Some types of functions to construct the associated calculi

Enas M. Shehata

Department of Mathematics, Faculty of Science, Menoufia University, Egypt

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Outline

1 Quantum calculus
2 Notes on the general quantum calculus.
3 Different types of the function $\beta$. 
Quantum calculus
Many problems in physics, economics, mechanics and life phenomena are characterized by functions that are not differentiable in the classical sense. The quantum calculus is an approach to deal with sets of non-differentiable functions. In literature, quantum calculus is known as calculus without limits, it substitutes the classical derivative by a difference operator. For more details we recommend the interesting book [8] by Kac and Cheung.
One type of quantum calculus is the Hahn calculus. In (1949) [7], Hahn introduced his difference operator, as a tool for constructing families of orthogonal polynomials, which is defined by

\[
D_{q, \omega} f(t) = \frac{f(qt + \omega) - f(t)}{t(q - 1) + \omega}, \quad t \neq \omega_0,
\]

where \( q \in (0, 1), \omega > 0 \) are fixed and \( \omega_0 = \frac{\omega}{1 - q} \). The derivative at \( t = \omega_0 \) is defined to be the classical derivative \( f'(\omega_0) \) whenever it exists.
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\[ D_{q,\omega} f(t) = \frac{f(qt + \omega) - f(t)}{t(q - 1) + \omega}, \quad t \neq \omega_0, \tag{1} \]

where \( q \in (0, 1), \omega > 0 \) are fixed and \( \omega_0 = \frac{\omega}{1 - q} \). The derivative at \( t = \omega_0 \) is defined to be the classical derivative \( f'(\omega_0) \) whenever it exists. In [1], the inverse operator was constructed and a rigorous analysis of the calculus associated with \( D_{q,\omega} \) was given.
Hahn quantum difference operator unifies two important difference operators. The first is the Jackson $q$-difference operator which is defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{t(q - 1)}, \quad t \neq 0,$$

and $D_q f(0) = f'(0)$, where $q$ is a fixed number, $q \in (0, 1)$. The function $f$ is defined on a $q$-geometric set $\mathbb{A} \subseteq \mathbb{R}$ (or $\mathbb{C}$) such that whenever $t \in \mathbb{A}$, $qt \in \mathbb{A}$. 


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(2)

and $D_q f(0) = f'(0)$, where $q$ is a fixed number, $q \in (0, 1)$. The function $f$ is defined on a $q$-geometric set $A \subseteq \mathbb{R}$ (or $\mathbb{C}$) such that whenever $t \in A$, $qt \in A$. The second is the forward difference operator $D_\omega$ which is defined by

$$D_\omega f(t) = \frac{f(t + \omega) - f(t)}{\omega}, \quad t \in \mathbb{R},$$

(3)

where $\omega$ is a fixed number and $\omega > 0$. See [8]
Auch in his PhD [2] in 2013 (supervised by Lynn Erbe and Allan Peterson) introduced the forward difference operator

\[ \Delta_{a,b} f(t) = \frac{f(\sigma(t)) - f(t)}{\sigma(t) - t}, \]  

(4)

where \( \sigma(t) = at + b \) with \( a \geq 1, b \geq 0 \) and \( a + b > 1 \). He defined \( f \) on a mixed time scale \( \mathbb{T}_\alpha := \{\cdots, \rho^2(\alpha), \rho(\alpha), \alpha, \sigma(\alpha), \sigma^2(\alpha), \cdots \} \), \( \alpha > \frac{b}{1-a} \), which is a discrete subset of \( \mathbb{R} \).
General quantum difference operator

Let $\beta : I \subseteq \mathbb{R} \rightarrow I$ be a strictly increasing continuous function that has fixed points belonging to $I$ and $\beta(t) \in I$ for every $t \in I$. 

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Enas M. Shehata (Department of Mathematics, Faculty of Science, Menoufia University, Egypt)

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General quantum difference operator

Let $\beta : I \subseteq \mathbb{R} \rightarrow I$ be a strictly increasing continuous function that has fixed points belonging to $I$ and $\beta(t) \in I$ for every $t \in I$. This $\beta$-function had been used by [6], where a general quantum difference operator $D_\beta$ is defined to be

$$D_\beta f(t) = \begin{cases} \frac{f(\beta(t)) - f(t)}{\beta(t) - t} & \text{for } t \neq s_0, \\ f'(s_0) & \text{for } t = s_0, \end{cases}$$

where $f$ is an arbitrary function defined on $I$ and is ordinary differentiable at $t = s_0$ and $s_0$ is a fixed point of the function $\beta$. 
The general function $\beta$ may be linear or nonlinear. Then $\beta$ has many types according to the number of its fixed points in $I$. Every choice of the function $\beta$ gives a new difference operator. Thus, we can obtain a wide class of quantum difference operators with the corresponding quantum calculi.

This operator yields Jackson's $q$-difference operator with $\beta(t) = qt$, and Hahn's difference operator with $\beta(t) = qt + \omega$; $q \in (0,1)$ and $\omega > 0$, which are linear and each has only one fixed point $s_0 = 0$, $\omega_1 - q$ (resp.), that satisfy the inequality $(t - s_0)(\beta(t) - t) \leq 0$ for all $t \in I$.

(5) $\lim_{k \to \infty} \beta_k(t) = s_0$. Where $\beta_k(t) := \beta \circ \beta \circ \cdots \circ \beta$ $k$-times.
The general function $\beta$ may be linear or nonlinear. Then $\beta$ has many types according to the number of its fixed points in $I$. Every choice of the function $\beta$ gives a new difference operator. Thus, we can obtain a wide class of quantum difference operators with the corresponding quantum calculi. This operator yields Jackson’s $q$-difference operator with $\beta(t) = qt$, and Hahn’s difference operator with $\beta(t) = qt + \omega$; $q \in (0, 1)$ and $\omega > 0$, which are linear and each has only one fixed point $s_0 = 0, \frac{\omega}{1-q}$ (resp.), that satisfy the inequality

$$\begin{aligned} (t - s_0)(\beta(t) - t) &\leq 0 \text{ for all } t \in I. \tag{5} \\
\lim_{k \to \infty} \beta^k(t) &= s_0. 
\end{aligned}$$

Where

$$\beta^k(t) := \underbrace{\beta \circ \beta \circ \cdots \circ \beta(t)}_{k\text{-times}}$$
Return to [6], that used the general $\beta$-function with special case of $\beta$, having only one fixed point $s_0$ and with satisfying the same condition (5).
Also, a related $\beta$-calculus had been given, see [4,5,6,10]. In the following we present some of these results.
Lemma

If \( f : I \rightarrow \mathbb{R} \) is continuous at \( s_0 \), then the sequence \( \{f(\beta_k(t))\}_{k \in \mathbb{N}_0} \) converges uniformly to \( f(s_0) \) on every compact interval \( J \subseteq I \) containing \( s_0 \).
Lemma

If $f : I \to \mathbb{R}$ is continuous at $s_0$, then the sequence $\{f(\beta^k(t))\}_{k \in \mathbb{N}_0}$ converges uniformly to $f(s_0)$ on every compact interval $J \subseteq I$ containing $s_0$.

Theorem

If $f : I \to X$ is continuous at $s_0$, then the series
\[ \sum_{k=0}^{\infty} \|(\beta^k(t) - \beta^{k+1}(t)) f(\beta^k(t))\| \] is uniformly convergent on every compact interval $J \subseteq I$ containing $s_0$. 
Some clear properties of the $\beta$-difference operator.

(i) $D_\beta$ is a linear operator.

(ii) If $f$ is $\beta$-differentiable at $t$, then $f(\beta(t)) = f(t) + (\beta(t) - t)D_\beta f(t)$.

(iii) If $f$ is $\beta$-differentiable, then $f$ is continuous at $s_0$. 
Theorem

Assume that $f, g : I \rightarrow \mathbb{R}$ are $\beta$-differentiable functions at $t \in I$. Then:

(i) The product $fg : I \rightarrow \mathbb{R}$ is $\beta$-differentiable at $t$ and

$$D_\beta(fg)(t) = (D_\beta f(t))g(t) + f(\beta(t))D_\beta g(t)$$

$$= (D_\beta f(t))g(\beta(t)) + f(t)D_\beta g(t).$$

(ii) $f/g$ is $\beta$-differentiable at $t$ and

$$D_\beta(f/g)(t) = \frac{(D_\beta f(t))g(t) - f(t)D_\beta g(t)}{g(t)g(\beta(t))}, \quad g(t)g(\beta(t)) \neq 0.$$
Lemma

Let $f : I \rightarrow \mathbb{R}$ be $\beta$-differentiable and $D_\beta f(t) = 0$ for all $t \in I$, then $f(t) = f(s_0)$, $t \in I$. 
Lemma

Let $f : I \longrightarrow \mathbb{R}$ be $\beta$-differentiable and $D_\beta f(t) = 0$ for all $t \in I$, then $f(t) = f(s_0)$, $t \in I$.

Examples

1. $D_\beta t^n = \sum_{k=0}^{n-1} (\beta(t))^{n-k-1} t^k$, $t \in I$, $n \geq 1$.
2. For $t \neq 0$, $D_\beta \frac{1}{t} = -\frac{1}{t\beta(t)}$, $t \in I$, $\beta(t) \neq 0$. 
\( \beta \)-integration

**Definition**

Let \( f : I \rightarrow \mathbb{R} \) and \( a, b \in I \). We define the \( \beta \)-integral of \( f \) from \( a \) to \( b \) by

\[
\int_{a}^{b} f(t) \, d\beta t = \int_{s_{0}}^{b} f(t) \, d\beta t - \int_{s_{0}}^{a} f(t) \, d\beta t,
\]

(6)

where

\[
\int_{s_{0}}^{x} f(t) \, d\beta t = \sum_{k=0}^{\infty} \left( \beta^k(x) - \beta^{k+1}(x) \right) f(\beta^k(x)), \quad x \in I,
\]

(7)

provided that the series converges at \( x = a \) and \( x = b \). \( f \) is called \( \beta \)-integrable on \( I \) if the series converges at \( a, b \) for all \( a, b \in I \). Clearly, if \( f \) is continuous at \( s_{0} \in I \), then \( f \) is \( \beta \)-integrable on \( I \).
In the integral formulas (6) and (7), when $\beta(t) = qt, q \in (0, 1)$, we obtain Jackson $q$-integration

$$\int_a^b f(t) dq_t := \int_0^b f(t) dq_t - \int_0^a f(t) dq_t,$$

(8)

where

$$\int_0^x f(t) dq_t := x(1 - q) \sum_{k=0}^{\infty} q^k f(xq^k), \quad x \in I.$$

(9)
In the integral formulas (6) and (7), when $\beta(t) = qt$, $q \in (0, 1)$, we obtain Jackson $q$-integration

$$\int_{a}^{b} f(t) dq t := \int_{0}^{b} f(t) dq t - \int_{0}^{a} f(t) dq t,$$

where

$$\int_{0}^{x} f(t) dq t := x(1 - q) \sum_{k=0}^{\infty} q^k f(xq^k), \quad x \in I. \tag{9}$$

If $\beta(t) = qt + \omega$, $q \in (0, 1)$, $\omega > 0$, then (6) and (7) reduce to the Hahn integral

$$\int_{a}^{b} f(t) dq,\omega t := \int_{\omega_0}^{b} f(t) dq,\omega t - \int_{\omega_0}^{a} f(t) dq,\omega t,$$

where

$$\int_{\omega_0}^{x} f(t) dq,\omega t := (x(1 - q) - \omega) \sum_{k=0}^{\infty} q^k f(xq^k + \omega[k]_q), \quad x \in I, \tag{11}$$

where $\omega_0 = \frac{\omega}{1 - q}$ and $[k]_q = \frac{1 - q^k}{1 - q}$. 
Lemma

Let $f : I \rightarrow \mathbb{R}$ be $\beta$-integrable on $I$ and $a, b, c \in I$, then the following statements are true:

(i) The $\beta$-integral is a linear operator.

(ii) $\int_{a}^{a} f(t) d\beta t = 0,$

(iii) $\int_{a}^{b} f(t) d\beta t = - \int_{b}^{a} f(t) d\beta t,$

(iv) $\int_{a}^{b} f(t) d\beta t = \int_{a}^{c} f(t) d\beta t + \int_{c}^{b} f(t) d\beta t.$
Let $f : I \rightarrow \mathbb{R}$ be continuous at $s_0$. Define the function

$$F(x) = \int_{s_0}^{x} f(t)d_\beta t, \quad x \in I.$$  \hspace{1cm} (12)

Then $F$ is continuous at $s_0$, $D_\beta F(x)$ exists for all $x \in I$ and $D_\beta F(x) = f(x)$. 

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Enas M. Shehata (Department of Mathematics, Faculty of Science, Menoufia University, Egypt)

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Theorem

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F(x) = \int_{s_0}^{x} f(t) d_\beta t, \quad x \in I.
\]  

(12)

Then \( F \) is continuous at \( s_0 \), \( D_\beta F(x) \) exists for all \( x \in I \) and \( D_\beta F(x) = f(x) \).

Theorem

If \( f : I \rightarrow \mathbb{R} \) is \( \beta \)-differentiable on \( I \), then

\[
\int_{a}^{b} D_\beta f(t) d_\beta (t) = f(b) - f(a), \quad \text{for all } a, b \in I.
\]  

(13)
Notes on the $\beta$-calculus
Notes on the $\beta$-calculus

(i) When dealing with the $\beta$-derivative, $D_\beta f(t)$, the failure of some results that hold for ordinary differentiable functions occur.
The following example shows that the function \( f \) may be discontinuous but it is \( \beta \)-differentiable.

**Example**

Let \( f : [-1, 1] \rightarrow \mathbb{R} \) be such that

\[
f(t) = \begin{cases} 
  t, & t \in (-1, 0), \\
  -t, & t \in (0, 1), \\
  1, & t = 0, \\
  0, & t = 1, -1,
\end{cases}
\]

and let

\[
\beta(t) = \frac{1}{4} t + \frac{1}{4}.
\]

We see that the function \( f \) is discontinuous but it is \( \beta \)-differentiable, where

\[
D_\beta f(t) = \begin{cases} 
  1, & t \in (-1, 0), \\
  -1, & t \in (0, 1), \\
  -5, & t = 0, \\
  1, & t = 1, -1.
\end{cases}
\]
At an unconstrained extreme point of a differentiable function $f$, $D_\beta f(t)$ may not vanish. In our knowledge there does not exist in the $\beta$-calculus a necessary condition for the existence of the extreme points of a function $f$ as shown in the following examples.
Example

Let \( f(t) = 2t - t^2 \) and \( \beta(t) = 4t^3 \). Both \( f, \beta : \mathbb{R} \rightarrow \mathbb{R} \). \( \max f(t) \) holds at \( \hat{t} = 1 \) which is unique, since \( f(t) \) is strictly concave, where \( f'(1) = 0 \) and \( D_\beta f(1) \neq 0 \). So the familiar necessary condition does not hold.
The following example shows that the necessary condition may hold in the $\beta$-calculus, for some types of functions.

**Example**

Let $f(t) = 4t - t^2$ and $\beta(t) = \frac{1}{4}t^3$, where $f, \beta : \mathbb{R} \rightarrow \mathbb{R}$, $maxf(t)$ is attained at $\hat{t} = 2$ which is the only maximal point implying $f'(2) = 0$. $D_\beta f(2) = 0$ verifying the known necessary condition.
(ii) On the behavior of $\beta^k(t)$ with respect to the only fixed point $s_0$ of $\beta(t)$.

(1) $s_0$ is absorbing to the $\beta^k(t)$ for all $t \in I$ i.e.

$$(t - s_0)(\beta(t) - t) \leq 0 \text{ for all } t \in I.$$ 

It is equivalent to

$$\lim_{k \to \infty} \beta^k(t) = s_0.$$
(2) \( s_0 \) is expelling or \((t - s_0)(\beta(t) - t) \geq 0\) for all \( t \in I \), which is equivalent to

\[
\lim_{k \to \infty} \beta^k(t) = \infty \quad \text{for} \quad t > s_0
\]

and

\[
\lim_{k \to \infty} \beta^k(t) = -\infty \quad \text{for} \quad t < s_0.
\]
\[(t - s_0)(\beta(t) - t) \leq 0 \text{ for } t \leq s_0 \text{ and } (t - s_0)(\beta(t) - t) \geq 0 \text{ for } t \geq s_0, \text{ or the converse in the left part (see figures 1, 2).}
\]

This form shows either \( \beta^k(t) \to \infty \) or \( -\infty \) but not both.
Figure 1

Figure 2
(iv) By relaxing the conditions of $\beta(t)$ to be increasing and continuous from the right such that $\lim_{t \to \infty} \beta(t) = 1$ and $\lim_{t \to -\infty} \beta(t) = 0$. Also, by using the Lebesgue-Stieltjes measure of the interval $[a, b]$ to be $\beta(b) - \beta(a)$. The $\beta$-function becomes the probability distribution function, where $\mathbb{R}$ being the sample space with the probability measure equals 1.
Different types of the \( \beta \)-function
Different types of the $\beta$-function

Since we deal with the strictly increasing continuous function $\beta(t)$, and since the fixed points of $\beta$ play essential role in our study. Accordingly we introduce some special types of $\beta$, in view of the fixed points, beside concerning the natural Permanent effect of $+\infty, -\infty$. 
Type I $\beta(t)$ has one fixed point $s_0$.

(i) The function $\beta(t)$ is linear.

We recall Hahn’s $\beta(t) = qt + \omega; \omega > 0, q \in (0, 1)$ and $s_0 = \frac{\omega}{1-q}$.

Also, Jackson’s $\beta(t) = qt$ with $s_0 = 0$, where $s_0$ is absorbing to the $\beta^k(t)$. We may also consider $\beta(t) = qt - \omega; \omega > 0$ and $q \in (0, 1)$ with $s_0 = \frac{-\omega}{1-q}$.
(ii) \( \beta : \mathbb{R} \to \mathbb{R} \) is non-linear differentiable strictly convex function with one fixed point \( s_0 \).
Example

Let $\beta(t) = e^t - 1$, $\beta : \mathbb{R} \rightarrow (-1, \infty)$, with the fixed point $s_0 = 0$ is (absorbing-expelling).
Figure: 3
Type II $\beta : I \subseteq \mathbb{R} \rightarrow I$ is non-linear differentiable function with two fixed points. We consider two models.

Model 1

$\beta : [0, \infty) \rightarrow [0, \infty)$, and $\beta(t) = qt^n$, $q \in (0, 1)$ and $n$ is even number. $\beta$ is strictly convex function, with fixed points $s_0 = 0$, $s_1 = \left(\frac{1}{q}\right)^{\frac{1}{n-1}}$, where $s_0 = 0$ is absorbing and $s_1 = \left(\frac{1}{q}\right)^{\frac{1}{n-1}}$ is expelling.
Model 2 \( \beta(t) = \frac{2t+2}{t+3} \) is strictly increasing differentiable pseudolinear function where \( \beta : (-3, \infty) \to (-\infty, 2) \) with two fixed points \(-2, 1\) where 1 is absorbing fixed point and \(-2\) is expelling.
**Type III** $\beta(t)$ is non-linear differentiable function with three fixed points. The given model is $\beta(t) = qt^n + bt$; $q \in (0, 1)$; $b \in (0, 1)$ and $n$ is odd. Note: $n = 1$, $b = 0$ leads to Jackson $\beta(t) = qt$, (Type I). So, we assume $n > 1$ and $\beta : \mathbb{R} \rightarrow \mathbb{R}$. 
Proposition

\( \beta \) is differentiable strictly quasilinear function with the fixed points
\( s_0 = 0, \pm s_1 = \pm \left( \frac{1-b}{q} \right)^{\frac{1}{n-1}} \), where 0 is absorbing (form (ii)-(1)),
\( \pm \left( \frac{1-b}{q} \right)^{\frac{1}{n-1}} \) are expelling (form (ii)-(2)).
Special cases of the model:

1. Let $b = 1 - q$, then $\beta(t) = qt^n + (1 - q)t$; $q \in (0, 1)$, $n \geq 3$ and is odd. The fixed points are $0, \pm 1$. In this case 0 is absorbing and $\pm 1$ are expelling. See figure 4.
Figure: 4
2. Let $b = 0$, then $\beta(t) = qt^n; \; q \in (0, 1)$ and $n \geq 3$ is odd. $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing differentiable quasilinear function with the fixed points $0, \pm s_0 = \pm (\frac{1}{q})^{\frac{1}{n-1}}$. We have $0$ is absorbing and the others are expelling.
2. Let $b = 0$, then $\beta(t) = qt^n$; $q \in (0, 1)$ and $n \geq 3$ is odd. $\beta : \mathbb{R} \rightarrow \mathbb{R}$ is strictly increasing differentiable quasilinear function with the fixed points $0, \pm s_0 = \pm \left(\frac{1}{q}\right)^{\frac{1}{n-1}}$. We have $0$ is absorbing and the others are expelling.

3. Put $b = 0$ and $q = 1$, $n \geq 3$ is odd number then $\beta(t) = t^n$ which is quasilinear : $\mathbb{R} \rightarrow \mathbb{R}$ with the fixed points $0, \pm 1$ and have the same direction of the $\beta^k(t)$ as in (1.) above. So $\beta(t) = t^n$ and $\beta(t) = qt^n + (1 - q)t$, $n$ is odd $\geq 3$ has the same common fixed points with the common effect on the $\beta^k(t)$. 
Type IV All points of the domain of $\beta$ are fixed points i.e. $\beta(t) = t$. This type is not interesting, since $\beta^k(t) = t$ for all $t$ and the type is static.
Type V \( \beta(t) : I \subseteq \mathbb{R} \rightarrow \mathbb{R} \) has no fixed points. The effect on the \( \beta^k(t) \) is done by \( \infty, -\infty \). A given example is \( \beta(t) = t + \omega \) where \( \omega > 0 \) with \( \beta^k(t) = t + k\omega \rightarrow \infty \).
Type V \( \beta(t) : I \subseteq \mathbb{R} \to \mathbb{R} \) has no fixed points. The effect on the \( \beta^k(t) \) is done by \( \infty, -\infty \). A given example is \( \beta(t) = t + \omega \) where \( \omega > 0 \) with \( \beta^k(t) = t + k\omega \to \infty \).

Also \( \beta(t) = t - \omega \); \( \omega > 0 \) and implying \( \beta^k(t) \to -\infty \).
Type V \( \beta(t) : I \subseteq \mathbb{R} \to \mathbb{R} \) has no fixed points. The effect on the \( \beta^k(t) \) is done by \( \infty, -\infty \). A given example is \( \beta(t) = t + \omega \) where \( \omega > 0 \) with \( \beta^k(t) = t + k\omega \to \infty \).

- Also \( \beta(t) = t - \omega \); \( \omega > 0 \) and implying \( \beta^k(t) \to -\infty \).
- We also give another nonlinear example \( \beta(t) = e^t : \mathbb{R} \to [0, \infty) \) where clearly \( \beta(t) > t \) for every \( t \in \mathbb{R} \) implying \( \beta^k(t) \to \infty \).
References


Thank You