1. Review of AKNS-ZS system in the set-up of Beals-Coifman–D-Bar (idea/)method
2. NLS and DNLS with Quantum Potential and the Related Reaction-Diffusion Systems

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This talk will be presented by two parts. We will review some results.

PART I
At first, we will review some old results about 2 by 2 AKNS system with linear and quadratic spectral parameter and the related evolution equations and the inverse scattering method under the set-up of Beals-Coifman. For the forward problem, we will explain the D-Bar idea to get the scattering data in this set-up, which is also applicable in higher dimensional cases. We will mention the ODE system for the direct problem in this set-up. This could be solved via the estimate of L1 norm and L2 norm of the potentials. Here the inverse problem is formulated as a Riemann-Hilbert problem.

PART II—(old, a few pages will be shown here)
Then we will make a brief report on the nonlinear Schrödinger equations with quantum potential, derivative nonlinear Schrödinger equations and the related reaction-diffusion systems. We will discuss some possible applications.
Part II-A-new-in preparation
Then we will make a brief introduction on some recent papers based on this set-up. We will discuss some possible connection with other angles, further investigation of selfsimilar solutions of DNLS (by Derchyi Wu’s 2002 paper), compact formula of N-solutions of DNLS in the form of Belas-Coifman’s paper (in preparation)
PART II-B (will not be covered in this talk)
Then we will make a brief introduction on the some recent papers based on this set-up. We will discuss some possible applications.
PART I

At first, we will review some old results about 2 by 2 AKNS system and the related evolution equations and the solution methods. In this section, we review some results of $2 \times 2$ AKNS-ZS system with linear spectral parameter and quadratic spectral parameter in Beals-Coifman set-up. We consider the following $2 \times 2$ system:

$$\frac{dM}{dx} = z[J, M] + QM, \quad \text{Im}z \neq 0, \quad Q \in L^1(R, M_2(C)), \quad (1)$$

where $M(\cdot, z)$ is bounded and continuous and $M(x, z)$ is normalized as 1 as $z$ approaches $\infty$. $M(x, \cdot)$ is a meromorphic function with jump on $R = \{z : \text{Im}z = 0\}$. (Here $\det(M) = 1$ by a wronskian argument.)
We also consider a similar system with quadratic spectral parameter (revised Kaup-Newell system). We also consider the related evolution equations (e.g. DNLS, etc.) and the solutions methods (degenerate inverse scattering method, Hirota method, etc.).
PART II- brief introduction—-(old)-a few pages are shown here

Then we will make a brief report on the nonlinear Schrödinger equations with quantum potential, derivative nonlinear Schrödinger equations and the related reaction-diffusion systems.

\[ i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \Lambda |\psi|^2 \psi = s \frac{1}{|\psi|} \frac{\partial^2 |\psi|}{\partial x^2} \psi. \]  

A novel integrable version of the NLS equation with quantum potential has been termed the resonant nonlinear Schrödinger equation (RNLS). It can be regarded as a third version of the NLS, intermediate between the defocusing and focusing cases. Even though the RNLS is integrable for arbitrary values of the coefficient s.
The critical values $s = 1$ separates two distinct regions of behaviour. Thus, for $s < 1$ the model is reducible to the conventional NLS, (focusing for $\Lambda > 0$ and defocusing for $\Lambda < 0$). However, under some condition, for $s > 1$, it is reducible to a Reaction-Diffusion system, which can be transformed into Kaup-Broer system. In this case, the model exhibits novel solitonic phenomena. The RNLS can be interpreted as an NLS-type equation with an additional quantum potential $U_Q = |\psi|_{xx}/|\psi|$. Very recently it was shown that RNLS naturally appears in a reduced equation in the plasma physics.
Question Prof. Simon Ruijsenaars mentions there are 4-free parameters in focusing NLS one-soliton solution, and 2-free parameter in defocusing NLS dark-soliton, ..then what are the case for RNLS? (or for the related RD system the solution of $(Q^+ Q^-)$, via Madelung-like transform)
A Hirota bilinear representation of the Reaction-Diffusion system is given. Here some exact solutions are obtained by Hirota bilinear method. Recently we notice that the derivative Reaction-Diffusion system can be transformed into Reaction-Diffusion system. So in this sense, DNLS with 'quantum potential' is related NLS with 'quantum potential'. We will also consider a non-Madelung type hydrodynamic representation for resonant nonlinear Schrödinger type equations. New Broer-Kaup type systems of hydrodynamic equations are also derived from the derivative reaction-diffusion systems arising in $SL(2, R)$ Kaup-Newell hierarchy, represented in the non-Madelung hydrodynamic form. The relation with Kaup-Broer system will be shown here.
papers related to Part II (joint with OK Pashaev)


Solutions in (NLS), DNLS and (derivative) Reaction-Diffusion system (DRD)

- Part I
  \( (NLS) \)
  \( DNLS \)

- Part II—old
  \( DNLS \) with ”Quantum potential”
  \( \rightarrow \) derivative Reaction-Diffusion system (DRD)
  \( (NLS \text{ with } ”QP” \rightarrow RD), \) (joint work with Oktay Pashaev)
Part I

- Some exact solutions of DNLS (Derivative Nonlinear Schrödinger Eq.) and derivative Reaction-Diffusion System
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  Jyh-Hao Lee

- On the dissipative evolution equations associated with the Zakharov-Shabat system with a quadratic spectral parameter
  *Trans. AMS* (1989)
  Jyh-Hae Lee
1. Review on $2 \times 2$ AKNS-ZS system with linear spectral parameter in Beals-Coifman set-up

1.1 *Scattering problem and the associated evolution equations*

In this section, we review some results of $2 \times 2$ AKNS-ZS system with linear spectral parameter in Beals-Coifman set-up. We consider the following $2 \times 2$ system

$$
\frac{dM}{dx} = z[J, M] +QM, \quad \text{Im}z \neq 0, \quad Q \in L^1(R, M_2(C)), \quad (3)
$$

where $M(\cdot, z)$ is bounded and continuous and $M(x, z)$ is normalized as 1 at $z = \infty$. $M(x, \cdot)$ is a meromorphic function with jump on $R = \{z : \text{Im}z = 0\}$. (det(M)=1 by a wronskian argument.)
Remark:
In $n \times n$ AKNS-ZS system with linear spectral parameter, Beals-Coifman consider the following $n \times n$ system

$$\frac{dM}{dx} = z[J, M] + QM, \quad \text{Im}(z) \neq 0, \quad Q \in L^1(R, M_n(C)), \quad (4)$$

$$J = \text{diag}(id_1, id_2, \ldots, id_n), \quad d_1 < d_2 < \cdots < d_n, \quad i = \sqrt{-1}.$$ 

where $M(\cdot, z)$ is bounded and continuous and $M(x, z)$ is normalized as 1 at $z = \infty$. $M(x, \cdot)$ is a meromorphic function with jump on $R = \{z : \text{Im}(z) = 0\}$. (det(M)=1 by a wronskian argument.)

$M$ is solved by the integral equation:

$$M_{kl} = \delta_{kl} + (\int_{\alpha}^x \exp(xzi(d_k - d_l))(QM)_{kl}dy).$$

here $\alpha = \pm 1$, depends on the sign of $(d_k - d_l)$ to let the quantity in exponential be zero or negative.
$M(x, z)$ is solved when $Q$ has small $L_1$ norm, $M$ may have poles in $z$ when $Q$ does not have small norm. To solve $M$ with $Q$ in $L_2$ norm (which is a conservation quantity in several cases.), Beals-Coifman considered the wedge produce of columns of $M$, i.e. first column, wedge produce of the first column and 2nd column, etc., normalized at $-\infty$, n-th column, wedge product of n-1 column and n-th column, etc., normalized at $\infty$.
Let $D_z M = \frac{dM}{dx} - z[J, M]$ and $M_\pm(x, \cdot)$ be the limits of $M$ on
$C_+ = \{z : \text{Im} z > 0\}$, $C_- = \{z : \text{Im} z < 0\}$.

Then $D_z ((M_-)^{-1}M_+) = 0$. So, there exists $v(z)$ such that

$$M_+(x, z) = M_-(x, z)e^{xzJ}v(z)e^{-xzJ}. \tag{5}$$
In generic case, $M(x, \cdot)$ has finite number of poles $z_1, z_2, \cdots z_N$ on $C/R$. Let $N_j(x, z)$ be the regular part of $M(x, z)$ near $z = z_j$. There exists $v(z_j)$ such that

$$\text{Res}(M((x, \cdot), z_j)) = N_j(x, z_j)e^{xz_j}v(z_j)e^{-xz_j}.$$ (6)

$$(v(z), z_1, z_2, \cdots, z_N, v(z_1), v(z_2), \cdots, v(z_N))$$

is called scattering data of the potential $Q[2, 3]$. Symbolicall the scattering data is like $M^{-1}(D - \text{Bar}(M))$. 


Let $D_z M = \frac{dM}{dx} - z[J, M]$ and $D_z M = QM$, $D - Bar(M)$ also satisfies the same eq.

Here $D - Bar(M) = \partial/\partial \bar{z}(M)$. Then $D_z((M)^{-1}D - Bar(M)) = 0$. So, there exists $v(z)$ such that

$$D - Bar(M)(x, z) = M(x, z)e^{xzJ}v(z)e^{-xzJ}.$$ \hspace{1cm} (7)
** D-Bar method has been mentioned by Beals-Coifman in these Seminar papers*****

In Section 3.1 and 3.2 of this survey paper, it contains abstracts of the results of Beals and Coifman on scalar case (Novikov and Veselov eq.) and matrix case (Davey-Stewartson eq. and Ishimori eq.) of multi-dimensional inverse scattering problems. \( D - \text{Bar}(m) = Tm \)

(comment from Derchy Wu———1985 Beals-Coifman paper Multidimensional inverse scatterings and nonlinear partial differential equations by R Beals  RR Coifman is one of the papers where Beals and Coifman developed the higher dimensional IST theory. Despite it is old, what I(DC Wu) saw in 2017 Toronto workshop,several papers were still based on this theory largely
generalized Cauchy integral formular

\[ m(x; k) = \frac{1}{2\pi i} \int \int_R \frac{\partial m}{\partial \bar{z}}(x; z) \frac{dz \wedge d\bar{z}}{z - k} + \frac{1}{2\pi i} \int_C \frac{m(x; z)}{z - k} dz \]

Here one needs to prove from scratch depending on the properties of \( \partial m \). It holds under suitable conditions on \( m \) and the condition of the limit of \( m(x, \infty) \equiv \text{constant matrix} \). (comments from DC Wu)
Here, we introduce the case for NLS. For the case

\[ Q = \begin{pmatrix} 0 & q \\ \varepsilon q^* & 0 \end{pmatrix}, \]

\[ \varepsilon = \pm 1, \quad Q^* = \varepsilon Q, \quad J^* = -J. \]

We have the following constraints

\[ v^*(\bar{z}) = -\varepsilon v(z), \quad z \in R \]
\[ v^*(\bar{z}_j) = \varepsilon v(z_j), \quad z_j \notin R, \] (8)

(We pose this condition only for \( \varepsilon = -1 \)) where \( v(z_j) = \begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix} \), \( \text{Im} z_j > 0; \) \( v(z_j) = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix} \), \( \text{Im} z_j < 0. \)
If \( v(z, t), v(z_j, t) \) evolves as

\[
\begin{align*}
\frac{dv(z, t)}{dt} &= \alpha z^2 [J, v(z, t)] \\
\frac{dv(z_j, t)}{dt} &= \alpha z^2 [J, v(z_j, t)],
\end{align*}
\]

(9)

where \( z_1, z_2, \cdots, z_N \) fixed, then the associated \( M \) satisfies

\[
\begin{align*}
\frac{dM}{dx} &= z[J, M] + QM \\
\frac{dM}{dt} &= \alpha z^2 [J, M] + \alpha (zG_1 + G_2) M.
\end{align*}
\]

(10)
Let $\psi = M \exp(xzJ)$, $\psi$ satisfies

$$\begin{cases}
\frac{d\psi}{dx} = (zJ + Q)\psi = U\psi \\
\frac{d\psi}{dt} = \alpha(z^2J + zG_1 + G_2)\psi = V\psi.
\end{cases} \quad (11)$$

For $\alpha = 2$, $U_t - V_x + [U, V] = 0$ is equivalent to

$$q_t = iq_{xx} - 2i\varepsilon|q|^2q, \quad \varepsilon = \pm 1, \quad (12)$$

which is the nonlinear Schrödinger equation (NLS).
2. AKNS-ZS system with quadratice spectral parameter revised (Kaup-Newell) system

\[
\begin{align*}
\psi_x &= (z^2 J + zQ + P)\psi = U\psi \\
\psi_t &= \alpha(z^4 J + z^3 G_1 + z^2 G_2 + z G_3 + G_4)\psi = V\psi
\end{align*}
\]
"revised" Kaup-Newell System

\[ U_t - V_x + [U, V] = 0 \]

\[ \alpha = 2, \quad Q = \begin{pmatrix} 0 & q \\ \varepsilon q^* & 0 \end{pmatrix}, \quad J = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \quad \varepsilon = \pm 1, \quad P = Q(\text{ad}J)^{-1}Q, \quad \text{ad}J(A) = [J, A] \]
\[ q_t = iq_{xx} - \varepsilon q^2 q^* + \frac{i}{2} |q|^4 q \]

\[ u = q e^{\int_{-\infty}^x i \varepsilon q q^*} \]

\[ u_t = i u_{xx} - \varepsilon (u^2 u^*)_x, \]

which is DNLS considered by Kaup and Newell.


\[ \int |q(x, t)|^2 dx \] conserved in \( t \)
Exact Solutions of DNLS

- Degenerate Inverse Scattering
- Hirota Bilinear Method
- ...........,etc.
\[ \psi = Me^{xz^2J} \]

\[ \frac{dM}{dx} = z^2 [J, M] + zQM + PM \]

\[ \text{Im} z^2 \neq 0, \ P = Q(adJ)^{-1}Q \]

\[ adJ(A) \overset{\text{def}}{=} [J, A] \]

\( M_\pm(x, \cdot) \) is the limit of \( M(x, \cdot) \) from \( \Omega_\pm \) to \( \Sigma = \{ z : \text{Im} z^2 = 0 \} \).

\[ M_+(x, z) = M_-(x, z)e^{xz^2J}v(z)e^{-xz^2J} \]

Here \( M(x, z) \)is 1 when \( z \) approaches \( \infty \), so this is good to pose inverse problem.
Consider the case $M(x, \cdot)$ has finite number of poles $z_1, z_2, \cdots, z_n$. Let $N_j(x, z)$ be the regular part of $M(x, z)$ near $z = z_j$.

$$\text{Res}(M(x, z), z_j) = N_j(x, z_j)e^{xz_j^2J} \nu(z_j)e^{-xz_j^2J}$$

$$(\nu(z); z_1, z_2, \cdots, z_n; \nu(z_1), \nu(z_2), \cdots, \nu(z_n))$$

is called (mathematical) scattering data.
\[ Q = \begin{pmatrix} 0 & q \\ \varepsilon q^* & 0 \end{pmatrix}, \varepsilon = \pm 1, q^* = \bar{q} \]

\[ P = Q(adJ)^{-1} Q, \quad adJ(A) = [J, A], \quad Q^* = \varepsilon Q, \quad P^* = -P, \quad J^* = -J. \]

We have the following constraints:

\[ v^*(-\varepsilon \bar{z}) = v(z), \quad z^2 \in R, \quad \text{Im}(z^2) = 0 \]
\[ v^*(-\varepsilon \bar{z}) = -v(z_j), \quad z_j \notin R, \quad \varepsilon = -1 \]
\[ v^*(z) = v(z), \quad z^2 \in R, \]
\[ v^*(z_j) = -v(z_j), \quad z_j \notin R, \quad \varepsilon = -1 \]
Here $\sigma$ is an automorphism given by

$$
\begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}
\xrightarrow{\sigma}
\begin{pmatrix}
  a & -b \\
  -c & d
\end{pmatrix}
$$

$$(AB)^\sigma = A^\sigma B^\sigma$$

where $A = \begin{pmatrix}
  a & b \\
  c & d
\end{pmatrix}$, $K = \begin{pmatrix}
  \alpha & 0 \\
  0 & \beta
\end{pmatrix}$ and $\beta = -\frac{1}{\alpha}$, i.e.

$$(A)^\sigma = KAK^{-1}$$
If \( v(z, t), v(z_j, t) \) evolve as

\[
\begin{align*}
\frac{dv(z,t)}{dt} &= \alpha z^4 [J, v(z, t)] \\
\frac{dv(z_j,t)}{dt} &= \alpha z^4 [J, v(z_j, t)],
\end{align*}
\]

where \( z_1, z_2, \ldots, z_n \) fixed, then the associated \( M \) satisfies

\[
\begin{align*}
\frac{dM}{dx} &= z^2 [J, M] + (zQ + P)M \\
\frac{dM}{dt} &= \alpha z^4 [J, M] + \alpha (z_3 G_1 + z^2 G_2 + zG_3 + G_4)M.
\end{align*}
\]

Here

\[
MJM^{-1} \sim J + \frac{G_1}{z} + \frac{G_2}{z^2} + \cdots
\]

\( MJM^{-1} \) satisfis the eq. \( \frac{dMJM^{-1}}{dx} = z^2 [J, MJM^{-1}] + [(zQ + P), MJM^{-1}] \) and \( MJM^{-1} \) satisfies the characteristc polynomial of \( J \)

\[(\lambda + i)(\lambda - i) = 0\text{ (Caley-Hamilton Thm.)}\] Then we could compute the asymptotic terms \( G_j \).
Let $\psi = Me^{xz^2J}$, $\psi$ satisfies

$$\begin{align*}
\frac{d\psi}{dx} &= (z^2 J + zQ + P)\psi = U\psi \\
\frac{d\psi}{dt} &= \alpha z^4 J\psi + \alpha(z^3 G_1 + z^2 G_2 + zG_3 + G_4)\psi = V\psi. \\
\frac{d^2\psi}{dxdt} &= \frac{d^2\psi}{dt dx} \Rightarrow U_t - V_x + [U, V] = 0
\end{align*}$$

$$\begin{align*}
\psi_{xt} &= U_t\psi + U\psi_t = U_t\psi + UV\psi \\
\psi_{tx} &= V_x\psi + V\psi_x = V_x\psi + VU\psi
\end{align*}$$

$$U_t + UV = V_x + VU$$

$$U_t - V_x + [U, V] = 0$$
\( \alpha = 2, \ U_t - V_x + [U, V] = 0 \) is equivalent to

\[
q_t = iq_{xx} - \varepsilon q^2 q_x^* + \frac{i}{2} |q|^4 q, \ \varepsilon = \pm 1,
\]

\[
u = q e^{\int_{-\infty}^{x} i\varepsilon q q^*}, \text{ then } u \text{ satisfies}
\]

\[
 u_t = iu_{xx} - \varepsilon (u^2 u^*)_x, \ \varepsilon = \pm 1
\]
DNLS by Kaup-Newell, Gerjikov

In degenerate case

\[ M = I + \sum_{k=1}^{4N} \frac{a_k}{z - z_k} \]

\[ = \frac{a_j}{z - z_j} + N_j(x, z) \text{ near } z = z_j \]

\[ \text{Res}(M(x, \cdot), z_j) = a_j = N_j(x, z_j)e^{xz_j^2J} \nu(z_j)e^{-xz_j^2J} \]
\[ M \sim I + \frac{m_1}{z} + \frac{m_2}{z^2} + \cdots \]

\[ Q = -[J, m_1] = -[J, \sum_{k=1}^{4N} a_k] \]

\[ Q = \begin{pmatrix} 0 & q \\ -q^* & 0 \end{pmatrix}, \text{ N-Soliton } q(x, t) \]
\[ N = 1, \quad z_1 = \alpha + i\beta \]

\[ q(x, t) = 2i \frac{2l_0 e^{-2i(xz_1^2 + tz_1^4)}}{1 + (\frac{1}{2\beta} - \frac{i}{2\alpha})^2|m|^2}, \]

where

\[ m = l_0 e^{-2i(xz_1^2 + tz_1^4)}. \]
Examples
By a slight modification the argument in 3 of JH Lee's 1989 paper (Transaction AMS) works for the case

\[
\frac{dm}{dx} = z^2 [J, m] + zQm.
\]  

(4.1)

where

\[
J = \text{diag}(-Ni, i, i, \ldots, i), \quad Q = \begin{pmatrix}
0 & q_1 & q_2 & \cdots & q_n \\
r_1 & & & & \\
r_2 & & & & \\
\vdots & & & & \\
r_n & & & &
\end{pmatrix}, \quad r_i = \pm q_i^*
\]

The associated evolution is an \(N\)-component derivative nonlinear Schrödinger equation:

\[
(q_j)_i = i(q_j)_{xx} + \epsilon \alpha \left( \sum_k (q_k q_k^*) q_j \right), \quad \epsilon = \pm 1, \ j = 1, 2, \ldots, n.
\]  

(4.2)
Since \( \frac{d}{dt} \int \sum_j (q_j^* q_j) = 0 \), the \( L^2 \)-norm of \( Q \) is invariant under the evolution (4.2). We have global existence for the \( N \)-component derivative nonlinear Schrödinger equation if \( Q(x, 0) \) is generic and of Schwartz class. For \( n = 1 \), (4.1) becomes the derivative nonlinear Schrödinger equation (DNLS).

The global existence of DNLS was obtained in [11]. Let \( \psi = m \exp(xz^2 J) \); then (4.1) becomes

\[
(4.3) \frac{d\psi}{dx} = (z^2 J + zQ)\psi.
\]

For \( n = 2 \) Kaup and Newell obtained soliton solutions for the spectral problem (4.3) [7]. Gerzhikov et al. also considered this case [6]. Morris and Dodd considered the two-component derivative nonlinear Schrödinger equation using a larger scattering problem (i.e., \( n = 3 \)) [?]. Sasaki derived a Hamiltonian structure for the evolution (4.2) [35].
Benney system and dispersionless vector NLS

The system of Benney equations as a descriptive of long shallow water waves has been reduced to an infinite system of two-dimensional hydrodynamic equations by Zakharov [36]. In one dimension the Benney system is

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + \frac{\partial h}{\partial x} = 0, \tag{13}
\]

\[
\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x} \eta u = 0, \tag{14}
\]

\[
h = \int_{0}^{1} \eta(x, \xi, t) \, d\xi \tag{15}
\]

where the horizontal velocity \( u(x, \xi, t) \) is function of layers in the z-direction, enumerated by parameter \( \xi \) (\( 0 < \xi < 1 \)) and the fluid surface shape is \( h(x, t) \). In the special case, when the flow is divided into \( n \) layers,

\[
\eta(x, \xi, t) = \sum_{k=1}^{n} \eta_k(x, t) \delta(\xi - \xi_k) \Rightarrow h(x, t) = \sum_{k=1}^{n} \eta_k(x, t) \tag{16}
\]
the Beney system reduces to the multicomponent (vector) hydrodynamic system

\[
\frac{\partial u_k}{\partial t} + u_k \frac{\partial u_k}{\partial x} + \frac{\partial h}{\partial x} = 0,
\]

\[
\frac{\partial \eta_k}{\partial t} + \frac{\partial}{\partial x} \eta_k u_k = 0,
\]

\[
h = \sum_{k=1}^{n} \eta_k(x, t), \quad k = 1, 2, \ldots, n.
\]
The system (13),(14),(15) can be considered as a dispersionless limit of an infinite (discrete or continuous) one dimensional system of Schrödinger equations for function of two space variables \( \psi = \psi(x, \xi, t) \) [36], \( i\psi_t = \frac{1}{2}\psi_{xx} - h\psi \) with nonlocal self-interaction potential \( h = \int_0^1 |\psi|^2 \, d\xi \).

This type of nonlinear Schrödinger equation is generalization to a continual limit of the vector NLS, \( i\psi^a_t = \frac{1}{2}\psi^a_{xx} - (\sum_{b=1}^n |\psi^b|^2) \psi^a \) where \( a=1,2...,n, \psi(x, \xi_a, t) \equiv \psi^a(x, t) \), and integration is going in the domain of this parameter. In particular case of the discrete distribution with \( n \)-points, it becomes the vector NLS. Recently in [38] this equation has been considered with a domain of integration as a whole real line, and equation was called the nonlocal NLS in 2+1 dimensions. Moreover, in that paper the bilinear method of Hirota was applied to solve it for \( N \)-soliton solutions. By using the freedom of the \( \xi \) dependence, the authors constructed solitons localized in plane.
Integrability of such type extended NLS system is related with the linear problem,

$$
\Phi_x = U\Phi, \quad \Phi_t = V\Phi
$$  \hspace{1cm} (17)

where

$$
\Phi = \begin{pmatrix}
\varphi_0 \\
|\varphi >
\end{pmatrix}
$$  \hspace{1cm} (18)

$$
U = \begin{pmatrix}
-i\lambda & <\psi| \\
|\psi > & i\lambda I
\end{pmatrix}
$$  \hspace{1cm} (19)

$$
V = \begin{pmatrix}
-i\lambda^2 - \frac{i}{2} <\psi|\psi > & \lambda <\psi| + \frac{i}{2} <\psi_x| \\
\lambda |\psi > - \frac{i}{2} |\psi_x > & i\lambda^2 I + \frac{i}{2} |\psi >>\psi|
\end{pmatrix}
$$  \hspace{1cm} (20)

where \( \frac{\partial}{\partial x} |\psi(x, t) >= |\psi(x, t)_x >, \ \frac{\partial}{\partial t} |\psi(x, t) >= |\psi(x, t)_t >, \)
Compatibility condition for this linear system is equivalent to equations

\[ i|\psi_t > = \frac{1}{2}|\psi_{xx} > - < \psi |\psi > |\psi > \] (21)

\[ -i < \psi_t | = \frac{1}{2} < \psi_{xx} | - < \psi |\psi > < \psi | \] (22)

in the Dirac notations form. We consider "observable" \( \hat{\xi} \) as an operator with eigenvalue problem

\[ \hat{\xi}|\xi > = \xi'|\xi' > \] (23)

If the spectrum \( \xi \) is discrete, with eigenstates \( \{|\xi_n >\} \), finite \( N \), or infinite, then by completeness relation

\[ \sum_{n=1}^{\infty} |\xi_n > < \xi_n | = I \] (24)

we get the set of "wave function"

\[ < \xi_n |\psi > = \psi_n(x, t), n = 1, 2, \ldots \] (25)

finite or infinite.
For the continuous spectrum with identity relation

\[ \int_{a}^{b} d\xi |\xi > < \xi| = I \]  \hspace{1cm} (26)

we have wave function of two space variables \(< \xi|\psi > = \psi(x, \xi, t)\),

For the inner product in the first case we have
\(< \psi|\psi > = \sum_{n=1}^{N} |\psi_{n}(x, t)|^{2} \) which imply global symmetry \(U(N)\) for finite \(N\), and \(U(\infty)\) for infinite \(N\).

For continuous case the inner product is

\[ < \psi|\psi > = \int_{a}^{b} |\psi(x, \xi, t)|^{2} d\xi \]  \hspace{1cm} (27)
Hirota Bilinear Method

\[ D^m_x D^n_t a(x, t) \cdot b(x, t) = \lim \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(x, t)b(x', t'), \]

\[ x = x', t = t' \]
\[ q_t = iq_{xx} - \varepsilon q^2 q_x^* + \frac{i}{2} |q|^4 q, \quad \varepsilon = -1, \]

\[ q = \frac{G}{F} \]

\[ D_t(G \circ F) - iD^2_x(G \circ F) = 0 \]

\[ D^2_x(\bar{F} \circ F) - i\frac{\varepsilon}{2} D_x(\bar{G} \circ G) = 0 \]

\[ D_x(\bar{F} \circ F) + i\frac{\varepsilon}{2}(\bar{G} \circ G) = 0, \quad \varepsilon = -1 \]
(Remark: Here we list the example of NLS solutions via Hirota Bilinearization method)

\[ i\psi_t + \psi_{xx} + 2\alpha|\psi|^2\psi = 0, \quad \alpha = \pm 1 \]

\[ \psi = G/F \]

\[
\begin{align*}
(iD_t + D_x^2)G \cdot F &= \lambda GF \\
D_x^2 F \cdot F - 2\alpha|G|^2 &= \lambda F^2
\end{align*}
\]
$N = 1$, one soliton solution of DNLS

\[
G = e^{\eta_1}, \quad F = 1 + e^{\eta_1 + \bar{\eta}_1 + A},
\]

\[
e^A = \frac{-i \varepsilon \bar{k}_1}{2(k_1 + \bar{k}_1)^2}.
\]

\[
q = \frac{G}{F} = \frac{e^{\eta_1}}{1 + e^{\eta_1 + \bar{\eta}_1 + A}} = \frac{e^{-\frac{A}{2}} e^{\pm \left( \frac{-\nu}{2} x + (k^2 + \frac{\nu^2}{4}) t \right)}}{2 \cosh[k(x - vt - x_0)]}
\]

\[
k = \frac{k_1 + \bar{k}_1}{2}, \quad \nu = -(k_1 - \bar{k}_1).
\]
Nonlinear Schrödinger Eq (NLS) by Z.S. (Zakharov-Shabat),

\[ q_t = i q_{xx} - 2i \varepsilon |q|^2 q, \quad \varepsilon = \pm 1. \]

derivative NLS (DNLS),

\[ u_t = i u_{xx} - \varepsilon (u^2 u^*)_x, \quad \varepsilon = \pm 1. \]

\[ u = q \exp \{ i \int^x \varepsilon qq^* \}, \]

\[ q_t = i q_{xx} - \varepsilon q^2 q^*_x + \frac{i}{2} |q|^4 q, \quad \varepsilon = \pm 1. \]
4 Hirota Bilinear Method and Multi-Soliton Solutions of DNLS

Hirota introduced a new method for constructing multi-soliton solutions to integrable nonlinear evolution equations like Korteweg-de Vries, Sine-Gordon, nonlinear Schrödinger, derivative nonlinear Schrödinger equations, ⋯, etc. The main idea of Hirota bilinear method was to make a transformation into new variables, so that in these new variables, multi-soliton solutions appear in a particularly simple form. We can also derive multi-soliton solutions by other methods, e.g., by the inverse scattering transform. The advantage of Hirota’s method over others is that it is algebraic rather than analytic.
Now, we introduce new derivatives $D_x$, $D_t$, called Hirota bilinear operators (also called Hirota derivative operators) defined by

$$D_x^m D_t^n a(x, t) \cdot b(x, t) \equiv \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial x'} \right)^m \left( \frac{\partial}{\partial t} - \frac{\partial}{\partial t'} \right)^n a(x, t)b(x', t') \big|_{x=x', t=t'}, \quad (28)$$

or in another form

$$D_x^m D_t^n a(x, t) \cdot b(x, t) \equiv \lim_{\epsilon_1, \epsilon_2 \to 0} \frac{\partial^{m+n}}{\partial \epsilon_1^m \partial \epsilon_2^n} \left( a(x + \epsilon_1, t + \epsilon_2)b(x - \epsilon_1, t - \epsilon_2) \right), \quad (29)$$

where $a(x, t)$ and $b(x, t)$ are multi-variable functions, $m$, $n$ are integers.
Here, we consider DNLS as follows

\[ u_t = iu_{xx} + \epsilon(u^2u_x), \quad \epsilon = -1. \]  

(30)

By the transformation \( u = q \exp(\int_{-\infty}^{x} i\epsilon q\bar{q}) \), then we can transform as above into another type of DNLS as below:

\[ q_t = iq_{xx} + \epsilon q^2\bar{q}_x + \frac{i}{2} q|q|^4, \quad \epsilon = +1. \]  

(31)
Consider the dependent variable transformation as below:

\[ q = \frac{G}{F}, \quad (32) \]

where \( G \) and \( F \) are complex functions depend on \( x, t \). Applying Hirota bilinear operators on the equation of \( q \), we can transform it into the following bilinear forms:

\[ D_t(G \cdot F) - iD_x^2(G \cdot F) = 0, \quad (33a) \]
\[ D_x^2(F \cdot F) - i\varepsilon D_x(G \cdot G) = 0, \quad (33b) \]
\[ D_x(F \cdot F) + i\varepsilon (F \cdot F) = 0, \quad \varepsilon = -1. \quad (33c) \]
Since we have this bilinear form, we can write down soliton solutions mimicking A. Nakamura and H.-H. Chen’s results. Here we could guess that the exact N-soliton solution (omitted here):
Part II

The (Derivative) Nonlinear Schrödinger Equation with Quantum Potential ( (derivative)RNLS) and the (derivative) Reaction-Diffusion System

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RNLS = NLS with "QP"
transformed into
RD(Reaction-Diffusion System)

DNLS = "QP" (Quantum Potential)
transformed into
derivative Reaction-Diffusion System(DRD)
NLS = s"QP" ("Quantum Potential")

\[ i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\Lambda}{4} |\psi|^2 \psi = s \frac{\partial^2 |\psi|}{\partial x^2} \frac{\psi}{|\psi|}, \quad s > 1. \]

Madlung-like transformation, O.K. Pashaev, Chi-Kun Lin, J.H. Lee 2000:

\[ \psi = e^{R-iS} = e^R e^{-iS}, \quad e^+ = \exp(R + S), \quad e^- = \exp(R - S). \]

Reaction-Diffusion

\[
\begin{cases}
  -e^+_t + e^+_xx + \frac{\Lambda}{4} e^+ e^- e^+ &= 0 \\
  e^-_t + e^-_xx + \frac{\Lambda}{4} e^+ e^- e^- &= 0
\end{cases}
\]
\[
NLS \\
DNLS_1 \\
DNLS_2 \\
'' QP'' = \frac{|u_{xx}|}{|u|}
\]

\[
NLS = s'' QP'' u \rightarrow \text{Reaction-Diffusion system (RD)} \\
\text{(related to eq. of cold plasma)}
\]

\[\uparrow \text{transform} \]

\[
DNLS_1 = s'' QP'' u \rightarrow \text{derivative Reaction-Diffusion system (DRD)}
\]

\[
NLS + \left\{ \begin{array}{c} DNLS_1 \\ DNLS_2 \end{array} \right\} = s'' QP'' u \rightarrow \text{mixed type Reaction-Diffusion system}
\]
5 DRD system and Multi-Dissipatton Solutions

Now, we apply equivalent relations theorem [see [19], Appendix] to DNLS equation. Consider DNLS as follows:

\[ i u_t + u_{xx} + i \epsilon (u^2 \bar{u})_x = 0, \quad \epsilon = \mp 1, \quad (34) \]

with complex potential

\[ \varphi(u) = \varphi_R(u) + i \varphi_I(u) = -i \epsilon (u \bar{u}_x + 2 \bar{u} u_x), \quad (35) \]

where \( \varphi_R(u) = -\frac{e^i}{2} (u \bar{u}_x - \bar{u} u_x), \varphi_I(u) = -\frac{3 \epsilon}{2} (u \bar{u})_x. \)
Let $u = \exp(R - iS)$, $Q^+ = \exp(R + S)$, $Q^- = \exp(R - S)$, which is a Madelung-like transform. We know that $Q^+ Q^- = |u|^2$. By a simple calculation, we have a Derivative Reaction Diffusion (DRD) system

$$
\begin{align*}
-Q_t^+ + Q_{xx}^+ - \epsilon (Q^+ Q^- Q^+)_x - \Phi Q^+ &= 0, \\
Q_t^- + Q_{xx}^- + \epsilon (Q^+ Q^- Q^-)_x - \Phi Q^- &= 0,
\end{align*}
$$

(36)

where $\Phi = (\ln Q^+ Q^-)_{xx} + \frac{1}{2} [ (\ln Q^+ Q^-)_x ]^2$. 


For each soliton solutions of (34), we can construct dissipatation solutions of (36). Motivated by the results of Reaction-Diffusion system of Pashaev et al [19, 20, 21], we consider a simple DRD system as follows

\begin{align}
Q_t^+ - Q_{xx}^+ + \epsilon (Q^+ Q^- Q^+)_x &= 0, \\
Q_t^- + Q_{xx}^- + \epsilon (Q^+ Q^- Q^-)_x &= 0, \quad \epsilon = \pm 1. 
\end{align}

(37)
By using equivalent relations, we can transform (37) into a modified DNLS equation

\[ iu_t + u_{xx} + i\epsilon(u^2\bar{u})_x = 2\frac{|u|_{xx}}{|u|}u, \quad \epsilon = \mp 1, \quad (38) \]

which is called Resonance DNLS.

Obviously, we can find dissipaton solutions of (37) and associate the soliton solutions of (38) by equivalent relations. Here we use Hirota bilinear method to find its exact solutions.
We consider $\epsilon = 1$ in (37) and write down DRD system in the following form
\begin{align*}
Q_t^+ - Q_{xx}^+ + (Q^+ Q^- Q^+)_x &= 0, \\
Q_t^- + Q_{xx}^- + (Q^+ Q^- Q^-)_x &= 0,
\end{align*}
(39)
where $\{Q^+, Q^-\}$ is a pair of real functions. Consider a transformation as follows
\begin{align*}
q^+ &= Q^+ \exp(- \int_{-\infty}^{x} q^+ q^-), \\
q^- &= Q^- \exp(+ \int_{-\infty}^{x} q^+ q^-),
\end{align*}
(40)
where $\{q^+, q^-\}$ also be a pair of real functions.
Consider a transformation as follows

\[
q^+ = Q^+ \exp(- \int_{-\infty}^{x} q^+ q^-), \\
q^- = Q^- \exp(+ \int_{-\infty}^{x} q^+ q^-),
\]

(41)

where \{q^+, q^-\} also be a pair of real functions. By this transformation (41), we can transform (39) into a new DRD system.
After some computation and using the fact $q^+ q^- = Q^+ Q^-$, we have the following new DRD system

\begin{align*}
q_t^+ - q_{xx}^+ - q^+ q^+ q_{x}^- - \frac{1}{2}(q^+ q^-)^2 q^+ &= 0, \\
q_t^- + q_{xx}^- - q^- q^- q_{x}^+ + \frac{1}{2}(q^+ q^-)^2 q^- &= 0, \\
\end{align*}

(42)

which can be written as

\begin{align*}
q_t^\pm = q_{xx}^\pm - q^\pm q^\pm q_{x}^\mp \mp \frac{1}{2}(q^+ q^-)^2 q^\pm &= 0. \\
\end{align*}

(43)
REMARK. For the expression $q^\pm$ in (42), we called $q^+$ be positive term and $q^-$ be negative term. Consider a dependent variable transformation similarly as (32).

\[ q^\pm = \frac{G^\pm}{F^\pm}, \]  

(44)

where $G^\pm$, $F^\pm$ are real functions depending on $x$, $t$. 
Now, we transform (42) into bilinear forms:

\begin{align}
D_t(G^\pm \cdot F^\pm) &\mp D_x^2(G^\pm \cdot F^\pm) = 0, \quad (45a) \\
D_x^2(F^\mp \cdot F^\pm) &\mp \frac{1}{2} D_x(G^\mp \cdot G^\pm) = 0, \quad (45b) \\
D_x(F^\mp \cdot F^\pm) &\pm \frac{1}{2} (G^\mp \cdot G^\pm) = 0. \quad (45c)
\end{align}

We may follow the style of N-soliton solutions to write down the N-dissipatlon solutions of this bilinear system see [27].
In the same way in DNLS case, we show the 1-dissipaton solution of DRD system

\[ G^\pm = \pm e^{\eta_1^\pm}, \quad F^\pm = 1 + e^{\eta_1^\pm + \eta_1^\mp + A^\pm}, \]
\[ A^\pm = \frac{\mp k_1^\mp}{2(k^\pm + k_1^\mp)^2}, \]

where \( \eta_1^\pm \equiv (k_1^\pm)x \pm (k_1^\pm)^2t + \eta_1^\pm(0) \). Moreover,

\[ q^\pm(x, t) = \pm e^{-A^\pm/2} \frac{e^{\pm (\mp \nu x + k^2 + \nu_2/4t)}}{\cosh[k(x - \nu t - x_0)]}, \]

where \( k = (k_1^+ + k_1^-)/2, \nu = -(k_1^+ - k_1^-) \).
We may plot the graphs by the software Mathematica. The computing results will be published elsewhere. This section contains parts of Lee Yen-Ching’s 1999 Master Thesis and Lin Chien-Chih’s 2001 Master Thesis of National Taiwan University [13, 15]. Remark: This could be done again by other softwares to compare.
6 Summary

Here we survey some results of N-soliton solutions of DNLS and the $N$-dissipaton solutions of DRD system ($N=1,2$). The plots of one and two dissipatons of DRD system are similar as Reaction-Diffusion system of Pashaev et al [16]. But the resonance states are not so clear. We notice the difference, but it needs further investigation in the future.
The Reaction-Diffusion system

\[
(RD) \begin{cases}
Q_t^+ - Q_{xx}^+ + (Q^+ Q^- Q^+) = 0 \\
Q_t^- - Q_{xx}^- + (Q^+ Q^- Q^-) = 0
\end{cases}
\]

was considered by Matina, Pashaev, Soliani 1998[16]. Then Pashaev et al used Madelung-like transform to link the NLS with Quantum potentail (also called resonant nonlinear Schrödinger equation RNLS) with Reaction-Diffusion system. e.g. [26, 30]
This novel integrable version of the NLS equation is as below:

\[ i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + \frac{\Lambda}{4} |\psi|^2 \psi = s \frac{1}{|\psi|} \frac{\partial^2 |\psi|}{\partial x^2} \psi. \] (48)

has been termed the \textit{resonant nonlinear Schrödinger equation} (RNLS). It can be regarded as a third version of the NLS, intermediate between the defocusing and focusing cases.
Even though the RNLS is integrable for arbitrary values of the coefficient $s$, the critical value $s = 1$ separates two distinct regions of behaviour. Thus, for $s < 1$ the model is reducible to the conventional NLS, (focusing for $\Lambda > 0$ and defocusing for $\Lambda < 0$). However, for $s > 1$ it is not reducible to the usual NLS, but rather to a reaction-diffusion system. In this case, the model exhibits novel solitonic phenomena [30].
The RNLS can be interpreted as an NLS-type equation with an additional 'quantum potential' 
$U_Q = |\psi|_{xx}/|\psi|$. 
Since then, several cases have been considered, DNLS with quantum potential become derivative reaction-diffusion system e.g. Lee JH, Lee YC, Lin CC [27] :

$$(DRD_1) \begin{cases} 
    Q_t^+ - Q_{xx}^+ + (Q^+ Q^- Q^+)_x = 0 \\
    Q_t^- - Q_{xx}^- + (Q^+ Q^- Q^-)_x = 0
\end{cases}$$
Similar method could be used to get the exact solutions of DRD$_2$ as below:

\[
(DRD_2) \begin{cases} 
Q_+^t - Q_-^{xx} + Q^+ Q^- Q^+_x = 0 \\
Q^-_t - Q_-^{xx} + Q^+ Q^- Q^-_x = 0
\end{cases}
\]
s > 1,

- NLS = $s^{"QP"}u$, $"QP" = \frac{|u|_{xx}}{|u|}$
  $\rightarrow$ RD (Raction-Diffusion System)

- DNLS$_1 = s^{"QP"}u$
  $\rightarrow$ DRD$_1$ (derivative Raction-Diffusion System), which can be transformed into RD.

- DNLS$_2 = s^{"QP"}u$
  $\rightarrow$ DRD$_2$. 
Mixed type of the above systems is also considered in a paper of Pashev OK and JH Lee JH [28]. Very recently it was shown that RNLS naturally appears in the plasma physics, where it describes the propagation of one-dimensional long magnetoacoustic waves in a cold collisionless plasma subject to a transverse magnetic field[31]. A Hirota bilinear representation of the Reaction-Diffusion system with non-zero boundary condition is given in that paper.
Here one dissipaton and two-dissipaton exact solutions are obtained by Hirota bilinear method and their mutual interactions are studied. Then it becomes necessary to consider the NLS with quantum potential, also called resonant NLS with nontrivial boundary conditions. The results of some exact solutions of Reaction-Difusion system with nontrivial boundary conditions are in the following paper [32].
Recently we found that the relation between DRD (derivative Reaction Diffusion System) and RD (Reaction-Diffusion system). So in this sense, DNLS with 'quantum potential' is related NLS with 'quantum potential'. So in this way, we can construct more exact solutions for RD.
Some remarks:: Burger’s Equation

\[ u_t + uu_x = \mu u_{xx} \]

Cole-Hopf Transformation (F. John, PDE)

\[ u = -2\mu (\log T)_x = -2\mu \frac{T_x}{T} \]

\[ T_t = \mu T_{xx} \]

Hopf, E., The partial differential equation

\[ u_t + uu_x = \mu u_{xx} \]


It is said that this transformation was found in Forstho’s PDE books (6 vol.) as an exercise around 1900-1902.
Madelung-like transform:

\[ i \frac{\partial \varphi}{\partial t} + \frac{\partial^2 \varphi}{\partial x^2} + \frac{\nu}{4} \varphi^2 \varphi = s \frac{\partial^2 |\varphi|}{\partial x^2} |\varphi| \]

where \( \varphi = e^{R-iS} \) and \( s > 1 \).

\[ E^+ = e^{R+S}, E^- = e^{R-S} \]

\[ E^+_t = E^+_{xx} + 2\nu E^+ E^- E^+ \]

\[ -E^-_t = E^-_{xx} + 2\nu E^+ E^- E^- \]

\[ \nu^+ = (\log E^+)_x, \nu^- = (\log E^-)_x \]

\[ \rho = E^+ E^-, \rho_x = \rho \nu^+ + \rho \nu^- \]
Broer (1975), Kaup (1975)

\[
\begin{aligned}
\{ & v^+_t = (v^+_x + (v^+)^2)_x + 2\nu \rho_x \\
& \rho_t + \rho_{xx} = (2\rho v^+)_x \\
\}
\]

\[
\begin{aligned}
\{ & -v^-_t = (v^-_x + (v^-)^2)_x + 2\nu \rho_x \\
& -\rho_t + \rho_{xx} = (2\rho v^-)_x \\
\}
\]
\[ \rho = E^+ E^- \]

\[ \rho_t = E^+_t E^- + E^+ E^-_t \]

\[ = (E^+_{xx} + 2\nu E^+ E^- E^+) E^- + E^+ (-E^-_{xx} - 2\nu E^+ E^- E^-) \]

\[ = E^+_{xx} E^- - E^+ E^-_{xx} \]

\[ = (E^+_x E^-)_x - (E^+ E^-_x)_x \]
\[ v^- = (\log E^-)_x = \frac{E^-_x}{E} \]

\[ (v^-)_t = (\log E^-)_{xt} = (\log E^-)_{tx} \]

\[ = (\log E^-)_{tx} \]

\[ = \left( \frac{E^-_t}{E^-} \right)_x \]

\[ = \left( \frac{-E^-_{xx} - 2\nu E^+ E^- E^-}{E^-} \right)_x \]
J. Satsuma, K. Kajiwara, J. Matsukidaira, J. Hietarinta
"Solutions of the Broer-Kaup System through Its Trilinear form"
J. of the physical Society of Japan, 1992

\[ h_t = (h_x + 2uh)_x \]
\[ u_t = (u^2 + 2h - u_x)_x \]
\[ h = \rho, \ u = -v \]

This is related to Nonlinear Schrodinger Eq. with quantum Potential and Reaction-Diffusion System.
References I


[27] Lee Jyh-hao, Lee Yen-Ching and Lin Chien-Chih, Exact solutions of DNLS and derivative Reaction-Diffusion system, *J. of Nonlinear Mathematical Physics* 2002 Vol. 9 Supplement 1, pp.87-97


