/2})_{\infty}}{(t; q)_{\infty}} = (tq^{-j/2}; q^{1/2})_{j-1}(tq^{-1/2}; q)_{\infty}. \]

It is clear that (4.13) is the special case $j = 1$, with $w \to wq$.

**Theorem 4.3.** For $|c/(ab)| < 1$ we have
\[ \sum_{n=0}^{\infty} \frac{(b; q)_n q^{n^2/2} S_n (aq^{-n-1/2}; q)}{(c; q)_n} \frac{(c)}{ab}^n = \frac{(c/b; q)_{\infty}}{(c; q)_{\infty}} \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q; q)_n} \frac{(c)}{ab}^n \frac{A_q \left( \frac{cq^{n-1/2}}{a} \right)}{(c; q)_{\infty}} \frac{1}{\phi_1 (b; a; q, -cq^{1/2}/(ab)).} \]

In particular, for $|cz| < q^{1/2}$ we have
\[ \sum_{n=0}^{\infty} \frac{q^{n^2/2}}{(q; q)_n} (cz)^n A_q (cq^{n-1}) = \frac{(c; q)_{\infty}}{(cz; q)_{\infty}} \sum_{n=0}^{\infty} q^{n^2} \prod_{j=0}^{2n-1} (z - q^2) (q^{2j}) \frac{(-c)^n}{(q^{2j}, c, cq^2; q^2)_{\infty}} \]
and
\[ A_q (c) = \sum_{n=0}^{\infty} q^{3n^2+n} \frac{(-c)^n}{(q^{2}, cq, cq^2; q^2)_{\infty}}. \]

Similarly, for $|c^2 z| < q^{-1}$ we have
\[ \sum_{n=0}^{\infty} \frac{q^{n^2+n} (-c^2 z)^n}{(q^{2}; q^{2})_{n}} A_q^2 (-c^2 q^{2n}) = \frac{(c^2 q; q^2)_{\infty}}{(c^2 z q; q^2)_{\infty}} \sum_{n=0}^{\infty} q^{n+1} \prod_{j=0}^{n-1} (z - q^{2j}) \frac{(-c)^n}{(q, cq^{1/2}, -cq^{1/2}; q)_{n}}. \]
and
\[ A_q^2 (-c^2) = \frac{(c^2 q; q^2)_{\infty}}{(q; q^2, -cq^{1/2}; q)_{\infty}} \sum_{n=0}^{\infty} q^{(3n^2-n)/2} c_{2n} \frac{(-c_{2n}^2)^n}{(q, cq^{1/2}, -cq^{1/2}; q)_{n}}. \]
Proof. The coefficient of $a^{-m}$ on the left-hand side of (4.20) is

\[
\sum_{n-k=m} \frac{(b; q)_n q^{n^2/2}(-1)^k}{(c; q)_n(q; q)_k(q; q)_{n-k}} (c/b)^n q^{k^2-k(n+1)/2} = \frac{(b; q)_m (c/b)^m}{(c; q)_m (q; q)_m} \phi_1(bq^m; cq^m; q, c/b) = \frac{(b; q)_m(c/b; q)_\infty}{(q; q)_m(c; q)_\infty} \left(\frac{c}{b}\right)^m q^{m^2/2}
\]

where we used (1.6). This is the same as the same coefficient of the right-hand side. Similarly the coefficient of $a^{-m}(c; q)_\infty/(c; q)_\infty$ of the middle term in (4.20) is

\[
\sum_{k+n=m} \frac{(-c)^k q^{n^2/2}}{(q; q)_k(q; q)_n} \left(\frac{c}{b}\right)^n q^{k(n+1)/2} = \frac{(c/b)_m}{(c; q)_m} \sum_{k=0}^m \left[\frac{m}{k}\right] q^{(k)_2} (-b)^k = \frac{(c/b)^m}{(q; q)_m(b; q)_m},
\]

which gives (4.20).

Let $a = q^{1/2}$ in (4.20) under the condition $|c/b| < q^{1/2}$ by applying (4.2) to get

\[
\frac{(c/b; q)_\infty}{(c; q)_\infty} \sum_{n=0}^\infty \frac{q^{(2)_n}}{(q; q)_n} \left(\frac{c}{b}\right)^n A_2 (cq^{n-1}) = \sum_{n=0}^\infty \frac{(b; q)_n q^{(3)_n} S_n (q^{n-1}; q)}{(c; q)_n} (c/b)^n = \sum_{n=0}^\infty \frac{(-1)^n (b; q)_2n q^{2n}_2}{(c; q)_2n (q^2; q^2)_n} (c/b)^{2n} = \sum_{n=0}^\infty q^{n^2} \prod_{j=0}^{2n-1} \left(1/b - q^j\right) (-c^2)^n \prod_{j=0}^{2n-1} \left(1/b - q^j\right) (c/b)^n,
\]

which gives (4.21). Let $b \to \infty$ in the above equation or $z = 0$ in (4.21) to obtain (4.22). Let $a = -1$ in (4.20) by (13) to obtain

\[
\frac{(c/b; q)_\infty}{(c; q)_\infty} \sum_{n=0}^\infty \frac{q^{n^2/2}}{(q; q)_n} \left(-\frac{c}{b}\right)^n A_2 (-cq^{n-1/2}) = \sum_{n=0}^\infty \frac{(b; q)_n q^{n^2/2} S_n (q^{n-1/2}; q)}{(c; q)_n} (-c)^n = \sum_{n=0}^\infty \frac{q^{(2)_n/2} \prod_{j=0}^{n-1} (1/b - q^j) (-c^n)}{(q^{1/2}; c^{1/2}, -c^{1/2}; q^{1/2})_n} = \sum_{n=0}^\infty q^{(2)_n/2} \prod_{j=0}^{n-1} (1/b - q^j) (-c^n)
\]

which gives (4.23), taking $b \to \infty$ in the above equation to get (4.24).
and
\[(4.26)\]
\[
\sum_{n=0}^{\infty} \frac{q^{(3n^2-n)/2+2mn}}{(q, q^{m+1/2}, -q^{m+1/2}; q)_n} = (-1)^m q^{-m(m-1)} \left\{ \frac{a_m(q^2)}{(q^2, q^8; q^{10})_\infty} - \frac{b_m(q^2)}{(q^4, q^8; q^{10})_\infty} \right\}.
\]

Proof. Equation (4.25) is obtained by letting \(c = -q^m, \ m = 0, 1, \ldots \) in (4.22), while equation (4.26) is followed by setting \(c = q^m, \ m = 0, 1, \ldots \) in (4.24).

\[\square\]

**Theorem 4.5.** For \(m \in \mathbb{Z}\), let
\[(4.27)\]
\[
u_m(a; q) = \sum_{n=-\infty}^{\infty} \frac{q^{n^2+mn}}{(aq; q)_n},
\]
as in \[11\]. Then,
\[(4.28)\]
\[
\frac{u_m(a; q)}{(q; q)_\infty} = \int_{-\infty}^{\infty} \exp\left(\frac{x^2}{\log q^2}\right) \frac{(-q^{m+1/2} e^{ix} - q^{1/2 - m} e^{-ix}; q)_\infty}{(aq, -aq^{1/2} e^{ix}; q)_\infty} \sqrt{\pi \log q^{-2}} dx.
\]

Moreover, we have
\[(4.29)\]
\[
u_m(a; q) = \frac{(q; q)_\infty}{(aq; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^m/a; q)_k}{(q; q)_k} q^{(k+1)/2} \frac{A_q(-q^{-m-k})}{(-a)^k}
\]
and
\[(4.30)\]
\[
u_m(a; q) = \frac{(q^2, -q, -q; q^2)_\infty}{(aq; q)_\infty q^{m^2/4}} \sum_{2(1+m)} \frac{(-a)^k}{(q; q)_k} q^{k^2/4 + (1-m)k/2}
\]
\[
+ \frac{2q^{1/4} (q^2, -q^2, -q^2; q^2)_\infty}{(aq; q)_\infty q^{m^2/4}} \sum_{2(1+m)} \frac{(-1)^k}{(q; q)_k} q^{k^2/4 + (1-m)k/2}.
\]

In particular, for \(a = 1\) we have
\[(4.31)\]
\[
\frac{q^{n^2/2}}{(aq^{n+1}; q)_\infty} = \int_{-\infty}^{\infty} \frac{\exp\left(\frac{x^2}{\log q^2} + inx\right)}{(-aq^{1/2} e^{ix}; q)_\infty} \sqrt{\pi \log q^{-2}} dx,
\]

Then, by applying (II.28) of \[5\] and \[17\] we get
\[
u_m(a; q) = \sum_{n=-\infty}^{\infty} \frac{q^{n^2+mn}}{(aq; q)_n} = \frac{1}{(aq; q)_\infty} \sum_{n=-\infty}^{\infty} (aq^{n+1}; q)_\infty q^{n^2+mn}
\]
\[
= \frac{1}{(aq; q)_\infty} \int_{-\infty}^{\infty} \frac{\exp\left(\frac{x^2}{\log q^2}\right) dx}{(-aq^{1/2} e^{ix}; q)_\infty \sqrt{\pi \log q^{-2}}} \left\{ \sum_{n=-\infty}^{\infty} q^{n^2/2} \left(q^m e^{ix}\right)_n \right\}
\]
\[
= \frac{(q; q)_\infty}{(aq; q)_\infty} \int_{-\infty}^{\infty} \frac{\exp\left(\frac{x^2}{\log q^2}\right) (-q^{m+1/2} e^{ix} - q^{1/2 - m} e^{-ix}; q)_\infty}{(-aq^{1/2} e^{ix}; q)_\infty \sqrt{\pi \log q^{-2}}} dx.
\]
in \((4.28)\) we get
\[
\left( -q^{m+1/2}e^{ix}; q \right)_\infty \left/ \left( -aq^{1/2}e^{ix}; q \right)_\infty \right.
\]
in \((4.28)\) we get
\[
u_m(a; q) = \frac{(q; q)_\infty}{(aq; q)_\infty} \sum_{k=0}^{\infty} \frac{(q^m/a; q)_k}{(q; q)_k} \left( -aq^{1/2} \right)^k \times \int_{-\infty}^{\infty} \exp \left( x^2 / \log q^2 + ikx \right) \left( -q^{1/2-m} e^{-ix}; q \right)_\infty dx
\]
\[
= \left( q; q \right)_\infty \sum_{k=0}^{\infty} \frac{(q^m/a; q)_k}{(q; q)_k} \left( -aq^{1/2} \right)^k q^{k^2/2} A_q \left( -q^{-m-k} \right),
\]
which is \((4.29)\). Applying \((II.1)\) of \([5]\) to \(1/(aq^{1/2}e^{ix}; q)_\infty\) in \((4.28)\) we get
\[
u_m(a; q) = \sum_{k=0}^{\infty} \frac{(-aq^{1/2})^k}{(q; q)_k} \times \int_{-\infty}^{\infty} \exp \left( x^2 / \log q^2 + ikx \right) \left( q, -q^{m+1/2}e^{ix}, -q^{1/2-m} e^{-ix}; q \right)_\infty dx
\]
\[
= \sum_{k=0}^{\infty} \frac{(-aq^{1/2})^k}{(q; q)_k} \sum_{n=-\infty}^{\infty} q^{n^2/2+mn} \int_{-\infty}^{\infty} \frac{\exp \left( x^2 / \log q^2 + i(k+n)x \right) dx}{(aq; q)_\infty \sqrt{\pi \log q^2}}
\]
\[
= \sum_{k=0}^{\infty} \frac{(-aq^{1/2})^k q^{k^2/2}}{(q; q)_k (aq; q)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2+(m+k)n}
\]
\[
= \sum_{k=0}^{\infty} \frac{(-aq^{1/2})^k q^{k^2/2}}{(aq; q)_\infty (q; q)_k} \left( q^2, -q^{m+k+1}, -q^{1-m-k}; q^2 \right)_\infty.
\]
For \(q = e^{\pi i \tau}, \Im(\tau) > 0, z = e^{2\pi i v},\) recall that
\[
\vartheta_3(v|\tau) = \sum_{k=-\infty}^{\infty} q^{k^2} e^{2k\pi iv} = \left( q^2, -q^{2\pi i v}, -q e^{-2\pi i v}; q^2 \right)_\infty,
\]
\[
\vartheta_2(v|\tau) = \sum_{k=-\infty}^{\infty} q^{(k+1/2)^2} e^{(2k+1)\pi iv} = 2q^{1/4} \cos \pi v \left( q^2, -q^{2\pi i v}, -q^{2\pi i v}; q^2 \right)_\infty,
\]
and
\[
\vartheta_3 \left( v + \frac{2n+1}{2} \right| \tau) = e^{-2n\pi iv} \vartheta_2 (v|\tau)
\]
\[
\vartheta_3 (v + n\tau|\tau) = q^{-n^2} e^{-2n\pi iv} \vartheta_3 (v|\tau).
\]
Then for \(2 \mid (m+k)\) we have
\[
\left( q^2, -q^{m+k+1}, -q^{1-m-k}; q^2 \right)_\infty = \vartheta_3 \left( q^{m+k}; q \right) = \vartheta_3 \left( \frac{m+k}{2} \right| \tau)
\]
\[
= q^{-(m+k)^2/4} \vartheta_3(0|\tau) = q^{-(m+k)^2/4} \left( q^2, -q; q^2 \right)_\infty.
\]
and
\[
(q^2, -q^{m+k+1}, -q^{1-m-k}; q^2)_\infty = \vartheta_3(q^{m+k}; q) = \vartheta_3\left(\frac{m+k}{2} \tau \right).
\]

Then,
\[
(aq; q)_\infty u_m(a; q) = \sum_{2|(k+m)} \frac{(-a)^k q^{k+1/2}}{(q; q)_k} \left(q^2, -q^{m+k+1}, -q^{1-m-k}; q^2\right)_\infty
\]
\[
+ \sum_{2|(k+m)} \frac{(-a)^k q^{k+1/2}}{(q; q)_k} \left(q^2, -q^{m+k+1}, -q^{1-m-k}; q^2\right)_\infty
\]
\[
= \frac{(q^2, -q, -q; q^2)_\infty}{q^{m^2/4}} \sum_{2|(k+m)} (-a)^k q^{2k/4+(1-m)k/2}
\]
\[
+ 2q^{1/4} \left(q^2, -q^2, -q^2; q^2\right)_\infty \sum_{2|(k+m)} (-a)^k q^{k^2/4+(1-m)k/2}.
\]

\[
\square
\]

References


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