Divisibility of binomials sums and $q$-congruences

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Define the Apéry number

\[ A_n := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2. \]

With help of those numbers, Apéry proved that \( \zeta(3) \) is irrational. The Apéry numbers also have some interesting arithmetic properties. For example, Gessel proved that

\[ A_{np} \equiv A_n \pmod{p^3} \]

for any \( n \geq 1 \) and any prime \( p \geq 5 \). Ahlgren and Ono proved that

\[ A_{\frac{p-1}{2}} \equiv a(p) \pmod{p^2} \]

for any odd prime, where \( a(p) \) is the Fourier coefficient of \( q^n \) in

\[ \eta(2z)^4 \eta(4z)^4. \]
Sun defined the Apéry polynomial

$$A_n(x) := \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n + k}{k}^2 x^k.$$ 

Sun proved that

$$\sum_{k=0}^{n-1} (2k + 1)A_k(x) \equiv 0 \pmod{n}.$$ 

In fact, he found that

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1)A_k(x) = \sum_{k=0}^{n-1} \binom{n-1}{k} \binom{n + k}{k} \binom{n + k}{2k + 1} \binom{2k}{k} x^k.$$
The Apéry polynomial

Sun conjectured that

\[ \sum_{k=0}^{n-1} \epsilon^k (2k + 1) A_k(x)^m \equiv 0 \pmod{n} \]

for any \( m \geq 1 \), where \( \epsilon \in \{1, -1\} \). When \( m = 1 \) and \( \epsilon = -1 \), Guo and Zeng proved that

\[
\frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k + 1) A_k(x)
\]

\[= (-1)^{n-1} \sum_{k=0}^{n-1} \binom{2k}{k} x^k \sum_{j=0}^k \binom{k}{j} \binom{k+j}{j} \binom{n-1}{k+j} \binom{n+k+j}{k+j}. \]
The Apéry polynomial

Sun also considered the Delannoy polynomial

$$D_n(x) := \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k,$$

and showed that

$$\frac{1}{n} \sum_{k=0}^{n-1} (2k + 1) D_k(x) = \sum_{k=0}^{n-1} \binom{n}{k+1} \binom{n+k}{k} x^k.$$

He conjectured that

$$\sum_{k=0}^{n-1} (2k + 1) D_k(x)^m \equiv 0 \pmod{n}$$

for any $m \geq 1$. 
The Apéry polynomial

Guo and Zeng called polynomial

\[ A_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n}{k} \alpha (n + k)^\alpha x^k, \]

the Schmidt polynomial. They found the explicit formulas for

\[ \frac{1}{n} \sum_{k=0}^{n-1} (2k + 1) A_k^{(\alpha)}(x) \quad \text{and} \quad \frac{1}{n} \sum_{k=0}^{n-1} (-1)^k (2k + 1) A_k^{(\alpha)}(x) \]

for any \( \alpha \geq 1 \). They conjectured that

\[ \sum_{k=0}^{n-1} \epsilon^k (2k + 1) A_k^{(\alpha)}(x)^m \equiv 0 \pmod{n}, \quad \epsilon \in \{1, -1\}, \]

for any \( m \geq 1 \).
we need to use \( q \)-conjecture to prove the conjecture of Guo and Zeng. For \( n \in \mathbb{N} \), define the \( q \)-integer

\[
[n]_q := \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}.
\]

Define the \( q \)-binomial coefficient

\[
\binom{n}{m}_q = \frac{[n]_q[n-1]_q \cdots [n-m+1]_q}{[m]_q[m-1]_q \cdots [1]_q}.
\]

Since

\[
\binom{n+1}{m}_q = q^m \binom{n}{m}_q + \binom{n}{m-1}_q,
\]

we have \( \binom{n}{m}_q \) is a polynomial in \( q \) with integral coefficients.
\( q \)-congruences

Define

\[
A^{(\alpha)}_n(x; q) := \sum_{k=0}^{n} q^{\alpha(\binom{k}{2} - nk)} \left[ \binom{n}{k} q^{\binom{n+k}{k}} \right] \alpha \cdot x^k.
\]

**Theorem (P.)**

\[
\sum_{k=0}^{n-1} q^{n-1-k}[2k + 1]q A^{(\alpha)}_k(x; q)^m \equiv 0 \pmod{[n]_q},
\]

where the \( q \)-congruence is considered over the ring of the polynomials in \( q \).

The greatest common divisor of the coefficients of \([n]_q\) is 1. By a result of Gauss, the left side of (*) coincides with \([n]_q \cdot H(x; q)\), where \( H(x; q) \) is a polynomial with integral coefficients. Letting \( q = 1 \) in (*),

\[
\sum_{k=0}^{n-1} (2k + 1) A^{(\alpha)}_k(x)^m = n \cdot H(x; 1) \equiv 0 \pmod{n}.
\]
In fact, the $q$-congruences maybe are easier to prove. Let

$$\Phi_d(q) = \prod_{1 \leq j \leq d, (j,d)=1} (q - e^{2\pi ij})$$

be the $d$-th cyclotomic polynomial. We know that

$$[n]_q = \prod_{d|n, d \geq 2} \Phi_d(q).$$

Recall that the cyclotomic polynomials are irreducible. In order to prove

$$\sum_{k=0}^{n-1} q^{n-1-k} [2k + 1]_q A_k^{(\alpha)}(x; q)^m \equiv 0 \pmod{[n]_q},$$

we only need to show that

$$\sum_{k=0}^{n-1} q^{n-1-k} [2k + 1]_q A_k^{(\alpha)}(x; q)^m \equiv 0 \pmod{\Phi_d(q)}$$

for each divisor $d \geq 2$ of $n$. 
A general result

Theorem

Let \( \nu_0(q), \nu_1(q), \ldots \) be a sequence of polynomials with integral coefficients. Suppose that for each \( d \geq 2 \),

(i) for any non-integers \( s, t \) with \( 0 \leq t \leq d - 1 \),

\[
\nu_{sd+t}(q) \equiv \mu_s^{(d)}(q)\nu_t(q) \pmod{\Phi_d(q)},
\]

where the polynomial \( \mu_s^{(d)}(q) \) with integral coefficients depends on \( s, d \);

(ii) \[ d-1 \sum_{k=0}^{d-1} \nu_k(q) \equiv 0 \pmod{\Phi_d(q)}. \]

Then for each integer \( n \geq 2 \),

\[
\sum_{k=0}^{n-1} \nu_k(q) \equiv 0 \pmod{[n]_q}.
\]
A general result

Proof.

Suppose that $d \mid n$ and $d \geq 2$. Then letting $u = n/d$,

$$
\sum_{k=0}^{n-1} \nu_k(q) = \sum_{s=0}^{u-1} \sum_{t=0}^{d-1} \nu_{sd+t}(q) \equiv \sum_{s=0}^{u-1} \mu_s^{(d)}(q) \sum_{t=0}^{d-1} \nu_t(q) \equiv 0 \pmod{\Phi_d(q)}.
$$

Let us return to $A_n^{(\alpha)}(x; q)$. We need to verify that for each $d \geq 2$,

(i) \[ A_{sd+t}^{(\alpha)}(x; q) \equiv B_s^{(d)}(x; q)A_t^{(\alpha)}(x; q) \pmod{\Phi_d(q)}, \]

for some polynomial $B_s^{(d)}(q)$ with integral coefficients.

(ii) \[ \sum_{k=0}^{d-1} q^{d-1-k}[2k+1]_q A_k^{(\alpha)}(x; q)^m \equiv 0 \pmod{\Phi_d(q)}. \]
We need the $q$-Lucas congruence to verify the condition (i) for $A_n^{(\alpha)}(x; q)$.

**Theorem**

Let $d \geq 2$. For any integers $s, t, u, v$ with $0 \leq t, v \leq d - 1$,

\[
\begin{bmatrix} sd + t \\ ud + v \end{bmatrix}_q \equiv \binom{s}{u} \binom{t}{v}_q \pmod{\Phi_d(q)}.
\]

In fact, using an easy induction on $s$, it suffices to show that

\[
\begin{bmatrix} n + d \\ m \end{bmatrix}_q \equiv \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m - d \end{bmatrix}_q \pmod{\Phi_d(q)},
\]

which easily follows from the $q$-Chu-Vandemonde identity

\[
\begin{bmatrix} n + d \\ m \end{bmatrix}_q = q^{dm} \begin{bmatrix} n \\ m \end{bmatrix}_q + \begin{bmatrix} n \\ m - d \end{bmatrix}_q + \sum_{k=1}^{d-1} q^{(d-k)(m-k)} \binom{d}{k}_q \binom{n}{m-k}_q.
\]

Further, Sagan also gave a combinatorial proof (*) by using group actions.
Lucas-type congruence for $A_k^{(\alpha)}(x; q)$

With help of the $q$-Lucas congruence, we can obtain that for each $d \geq 2$,

$$A_{sd+t}^{(\alpha)}(x; q) = \sum_{k=0}^{sd+t} (-1)^{\alpha k} q^{\alpha k^2} \left[ \begin{array}{c} sd + t \\ k \end{array} \right]_q \alpha \left[ \begin{array}{c} -sd - t - 1 \\ k \end{array} \right]_q \cdot x^k$$

$$= \sum_{u=0}^{s} \sum_{v=0}^{d-1} (-1)^{\alpha (ud+v)} q^{\alpha (ud+v)^2} \left[ \begin{array}{c} sd + t \\ ud + v \end{array} \right]_q \alpha \left[ \begin{array}{c} -sd - t - 1 \\ ud + v \end{array} \right]_q \cdot x^{ud+v}$$

$$\equiv \sum_{u=0}^{s} (-1)^{\alpha ud} x^{ud} \binom{s}{u}^\alpha (-s - 1)^\alpha \sum_{v=0}^{d-1} (-1)^{\alpha v} q^{\alpha v^2} x^v \left[ \begin{array}{c} t \\ v \end{array} \right]_q \alpha \left[ \begin{array}{c} d - t - 1 \\ v \end{array} \right]_q$$

$$\equiv A_t^{(\alpha)}(x; q) \cdot \sum_{u=0}^{s} (-1)^{\alpha ud} x^{ud} \binom{s}{u}^\alpha (-s - 1)^\alpha \pmod{\Phi_d(q)}.$$

Thus the condition (i) for $A_k^{(\alpha)}(x; q)$ is verified.
The condition (ii) for $A_k^{(\alpha)}(x; q)$

In order to verify the condition (ii) for $A_k^{(\alpha)}(x; q)$, we need to consider for a prime $p$, how to prove $\sum_{k=0}^{p-1}(2k + 1)A_k^{(\alpha)}(x)^m \equiv 0 \pmod{p}$, i.e., to prove Guo and Zeng’s conjecture for the case $n$ is prime.

In fact, it is not difficult to show that

$$A_{p-1-k}^{(\alpha)}(x) \equiv A_k^{(\alpha)}(x) \pmod{p}$$

for each $0 \leq k \leq p - 1$. So

$$(2k + 1)A_k^{(\alpha)}(x)^m + (2(p - 1 - k) + 1)A_{p-1-k}^{(\alpha)}(x)^m \equiv 0 \pmod{p}.$$ 

Giving a $q$-analogue of the above discussion, we may get

$$q^{d-1-k}[2k+1]_qA_k^{(\alpha)}(x; q)^m + q^k[2d-2k-1]_qA_{d-1-k}^{(\alpha)}(x; q)^m \equiv 0 \pmod{\Phi_d(q)}$$

for each $0 \leq k \leq d - 1$. Then the condition (ii) is valid, i.e.,

$$\sum_{k=0}^{d-1} q^{d-1-k}[2k + 1]_qA_k^{(\alpha)}(x; q)^m \equiv 0 \pmod{\Phi_d(q)}.$$
Sun’s conjectures

Sun discovered that there exists a connection between the convergent series concerning \( \pi \) and the divisibility of some binomial sums. For example, Sun conjectured that for each integer \( n \geq 2 \),

\[
\sum_{k=0}^{n-1} (5k + 1) \binom{2k}{k}^2 \binom{3k}{k} \cdot (-192)^{n-1-k} \equiv 0 \pmod{n \binom{2n}{n}},
\]

which corresponds to the identity of Ramanujan

\[
\sum_{k=0}^{\infty} \frac{5k + 1}{(-192)^k} \cdot \binom{2k}{k}^2 \binom{3k}{k} = \frac{4\sqrt{3}}{\pi}.
\]

In fact, Sun found almost all of Ramanujan-type convergent series might correspond to some congruences for binomial sums. Unfortunately, we don’t know how to prove Sun’s conjecture by using \( q \)-congruences.
The divisibility of some binomial sums

However, using \( q \)-congruences, recently Ni and P. proved a similar result on the divisibility of some binomial sums. For \( a \in \mathbb{Q} \) and \( n \in \mathbb{Z}^+ \), let

\[
N_{\alpha, n} = \text{the numerator of } n \cdot \left| \binom{-\alpha}{n} \right|
\]

where \( \binom{x}{n} = x(x-1) \cdots (x-n+1)/n! \).

**Theorem (Ni and P.)**

Suppose that \( n \geq 2 \) is an integer and \( \alpha \in \mathbb{Q} \). Then for each \( \rho \geq 1 \),

\[
\sum_{k=0}^{n-1} (2k + \alpha) \cdot \binom{-\alpha}{k}^\rho \equiv 0 \pmod{N_{\alpha, n}}.
\]

In particular, setting \( \alpha = 1/2 \), we get that for \( \rho \geq 2 \),

\[
\sum_{k=0}^{n-1} (4k + 1) \binom{2k}{k}^\rho \cdot (-4)^{n-1-k} \equiv 0 \pmod{n \binom{2n}{n}}.
\]
Two polynomials

Suppose that \( r \in \mathbb{Z}, \ m \in \mathbb{Z}^+ \) and \((r, m) = 1\). For each positive integer \( d \) with \((d, m) = 1\), let \( \lambda_{r,m}(d) \) be the integer lying in \( \{0, 1, \ldots, d-1\} \) such that \( r + \lambda_{r,m}(d)m \equiv 0 \pmod{d} \). Let

\[
S_{r,m}(n) = \left\{ d \geq 2 : (d, m) = 1, \left\lfloor \frac{n-1 - \lambda_{r,m}(d)}{d} \right\rfloor = \left\lfloor \frac{n}{d} \right\rfloor \right\},
\]

where \( \lfloor \cdot \rfloor \) denotes the floor function. Let

\[
A_{r,m,n}(q) = \prod_{d \in S_{r,m}(n)} \Phi_d(q), \quad C_{m,n}(q) = \prod_{d | n, (d,m) = 1} \Phi_d(q).
\]

Clearly, if \( d \mid n \), then we can’t have \( d \in S_{r,m}(n) \). So \( A_{r,m,n}(q) \) and \( C_{m,n}(q) \) are co-prime. In fact, it is not difficult to prove that

\[
A_{r,m,n}(1)C_{m,n}(1) = N_{r,m,n}.
\]
A general result

**Theorem**

Let $r \in \mathbb{Z}$ and $m \in \mathbb{Z}^+$ with $(r, m) = 1$. Let $\nu_0(q), \nu_1(q), \ldots$ be a sequence of rational functions. Suppose that for each $d \geq 2$ with $(d, m) = 1$,

(i) for each $k \geq 0$, the denominator of $\lambda_k(q)$ is prime to $\Phi_d(q)$;

(ii) $\nu_{sd+t}(q) \equiv \mu_s^{(d)}(q)\nu_t(q) \pmod{\Phi_d(q)}$ for any non-integers $s, t$ with $0 \leq t \leq d - 1$.

(iii) \[
\sum_{k=0}^{d-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv 0 \pmod{\Phi_d(q)},
\]

where $(x; q)_k = (1 - x)(1 - xq) \cdots (1 - xq^{k-1})$.

Then for each integer $n \geq 2$,

\[
\sum_{k=0}^{n-1} \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \cdot \nu_k(q) \equiv 0 \pmod{A_{r,m,n}(q)C_{m,n}(q))}.
\]
The divisibility of some \( q \)-binomial sums

With help of the above result, we can prove that

**Theorem**

Suppose that \( r \in \mathbb{Z}, \ m \in \mathbb{Z}^+ \) and \((r, m) = 1\). For each \( \rho \geq 1 \) and \( n \geq 2 \),

\[
\sum_{k=0}^{n-1} (-1)^{\rho k} q^{mk+\rho(mhk-m(k))} \cdot [2mk + r]_q \cdot \frac{(q^r; q^m)_k^\rho}{(q^m; q^m)_k^\rho}
\]

is divisible by \( A_{r,m,n}(q)C_{m,n}(q) \).

Since

\[
\left. \frac{(q^r; q^m)_k}{(q^m; q^m)_k} \right|_{q=1} = (-1)^k \cdot \left( -\frac{r}{m} \right)_k,
\]

substituting \( q = 1 \), we get

\[
\sum_{k=0}^{n-1} (2km + r) \cdot \left( -\frac{r}{m} \right)_k^\rho \equiv 0 \pmod{N_{\frac{r}{m},n}}.
\]


