Gamma expansions of $q$-Narayana polynomials, pattern avoidance and the $(-1)$-phenomenon

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Joint work with Bin Han (Université Claude Bernard Lyon 1),
Dazhao Tang (Chongqing University) and Jiang Zeng (Université Claude
Bernard Lyon 1)
Plan of the talk

Introduction

Main results

\((-1)\)-phenomenon: motivation and application

Complete characterization for $\mathfrak{S}_n(\tau)$, $|\tau| = 3$

$\mathfrak{S}_n(2413, 3142)$ and $\mathfrak{S}_n(1342, 2431)$

Sketch of the proofs
Gamma positivity

A polynomial $f(x) = \sum_i a_i x^i \in \mathbb{R}[x]$ is called $\gamma$-positive if

$$f(x) = \sum_{i=0}^{\lfloor n/2 \rfloor} \gamma_i x^i (1 + x)^{n-2i}$$

for $n \in \mathbb{N}$ and nonnegative reals $\gamma_0, \gamma_1, \ldots, \gamma_{\lfloor n/2 \rfloor}$.
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Eulerian polynomials and their gamma expansions

For any permutation $\pi = \pi(1)\pi(2) \cdots \pi(n)$:

$\text{exc } \pi = |\{1 \leq i \leq n : \pi(i) > i\}|$,

$\text{wex } \pi = |\{1 \leq i \leq n : \pi(i) \geq i\}|$,

$\text{des } \pi = |\{1 \leq i \leq n : \pi(i) > \pi(i + 1)\}|$,

$\text{dd}^* \pi = |\{1 \leq i \leq n : \pi(i - 1) > \pi(i) > \pi(i + 1)\}|$,

where we let $\pi(0) = \pi(n + 1) = n + 1$. 

\[ A_n(t) := \sum_{\pi \in S_n} t^{\text{des } \pi} - \sum_{\pi \in S_n} t^{\text{exc } \pi} = \sum_{\pi \in S_n} t^{\text{wex } \pi - 1} = \left\lfloor \frac{n - 1}{2} \right\rfloor \sum_{k = 0}^\infty \gamma_{A_n} A_n k t^k n - 1 - 2k, \]

\[ A_1(t) = 1, \quad A_2(t) = 1 + t, \quad A_3(t) = 1 + 4t + t^2 = (1 + t)^2 + 2t, \]

\[ A_4(t) = 1 + 11t + 11t^2 + t^3 = (1 + t)^3 + 8t(1 + t), \]

\[ A_5(t) = 1 + 26t + 66t^2 + 26t^3 + t^4 = (1 + t)^4 + 22t(1 + t)^2 + 16t^2(1 + t), \]

\[ A_6(t) = 1 + 57t + 302t^2 + 302t^3 + 57t^4 + t^5 = (1 + t)^5 + 52t(1 + t)^3 + 136t^2(1 + t)^2, \]
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- valley: \( 1 \)
- double descent: \( t \)
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- \( \# \text{peak} + 1 = \# \text{valley} \)
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\[ N_n(t) := \sum_{\pi \in S_n(231)} t^{\text{des} \pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_n^N t^k (1 + t)^{n-1-2k}, \]

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Alternating permutations

A permutation is said to be *alternating (or up-down)* if it starts with an ascent and then descents and ascents come in turn. We denote by $\mathcal{S}_n$ (resp. $\mathcal{A}_n$) the set of permutations (resp. alternating permutations) of length $n$.

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For example, there are only two alternating permutations of length 3:

132 and 231.
Permutation pattern

Given two permutations $\pi \in S_n$ and $p \in S_k$, we say that $\pi$ contains the pattern $p$ if there exists a set of indices $1 \leq i_1 < i_2 < \cdots < i_k \leq n$ such that the subsequence $\pi(i_1)\pi(i_2)\cdots\pi(i_k)$ of $\pi$ is order-isomorphic to $p$. Otherwise, $\pi$ is said to avoid $p$. 

Dates back to Knuth’s study of stack-sortable permutations. (231-avoiding) Babson and Steingrímsson introduced vincular pattern. We use dash - to indicate possible spaces between letters of the pattern. For instance, 15324 contains 123 and 3-12, but avoids 231 and 12-3.
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\(q\)-Narayana polynomials

We define \(N_n(t, q)\) as the Taylor coefficients in the following continued fraction expansion

\[
\sum_{n=0}^{\infty} N_n(t, q) z^n = \frac{1}{1 - \frac{c_1 z}{1 - \frac{c_2 z}{1 - \frac{c_3 z}{1 - \frac{c_4 z}{1 - \ldots}}}}},
\]

where \(c_{2k-1} = q^{k-1}\) and \(c_{2k} = tq^{k-1}\) for \(k = 1, 2, \ldots\).
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$N_n(t, q) = \sum_{\pi \in \mathfrak{S}_n(321)} t^{\text{exc} \pi} q^{\text{inv} \pi - \text{exc} \pi}$.

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- \(N_n(t, q) = \sum_{\pi \in \mathfrak{S}_{n}(321)} t^{\text{exc} \pi} q^{\text{inv} \pi - \text{exc} \pi}\).

- \(N_n(tq, q^2) = \text{Dyck}(n; t, q) = \sum_{\rho \in \text{Dyck}(n)} t^{\text{rank} \rho} q^{\text{area} \rho}\).


Main result I: ten new interpretations for $N_n(t, q)$

Theorem (F.-Han-Tang-Zeng)

\[
N_n(t, q) = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des} \pi} q^{(31-2) \pi} = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des} \pi} q^{(13-2) \pi} = \sum_{\pi \in \mathfrak{S}_n(231)} t^{\text{des} \pi} q^{\text{adi} \pi}
\]

\[
= \sum_{\pi \in \mathfrak{S}_n(312)} t^{\text{des} \pi} q^{(2-31) \pi} = \sum_{\pi \in \mathfrak{S}_n(312)} t^{\text{des} \pi} q^{(2-13) \pi} = \sum_{\pi \in \mathfrak{S}_n(312)} t^{\text{des} \pi} q^{\text{adi} \pi}
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\]
Main result II: new interpretations for the gamma coefficients

Theorem (F.-Han-Tang-Zeng)

For \( n \geq 1 \), the following \( \gamma \)-expansions formula holds true

\[
N_n(t, q) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \gamma_{n,k}(q) t^k (1 + t)^{n-1-2k},
\]

(2)

where

\[
\gamma_{n,k}(q) = \sum_{\pi \in \tilde{S}_{n,k}(321)} q^{\text{inv } \pi - \text{exc } \pi} \quad \text{(3)}
\]

\[
= \sum_{\pi \in \tilde{S}_{n,k}(213)} q^{(31-2) \pi} = \sum_{\pi \in \tilde{S}_{n,k}(231)} q^{(2-13) \pi} \quad \text{(4)}
\]

\[
= \sum_{\pi \in \tilde{S}_{n,k}(132)} q^{(2-31) \pi} = \sum_{\pi \in \tilde{S}_{n,k}(312)} q^{(13-2) \pi}. \quad \text{(5)}
\]

Remark

Eq. (3) is due to Lin and Fu.

Theorem (F.-Han-Tang-Zeng)

We have

\[
\sum_{\pi \in \mathcal{G}_n(213)} t^{\text{des}} \pi q^{\text{adi} \pi} = \sum_{\pi \in \mathcal{G}_n(132)} t^{\text{des}} \pi q^{\text{adi}^* \pi} \tag{6}
\]

\[
= \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left( \sum_{\pi \in \mathcal{G}_n,k(213)} q^{\text{adi} \pi} \right) t^k (1 + t)^{n-1-2k}, \tag{7}
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\[
= \left\lfloor \frac{n-1}{2} \right\rfloor \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} \left( \sum_{\pi \in \mathcal{G}_n,k(132)} q^{\text{adi}^* \pi} \right) t^k (1 + t)^{n-1-2k}. \tag{8}
\]
Classical results

For $n$ odd, the $-1$ evaluation of Eulerian polynomials gives tangent numbers $E_{2n+1}$:

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\sum_{\pi \in S_n} (-1)^{\text{des} \pi} = \sum_{\pi \in S_n} (-1)^{\text{exc} \pi} = \begin{cases} 
0 & \text{if } n \text{ is even,} \\
(-1)^{\frac{n-1}{2}} E_n & \text{if } n \text{ is odd.}
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While for $n$ even, the $-1$ evaluation of Roselle polynomials gives secant numbers $E_{2n}$:

$$\sum_{\pi \in D_n^*} (-1)^{\text{des} \, \pi} = \sum_{\pi \in D_n} (-1)^{\text{exc} \, \pi} = \begin{cases} (-1)^{\frac{n}{2}} E_n & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

(10)

where $D_n$ (resp. $D_n^*$) denotes the set of derangements (resp. coderangements) of length $n$. q-analogues of above results have been obtained by Foata and Han, Josuat-Vergès, Shin and Zeng.
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For $n$ odd, the $-1$ evaluation of Eulerian polynomials gives tangent numbers $E_{2n+1}$:

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$q$-analogues of above results have been obtained by Foata and Han, Josuat-Vergès, Shin and Zeng.
For a given subset $\mathcal{S}_n(p_1, p_2, \cdots, p_m) \subset \mathcal{S}_n$ arising from pattern avoidance, we do the following things.

1. Enumerate $\mathcal{A}_n(p_1, p_2, \cdots, p_m)$.

2. Derive the generating function of des (resp. exc) over $\mathcal{S}_n(p_1, p_2, \cdots, p_m)$, say $X_n(t)$, then evaluate $X_n(-1)$.

3. Derive the generating function of des (resp. exc) over $\mathcal{D}_n^*(p_1, p_2, \cdots, p_m)$ (resp. $\mathcal{D}_n(p_1, p_2, \cdots, p_m)$), say $Y_n(t)$, then evaluate $Y_n(-1)$.

If the result of 1. (up to an index shift) matches with either that of 2. in the sense of Eq. (9), or that of 3. in the sense of Eq. (10), we say $\mathcal{S}_n(p_1, p_2, \cdots, p_m)$ exhibits the $(-1)$-phenomenon. If we get a double match, then we call it the strong $(-1)$-phenomenon.
(-1)-phenomenon

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This approach was already used by Foata-Schützenberger who first derived (9) and (10) via \( \gamma \)-expansions of the Eulerian polynomials.
Theorem (F.-Han-Tang-Zeng)
For any \( n \geq 1 \),

\[
N_n(-1, q) = \sum_{\pi \in S_n(321)} (-1)^{\text{exc} \ \pi} q^{\text{inv} \ \pi - \text{exc} \ \pi} = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
(-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}}(q^2) & \text{if } n \text{ is odd},
\end{cases}
\]

\[\tag{11}\]

\[
\sum_{\pi \in D_n(321)} (-1)^{\text{exc} \ \pi} q^{\text{inv} \ \pi} = \begin{cases} 
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\]

\[\tag{12}\]

Where \( C_n(q) := N_n(q, q^2) \) is Carlitz's \( q \)-Catalan number, a polynomial of degree \( \binom{n}{2} \).
Three tables

**Table**: The enumeration of $\mathcal{A}_n(\tau)$, for $n \geq 3$ odd and even.

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### Table: The $(-1)$-evaluation over $\mathcal{S}_n(\tau)$ and $\mathcal{D}_n^*(\tau)$ with respect to des. The signs $(-1)^n$ have all been removed.

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<thead>
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<th>des $\setminus \tau$</th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{S}_{2n+1}$</td>
<td>$\ast$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$\ast$</td>
</tr>
<tr>
<td>$\mathcal{D}_{2n}^*$</td>
<td>$\ast$</td>
<td>$C_n$</td>
<td>$C_{n-1}$</td>
<td>$C_n$</td>
<td>$\ast$</td>
<td>$\ast$</td>
</tr>
</tbody>
</table>
Three tables

Table: The enumeration of $\mathcal{A}_n(\tau)$, for $n \geq 3$ odd and even.

<table>
<thead>
<tr>
<th>$\tau$</th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{A}_{2n+1}(\tau)$</td>
<td>$C_{n+1}$</td>
<td>$C_n$</td>
<td>$C_{n+1}$</td>
<td>$C_n$</td>
<td>$C_{n+1}$</td>
<td>$C_{n+1}$</td>
</tr>
<tr>
<td>$\mathcal{A}_{2n}(\tau)$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$C_{n+1}$</td>
</tr>
</tbody>
</table>

Table: The $(-1)$-evaluation over $\mathcal{S}_n(\tau)$ and $\mathcal{D}^*_n(\tau)$ with respect to des. The signs $(-1)^n$ have all been removed.

<table>
<thead>
<tr>
<th>$\text{des} \backslash \tau$</th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{S}_{2n+1}$</td>
<td>*</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>*</td>
</tr>
<tr>
<td>$\mathcal{D}^*_{2n}$</td>
<td>*</td>
<td>$C_n$</td>
<td>$C_{n-1}$</td>
<td>$C_n$</td>
<td>*</td>
<td>*</td>
</tr>
</tbody>
</table>

Table: The $(-1)$-evaluation over $\mathcal{S}_n(\tau)$ and $\mathcal{D}_n(\tau)$ with respect to exc. The signs $(-1)^n$ have all been removed.

<table>
<thead>
<tr>
<th>$\text{exc} \backslash \tau$</th>
<th>123</th>
<th>132</th>
<th>213</th>
<th>231</th>
<th>312</th>
<th>321</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathcal{S}_{2n+1}$</td>
<td>*</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>*</td>
<td>*</td>
<td>$C_n$</td>
</tr>
<tr>
<td>$\mathcal{D}_{2n}$</td>
<td>$F_n$</td>
<td>$C_n$</td>
<td>$C_n$</td>
<td>*</td>
<td>*</td>
<td>$C_n$</td>
</tr>
</tbody>
</table>
For all $1 \leq i \leq n$, the entry $\pi(i)$ is called a *nondescent top* (resp. *nonexcedance top*) of $\pi$, if $\pi(i) < \pi(i + 1)$ (resp. $\pi(i) \leq i$), where $\pi(n + 1) = n + 1$. $\pi(i)$ is called a *left-to-right maximum* if $\pi(i) = \max\{\pi(1), \pi(2), \ldots, \pi(i)\}$. A nondescent top $\pi(i)$ ($i = 1, \ldots, n$) is called a *foremaximum* of $\pi$ if it is at the same time a left-to-right maximum. Denote the number of foremaximum of $\pi$ by $f_{\text{max}} \pi$. 

**Definition (Shin-Zeng)**: A permutation $\pi$ is called *coderangement* if $f_{\text{max}} \pi = 0$. Let $\mathcal{D}^*_n$ be the subset of $S_n$ of coderangements.
For all $1 \leq i \leq n$, the entry $\pi(i)$ is called a non-descent top (resp. non-excedance top) of $\pi$, if $\pi(i) < \pi(i + 1)$ (resp. $\pi(i) \leq i$), where $\pi(n + 1) = n + 1$. $\pi(i)$ is called a left-to-right maximum if $\pi(i) = \max \{\pi(1), \pi(2), \cdots, \pi(i)\}$. A non-descent top $\pi(i)$ ($i = 1, \cdots, n$) is called a foremaximum of $\pi$ if it is at the same time a left-to-right maximum. Denote the number of foremaximum of $\pi$ by $f_{\max} \pi$.

**Definition (Shin-Zeng)**

A permutation $\pi$ is called coderangement if $f_{\max} \pi = 0$. Let $\mathcal{D}_n^*$ be the subset of $\mathcal{S}_n$ of coderangements.
A key lemma for $\mathcal{D}^*(\tau)$

Lemma
Let $P_0(t, q) = Q_0(t, q) = R_1(t, q) = 1$, $P_1(t, q) = Q_1(t, q) = 0$, and for $n \geq 2$,

\[ P_n(t, q) := \sum_{\pi \in \mathcal{D}_n^* (231)} t^{\text{des } \pi} q^{(13-2) \pi}, \]
\[ Q_n(t, q) := \sum_{\pi \in \mathcal{D}_n^* (132)} t^{\text{des } \pi} q^{(2-31) \pi}, \]
\[ R_n(t, q) := \sum_{\pi \in \mathcal{D}_n^* (213)} t^{\text{des } \pi} q^{(31-2) \pi}. \]

Then for $n \geq 2$,

\[ P_n(t, q) = \sum_{m=0}^{n-2} t q^{n-m-1} P_m(t, q) N_{n-m-1}(t, q), \quad (13) \]
\[ Q_n(t, q) = \sum_{m=0}^{n-2} t q^m Q_m(t, q) N_{n-m-1}(t, q), \quad (14) \]
\[ R_n(t, q) = \sum_{m=1}^{n-1} t q^{n-m-1} R_m(t, q) N_{n-m-1}(t, q). \]
231-avoiding des-case

Theorem (F.-Han-Tang-Zeng)

For any $n \geq 1$,

$$
\sum_{\pi \in \mathcal{S}_n(231)} (-1)^{\text{des } \pi} q^{(31-2) \pi} = \sum_{\pi \in \mathcal{G}_n(231)} (-1)^{\text{des } \pi} q^{(13-2) \pi} = \begin{cases} 
0 & \text{even}, \\
(-q)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} (q^2) & \text{odd},
\end{cases}
$$

(16)

$$
\sum_{\pi \in \mathcal{D}_n^*(231)} (-q)^{\text{des } \pi} q^{(31-2) \pi} = \begin{cases} 
(-q)^{\frac{n}{2}} C_{\frac{n}{2}} (q^2) & \text{if } n \text{ is even}, \\
0 & \text{if } n \text{ is odd},
\end{cases}
$$

(17)

$$
\sum_{\pi \in \mathcal{D}_n^*(231)} (-1)^{\text{des } \pi} q^{(13-2) \pi} = \begin{cases} 
(-1)^{\frac{n}{2}} C_{\frac{n}{2}}^* (q) & \text{if } n \text{ is even}, \\
0 & \text{if } n \text{ is odd},
\end{cases}
$$

(18)

where $C_n^*(q) := \sum_{\pi \in \mathcal{A}_2 n(132)} q^{(2-31) \pi}$. For example,

$$
C_0^*(q) = C_1^*(q) = 1,
$$

$$
C_2^*(q) = 2q,
$$

$$
C_3^*(q) = 3q^2 + 2q^4,
$$

$$
C_4^*(q) = 4q^3 + 6q^5 + 2q^7 + 2q^9.
$$
Other des-cases for $n$ odd

**Theorem (F.-Han-Tang-Zeng)**

*For any $n \geq 1$,*

\[
\sum_{\pi \in S_n(132)} (-1)^{\text{des } \pi} q^{(2-31)} \pi = \sum_{\pi \in S_n(132)} (-1)^{\text{des } \pi} q^{(2-13)} \pi = \begin{cases} 0 & \text{even}, \\ (-q)^{n-1} C_{n-1}^{n-1} \left( q^2 \right) & \text{odd}, \end{cases}
\]

\[
\sum_{\pi \in S_n(213)} (-1)^{\text{des } \pi} q^{(31-2)} \pi = \sum_{\pi \in S_n(213)} (-1)^{\text{des } \pi} q^{(13-2)} \pi = \begin{cases} 0 & \text{even}, \\ (-q)^{n-1} C_{n-1}^{n-1} \left( q^2 \right) & \text{odd}, \end{cases}
\]

\[
\sum_{\pi \in S_n(312)} (-1)^{\text{des } \pi} q^{(2-31)} \pi = \sum_{\pi \in S_n(312)} (-1)^{\text{des } \pi} q^{(2-13)} \pi = \begin{cases} 0 & \text{even}, \\ (-q)^{n-1} C_{n-1}^{n-1} \left( q^2 \right) & \text{odd}. \end{cases}
\]
Other des-cases for $n$ even

Theorem (F.-Han-Tang-Zeng)

\[
\sum_{\pi \in \mathcal{D}_{n}^{\ast} (132)} (-1)^{\text{des} \pi} q^{(2\cdot31) \pi} = \begin{cases} 
(-1)^{n} \hat{C}_{n}^{\frac{1}{2}} (q) & \text{if } n \text{ is even,} \\
0 & \text{if } n \text{ is odd,}
\end{cases} \tag{22}
\]

\[
\sum_{\pi \in \mathcal{D}_{n}^{\ast} (132)} (-q)^{\text{des} \pi} q^{(31\cdot2) \pi} = \begin{cases} 
(-q)^{\frac{n}{2}} \overline{C}_{n}^{\frac{1}{2}} (q) & \text{if } n \text{ is even,} \\
0 & \text{if } n \text{ is odd,}
\end{cases} \tag{23}
\]

\[
\sum_{\pi \in \mathcal{D}_{n}^{\ast} (213)} (-1)^{\text{des} \pi} q^{(13\cdot2) \pi} = \begin{cases} 
(-1)^{\frac{n}{2}} q^{\frac{n-2}{2}} C_{\frac{n-2}{2}} (q^{2}) & \text{if } n \text{ is even,} \\
0 & \text{if } n \text{ is odd,}
\end{cases} \tag{24}
\]

\[
\sum_{\pi \in \mathcal{D}_{n}^{\ast} (213)} (-q)^{\text{des} \pi} q^{(31\cdot2) \pi} = \begin{cases} 
(-1)^{\frac{n}{2}} q^{\frac{3n-4}{2}} C_{\frac{n-2}{2}} (q^{2}) & \text{if } n \text{ is even,} \\
0 & \text{if } n \text{ is odd,}
\end{cases} \tag{25}
\]

where $\hat{C}_{n} (q) := \sum_{\pi \in \mathcal{A}_{2n} (231)} q^{(13\cdot2) \pi}$ and $\overline{C}_{n} (q) := \sum_{\pi \in \mathcal{A}_{2n} (231)} q^{(2\cdot13) \pi}$. 
Further lemmas

**Lemma**

For any \( n \geq 1 \),

\[
\sum_{\pi \in \mathcal{D}_n^*(213)} t^{\text{des } \pi} q^{(13-2) \pi} = t \sum_{\pi \in \mathfrak{S}_{n-1}(213)} t^{\text{des } \pi} q^{(13-2) \pi}.
\]

**Lemma**

For \( n \geq 1 \) and any \( \pi \in \mathfrak{S}_n \),

\[
\text{des } \pi + (31-2) \pi + 1 = \text{fl } \pi + (13-2) \pi,
\]

where \( \text{fl } \pi = \pi(1) \) is the first letter of \( \pi \).

**Lemma**

For any \( n \geq 2 \),

\[
\sum_{\pi \in \mathcal{D}_n^*(132)} t^{\text{des } \pi} q^{\text{fl } \pi} = \sum_{k=1}^{\left\lfloor \frac{n}{2} \right\rfloor} \left( \sum_{\pi \in \mathcal{D}_n^{*,k}(132)} q^{\text{fl } \pi} \right) t^k (1 + t)^{n-2k},
\]

where \( \mathcal{D}_n^{*,k}(132) := \{ \pi \in \mathcal{D}_n^*(132) : \text{dd}^* \pi = 1, \text{des } \pi = k \} \).
\( \overline{C}_n(q) \) and ballot numbers

**Definition**

Let \( \overline{C}_n(q) = q^{-n-1} \sum_{\pi \in A_{2n}(231)} q^{\text{fl} \pi^r} = \sum_{k=0}^{n-1} a_{n,k} q^k \), where

\[
a_{n,k} = \{ \pi \in A_{2n}(231) : \text{fl} \pi^r = n + k + 1 \} \text{ and } a_{n,k} = |a_{n,k}|.
\]
\( \overline{C}_n(q) \) and ballot numbers

Definition

Let \( \overline{C}_n(q) = q^{-n-1} \sum_{\pi \in \mathcal{A}_{2n}(231)} q^{\text{fl} \pi'} := \sum_{k=0}^{n-1} a_{n,k} q^k \), where

\[
a_{n,k} = \{ \pi \in \mathcal{A}_{2n}(231) : \text{fl} \pi' = n + k + 1 \} \text{ and } a_{n,k} = |a_{n,k}|.
\]

Recall that the ballot numbers \( f(n, k) \) satisfy the recurrence relation

\[
f(n, k) = f(n, k - 1) + f(n - 1, k), \quad (n, k \geq 0),
\]

where \( f(n, k) = 0 \) if \( n < k \) and \( f(0, 0) = 1 \), and have the explicit formula

\[
f(n, k) = \frac{n - k + 1}{n + 1} \binom{n + k}{k}, \quad (n \geq k \geq 0).
\]
$C_n(q)$ and ballot numbers

**Definition**

Let $C_n(q) = q^{-n-1} \sum_{\pi \in \mathcal{A}_{2n}(231)} q^{fl \pi^r} := \sum_{k=0}^{n-1} a_{n,k} q^k$, where

$$a_{n,k} = \{\pi \in \mathcal{A}_{2n}(231) : fl \pi^r = n + k + 1\} \text{ and } a_{n,k} = |a_{n,k}|.$$

Recall that the ballot numbers $f(n,k)$ satisfy the recurrence relation

$$f(n,k) = f(n,k-1) + f(n-1,k), \quad (n,k \geq 0), \quad (28)$$

where $f(n,k) = 0$ if $n < k$ and $f(0,0) = 1$, and have the explicit formula

$$f(n,k) = \frac{n-k+1}{n+1} \binom{n+k}{k}, \quad (n \geq k \geq 0).$$

**Proposition**

For $0 \leq k \leq n-1$,

$$a_{n,k} = f(n-1,k) = \frac{n-k}{n} \binom{n-1+k}{k}.$$
Other exc-cases

Theorem (F.-Han-Tang-Zeng)
For any $n \geq 1$,
\[
\sum_{\pi \in S_n(213)} (-1)^{\text{exc } \pi} = \sum_{\pi \in S_n(132)} (-1)^{\text{exc } \pi} = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
(-1)^{\frac{n-1}{2}} C_{\frac{n-1}{2}} & \text{if } n \text{ is odd}, 
\end{cases} 
\tag{29}
\]
\[
\sum_{\pi \in D_n(213)} (-1)^{\text{exc } \pi} = \sum_{\pi \in D_n(132)} (-1)^{\text{exc } \pi} = \begin{cases} 
(-1)^{\frac{n}{2}} C_{\frac{n}{2}} & \text{if } n \text{ is even}, \\
0 & \text{if } n \text{ is odd}. 
\end{cases} 
\tag{30}
\]

Lemma (Elizalde)
For any $n \geq 1$,
\[
\sum_{\pi \in S_n(321)} t^{\text{exc } \pi} y^{\text{fix } \pi} = \sum_{\pi \in S_n(132)} t^{\text{exc } \pi} y^{\text{fix } \pi}.
\]
A conjecture for $\mathcal{D}_n(123)$ with respect to exc

**Conjecture**

Define the polynomials $G_n(t) := \sum_{\pi \in \mathcal{D}_n(123)} t^{\text{exc } \pi}$ for $n \geq 1$. There is a sequence $\{F_n\}_{n \geq 1}$ of positive integers such that

$$\sum_{\pi \in \mathcal{D}_n(123)} (-1)^{\text{exc } \pi} = \begin{cases} (-1)^{\frac{n}{2}} F_{\frac{n}{2}} & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd,} \end{cases}$$

and the polynomials $G_n(t)$ are $\gamma$-positive.
A conjecture for $\mathfrak{S}_n(123)$ with respect to exc

**Conjecture**

Define the polynomials $G_n(t) := \sum_{\pi \in \mathfrak{S}_n(123)} t^{\text{exc } \pi}$ for $n \geq 1$. There is a sequence $\{F_n\}_{n \geq 1}$ of positive integers such that

$$\sum_{\pi \in \mathfrak{S}_n(123)} (-1)^{\text{exc } \pi} = \begin{cases} (-1)^{\frac{n}{2}} F_{\frac{n}{2}} & \text{if } n \text{ is even}, \\ 0 & \text{if } n \text{ is odd}, \end{cases} \quad (31)$$

and the polynomials $G_n(t)$ are $\gamma$-positive.

- We note that neither of the sequences $G_n(1)$ and $F_n$ ($n \geq 1$) is registered in OEIS. The first values are given by $G_n(1) = 0, 1, 2, 7, 20, 66, 218, 725, \ldots$ and $F_n = 1, 7, 58, 545, 5570, \ldots$. 
A conjecture for $\mathfrak{D}_n(123)$ with respect to exc

Conjecture

Define the polynomials $G_n(t) := \sum_{\pi \in \mathfrak{D}_n(123)} t^{\text{exc} \pi}$ for $n \geq 1$. There is a sequence $\{F_n\}_{n \geq 1}$ of positive integers such that

$$
\sum_{\pi \in \mathfrak{D}_n(123)} (-1)^{\text{exc} \pi} = \begin{cases} 
(-1)^{\frac{n}{2}} F_{\frac{n}{2}} & \text{if } n \text{ is even,} \\
0 & \text{if } n \text{ is odd,}
\end{cases}
$$

(31)

and the polynomials $G_n(t)$ are $\gamma$-positive.

- We note that neither of the sequences $G_n(1)$ and $F_n \ (n \geq 1)$ is registered in OEIS. The first values are given by $G_n(1) = 0, 1, 2, 7, 20, 66, 218, 725, \ldots$ and $F_n = 1, 7, 58, 545, 5570, \ldots$.

- The symmetry of $G_n(t)$ follows from the map $\pi \mapsto \pi^{rc}$, which is stable on $\mathfrak{S}_n(123)$ and $\mathfrak{D}_n(123)$, and satisfies $\text{exc}(\pi) = n - \text{exc}(\pi^{rc}) - \text{fix}(\pi)$. Thus, if $\pi \in \mathfrak{D}_n(123)$, we obtain the symmetry.
Two gamma expansions

\[ S_n(t) := \sum_{\pi \in \mathfrak{S}_n(2413, 3142)} t^{\text{des } \pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^S t^k (1 + t)^{n-1-2k}, \quad (32) \]

\[ Y_n(t) := \sum_{\pi \in \mathfrak{S}_n(1342, 2431)} t^{\text{des } \pi} = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^Y t^k (1 + t)^{n-1-2k}, \quad (33) \]

where

\[ \gamma_{n,k}^S = \# \{ \pi \in \mathfrak{S}_n(2413, 3142) : \text{dd}^* \pi = 0, \text{des } \pi = k \}, \]

\[ \gamma_{n,k}^Y = \# \{ \pi \in \mathfrak{S}_n(1342, 2431) : \text{dd}^* \pi = 0, \text{des } \pi = k \}. \]

Two algebraic equations for gamma polynomials due to Lin

For \( * = N, S, Y \), let

\[
\Gamma_*(x, z) := \sum_{n=1}^{\infty} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \gamma_{n,k}^* x^k z^n.
\]

Then we have

\[
\Gamma_S = z + z\Gamma_S + xz\Gamma_S^2 + x\Gamma_S^3,
\]

\[
\Gamma_Y = z + z\Gamma_Y + 2xz\Gamma_N\Gamma_Y + x\Gamma_N^2(\Gamma_Y - z).
\]
Theorem

Let \( r_n := |A_{2n+1}(2413, 3142)| \), \( R(x) := \sum_{n=1}^{\infty} r_n x^n \), then we have

\[
R(x) = x(R(x) + 1)^2 + x(R(x) + 1)^3. \tag{34}
\]

Consequently, \( r_0 = 1 \) and for \( n \geq 1 \),

\[
r_n = \frac{2}{n} \sum_{i=0}^{n-1} 2^i \binom{2n}{i} \binom{n}{i+1}. \tag{35}
\]
Theorem
Let $r_n := |\mathcal{A}_{2n+1}(2413, 3142)|$, $R(x) := \sum_{n=1}^{\infty} r_n x^n$, then we have

\[ R(x) = x(R(x) + 1)^2 + x(R(x) + 1)^3. \]  \hspace{1cm} (34)

Consequently, $r_0 = 1$ and for $n \geq 1$,

\[ r_n = \frac{2}{n} \sum_{i=0}^{n-1} 2^i \binom{2n}{i} \left( \binom{n}{i+1} \right). \]  \hspace{1cm} (35)

Theorem
For $n \geq 1$, let $t_n := |\mathcal{A}_{2n}(2413, 3142)|$, $n \geq 1$, $T(x) := \sum_{n=1}^{\infty} t_n x^n$, then we have

\[ \frac{1}{2} R(x) = \frac{1}{2} R(x) \cdot T(x) + T(x). \]  \hspace{1cm} (36)

Consequently, $t_1 = 1$ and for $n \geq 2$,

\[ t_n = \frac{4}{n-1} \sum_{i=0}^{n-2} 2^i \binom{2n-1}{i} \left( \binom{n-1}{i+1} \right). \]  \hspace{1cm} (37)
Theorem

Let \( u_n := |A_{2n+1}(1342, 2431)| \), \( U(x) := \sum_{n=0}^{\infty} u_n x^n \), then we have

\[
U(x) = \frac{\sqrt{1 - 4x}}{\sqrt{1 - 4x - 2x}} = \frac{1}{1 - \frac{2x}{1 - \frac{2x}{1 - \frac{x}{1 - \frac{x}{\ddots}}}}}. \tag{38}
\]

(1342, 2431)-avoiding alternating permutations

Remark

The sequence \( \{u_n\}_{n \geq 0} \) is also on OEIS (see oeis:A084868), but there is no simple sum formula for \( u_n \). A multiple sum formula can be derived as

\[
u_n = \sum_{m=0}^{n} \sum_{k_1} \cdots \sum_{k_m} \cdots \sum_{k_m} \prod_{i=1}^{m} (2k_i - k_i). \tag{39}\]
Theorem
Let \( u_n := |\mathcal{A}_{2n+1}(1342, 2431)| \), \( U(x) := \sum_{n=0}^{\infty} u_n x^n \), then we have

\[
U(x) = \frac{\sqrt{1 - 4x}}{\sqrt{1 - 4x - 2x}} = \frac{1}{1 - \frac{2x}{1 - \frac{x}{1 - \frac{x}{\ddots}}}}.
\]

Remark
The sequence \( \{u_n\}_{n \geq 0} \) is also on OEIS (see oeis:A084868), but there is no simple sum formula for \( u_n \). A multiple sum formula can be derived as

\[
u_n = \sum_{m=0}^{n} 2^m \sum_{k_1 + \cdots + k_m = n-m} \prod_{i=1}^{m} \binom{2k_i}{k_i}.
\]

Our result above seems to be the first combinatorial interpretation for \( u_n \).
\((-1)\)-phenomenon for free

**Theorem**

*For any* \(n \geq 1\), *there holds*

\[
S_n(-1) = \sum_{\pi \in \mathcal{S}_n(2413, 3142)} (-1)^{\text{des} \pi} = \begin{cases} 0 & \text{if } n \text{ is even}, \\ (-1)^{\frac{n-1}{2}} \frac{r_{n-1}}{2} & \text{if } n \text{ is odd}. \end{cases} \tag{40}
\]

**Theorem**

*For any* \(n \geq 1\), *there holds*

\[
Y_n(-1) = \sum_{\pi \in \mathcal{S}_n(1342, 2413)} (-1)^{\text{des} \pi} = \begin{cases} 0 & \text{if } n \text{ is even}, \\ (-1)^{\frac{n-1}{2}} \frac{u_{n-1}}{2} & \text{if } n \text{ is odd}. \end{cases} \tag{41}
\]

\*Remark*

Both \(S_n(2413, 3142)\) and \(S_n(1342, 2413)\) exhibit only \((-1)\)-phenomenon (not strong). This should not come as a surprise in view of the reversal relations \((2413) \stackrel{r}{\to} 3142\), \((1342) \stackrel{r}{\to} 2431\), and the fact that the definition of coderangements is incompatible with the reverse map.
Theorem
For any $n \geq 1$, there holds
\[
S_n(-1) = \sum_{\pi \in \mathcal{S}_n(2413,3142)} (-1)^{\text{des} \, \pi} = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
(-1)^{n-1} \frac{r_{n-1}}{2} & \text{if } n \text{ is odd}.
\end{cases}
\] (40)

Theorem
For any $n \geq 1$, there holds
\[
Y_n(-1) = \sum_{\pi \in \mathcal{S}_n(1342,2413)} (-1)^{\text{des} \, \pi} = \begin{cases} 
0 & \text{if } n \text{ is even}, \\
(-1)^{n-1} \frac{u_{n-1}}{2} & \text{if } n \text{ is odd}.
\end{cases}
\] (41)

Remark
Both $\mathcal{S}_n(2413, 3142)$ and $\mathcal{S}_n(1342, 2431)$ exhibit only $(-1)$-phenomenon (not strong). This should not come as a surprise in view of the reversal relations between the two patterns that we avoid, namely $(2413)^r = 3142, (1342)^r = 2431$, and the fact that the definition of coderangements is incompatible with the reverse map.
Definition

The statistic \( \text{MAD} \), the number of fixed points, weak excedances, the inversion number, crossing number and inverse crossing number, nesting number and inverse nesting number of \( \pi \in \mathfrak{S}_n \) are defined by

\[
\begin{align*}
\text{MAD} \, \pi &= \text{des} \, \pi + (31-2) \, \pi + 2(2-31) \, \pi, \\
\text{fix} \, \pi &= \sum_{1 \leq i \leq n} \chi\left( \pi(i) = i \right), \\
\text{wex} \, \pi &= \text{exc} \, \pi + \text{fix} \, \pi, \\
\text{inv} \, \pi &= \sum_{1 \leq i < j \leq n} \chi\left( \pi(i) > \pi(j) \right), \\
\text{cros} \, \pi &= \#\{(i,j) : i < j \leq \pi(i) < \pi(j) \quad \text{or} \quad \pi(i) < \pi(j) < i < j\}, \\
\text{nest} \, \pi &= \#\{(i,j) : i < j \leq \pi(j) < \pi(i) \quad \text{or} \quad \pi(j) < \pi(i) < i < j\}, \\
\text{icr} \, \pi &= \text{cros} \, \pi^{-1}, \\
\text{ine} \, \pi &= \text{nest} \, \pi^{-1},
\end{align*}
\]

where \( \chi(A) = 1 \) if \( A \) is true and 0 otherwise.
The Clarke-Steingrímsson-Zeng bijection linking des based statistics with exc based ones is the composition of two famous bijections. It implies the following equidistribution result.

**Lemma (Shin-Zeng)**

*For* \( n \geq 1 \), *there is a bijection* \( \Phi \) *on* \( S_n \) *such that*

\[
(\text{des}, \text{fmax}, 31-2, 2-31, \text{MAD}) \pi = (\text{exc}, \text{fix}, \text{icr}, \text{ine}, \text{inv}) \Phi(\pi) \quad \text{for all } \pi \in S_n.
\]
Lemma (Shin-Zeng)

Let

\[ A_n(x, y, q, p, s) := \sum_{\pi \in \mathcal{S}_n} x^{\text{des} \pi} y^{\text{fmax} \pi} q^{(31-2) \pi} p^{(2-31) \pi} s^{\text{MAD} \pi} \]

\[ = \sum_{\pi \in \mathcal{S}_n} x^{\text{exc} \pi} y^{\text{fix} \pi} q^{\text{icr} \pi} p^{\text{ine} \pi} s^{\text{inv} \pi}. \]  

(42)

Then we have

\[
1 + \sum_{n=1}^{\infty} A_n(x, y, q, p, s)t^n = \frac{1}{1 - b_0 t - \frac{a_0 c_1 t^2}{1 - b_1 t - \frac{a_1 c_2 t^2}{1 - b_2 t - \ddots}}},
\]

(43)

where, for \( h \geq 0 \),

\[
a_h = s^{2h+1}[h + 1]_{q,ps}, \quad b_h = yp^h s^{2h} + (x + q)s^h[h]_{q,ps},
\]

and

\[
c_h = x[h]_{q,ps}, \quad [h]_{u,v} := (u^h - v^h)/(u - v).
\]
A key observation

Lemma
For any $n \geq 1$,

\[
\mathcal{S}_n(2-13) = \mathcal{S}_n(213), \quad \mathcal{S}_n(31-2) = \mathcal{S}_n(312), \\
\mathcal{S}_n(13-2) = \mathcal{S}_n(132), \quad \mathcal{S}_n(2-31) = \mathcal{S}_n(231).
\]

Moreover, the mapping $\Phi$ has the property that $\Phi(\mathcal{S}_n(231)) = \mathcal{S}_n(321)$. Consequently, $\pi \in \mathcal{S}_n(321)$ if and only if nest $\pi = \text{ine} \pi = 0$. 
A key observation

Lemma

For any $n \geq 1$,

\[
\mathfrak{S}_n(2-13) = \mathfrak{S}_n(213), \; \mathfrak{S}_n(31-2) = \mathfrak{S}_n(312),
\]

\[
\mathfrak{S}_n(13-2) = \mathfrak{S}_n(132), \; \mathfrak{S}_n(2-31) = \mathfrak{S}_n(231).
\]

Moreover, the mapping $\Phi$ has the property that $\Phi(\mathfrak{S}_n(231)) = \mathfrak{S}_n(321)$.

Consequently, $\pi \in \mathfrak{S}_n(321)$ if and only if $\text{nest } \pi = \text{ine } \pi = 0$.

By the above observation, the special $p = 0$, $q = 1$ case yields a result of Cheng et al.

Lemma (Cheng et al.)

\[
\sum_{n=0}^{\infty} \left( \sum_{\pi \in \mathfrak{S}_n(321)} q^{\text{inv } \pi} t^{\text{exc } \pi} y^{\text{fix } \pi} \right) z^n = \frac{1}{1 - yz - t q z^2 - t q^3 z^2 - t q^5 z^2 - \cdots}
\]
Modified Foata-Strehl action

Let $\pi \in \mathcal{S}_n$, for any $x \in [n]$, the $x$-factorization of $\pi$ reads $\pi = w_1 w_2 x w_3 w_4$, where $w_2$ (resp. $w_3$) is the maximal contiguous subword immediately to the left (resp. right) of $x$ whose letters are all larger than $x$. Following Foata and Strehl we define the action $\varphi_x$ by

$$\varphi_x(\pi) = w_1 w_3 x w_2 w_4.$$  

Brändén modified $\varphi_x$ to be

$$\varphi'_x(\pi) := \begin{cases} 
\varphi_x(\pi), & \text{if } x \text{ is a double ascent or double descent of } \pi; \\
\pi, & \text{if } x \text{ is a valley or a peak of } \pi.
\end{cases}$$

For any subset $S \subseteq [n]$ we define the function $\varphi'_S : \mathcal{S}_n \to \mathcal{S}_n$ by

$$\varphi'_S(\pi) = \prod_{x \in S} \varphi'_x(\pi).$$

The group $\mathbb{Z}_2^n$ acts on $\mathcal{S}_n$ via the functions $\varphi'_S$, $S \subseteq [n]$. 
Fig.: MFS-actions on 596137428
A new variant of MFS

Lemma

For any $n \geq 2$,

$$
\sum_{\pi \in D_n^*(132)} t^{\text{des} \pi} q^{\text{fl} \pi} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left( \sum_{\pi \in \overline{D}_{n,k}^*(132)} q^{\text{fl} \pi} \right) t^k (1 + t)^{n-2k}, \tag{44}
$$

where $\overline{D}_{n,k}^*(132) := \{\pi \in D_n^*(132) : \text{dd}^* \pi = 1, \text{des} \pi = k\}$. 

ϕx(π) := {π, if x is a valley, a peak, or a left-to-right maximum of π; ϕx(π), otherwise.
A new variant of MFS

Lemma
For any \( n \geq 2 \),

\[
\sum_{\pi \in \mathcal{D}_n^* (132)} t^{\text{des } \pi} q^{\text{fl } \pi} = \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \left( \sum_{\pi \in \mathcal{D}_n^* (132), k} q^{\text{fl } \pi} \right) t^k (1 + t)^{n-2k}, \tag{44}
\]

where \( \mathcal{D}_n^* (132), k \) := \( \{ \pi \in \mathcal{D}_n^* (132) : \text{dd}^* \pi = 1, \text{des } \pi = k \} \).

\( \overline{\varphi}_x (\pi) := \begin{cases} 
\pi, & \text{if } x \text{ is a valley, a peak, or a left-to-right maximum of } \pi; \\
\varphi_x (\pi), & \text{otherwise.}
\end{cases} \)
Definition (Shareshian and Wachs)

Let $\pi = \pi(1)\pi(2) \cdots \pi(n)$ be a permutation of $S_n$ and $\pi(0) = \pi(n + 1) = 0$. An admissible inversion of $\pi$ is an inversion pair $(\pi(i), \pi(j))$, i.e., $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$, satisfying either of the following conditions:

- $\pi(j) < \pi(j + 1)$ or
- there is some $l$ such that $i < l < j$ and $\pi(j) > \pi(l)$.

Definition (Lin and Zeng)

Let $\pi = \pi(1)\pi(2) \cdots \pi(n)$ be a permutation of $S_n$ and $\pi(0) = \pi(n + 1) = n + 1$. A star admissible inversion of $\pi$ is a pair $(\pi(i), \pi(j))$ such that $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$ and satisfies either of the following conditions:

- $\pi(i - 1) < \pi(i)$ or
- there is some $l$ such that $i < l < j$ and $\pi(i) < \pi(l)$. 

Definition (Shareshian and Wachs)
Let $\pi = \pi(1)\pi(2) \cdots \pi(n)$ be a permutation of $\mathfrak{S}_n$ and $\pi(0) = \pi(n+1) = 0$. An admissible inversion of $\pi$ is an inversion pair $(\pi(i), \pi(j))$, i.e., $1 \leq i < j \leq n$ and $\pi(i) > \pi(j)$, satisfying either of the following conditions:
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Definition (Lin and Zeng)
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- $\pi(i-1) < \pi(i)$ or
- there is some $l$ such that $i < l < j$ and $\pi(i) < \pi(l)$.

Remark
The initial condition $\pi(0) = \pi(n+1) = 0$, the definition of adi, and the construction of the MSF-action, are all dual to those used by Lin-Zeng. When patterns $\{231, 132, 2\,\text{31}, 1\,\text{32}\}$ are concerned, we use Lin-Zeng’s version, while for patterns $\{213, 312, 2\,\text{13}, 31\,\text{2}\}$ we use our current version.
Lemmas for main result II

Lemma
\[ \text{adi } \pi = (2\text{-}13) \pi, \text{ if } \pi \in \mathcal{S}_n(312), \]
\[ \text{adi}^* \pi = (13\text{-}2) \pi, \text{ if } \pi \in \mathcal{S}_n(231). \]

Lemma
Let \( \pi \in \mathcal{G}_n. \) For each \( x \in [n], \) we have \( \text{adi } \pi = \text{adi } \varphi'_x(\pi). \)

Lemma
The statistics \((2\text{-}31), (13\text{-}2), (2\text{-}13)\) and \((31\text{-}2)\) are constant on any orbit under the MFS-action.

Lemma
The MFS-action preserves the pattern \(213, 312, 132\) and \(231, \) i.e., the map \( \varphi'_S \)
is closed on the subsets \( \mathcal{S}_n(\tau), \) for \( \tau = 213, 312, 132, 231. \)
Lemma (Shin-Zeng)

The following four polynomials are equal

$$
\sum_{\pi \in \mathcal{S}_n} t^\text{des} \pi p^{(2-13)} \pi q^{(31-2)} \pi = \sum_{\pi \in \mathcal{S}_n} t^\text{des} \pi p^{(31-2)} \pi q^{(2-13)} \pi \\
= \sum_{\pi \in \mathcal{S}_n} t^\text{des} \pi p^{(2-31)} \pi q^{(31-2)} \pi = \sum_{\pi \in \mathcal{S}_n} t^\text{des} \pi p^{(31-2)} \pi q^{(2-31)} \pi.
$$
Lemma (Shin-Zeng)

The following four polynomials are equal

\[
\sum_{\pi \in S_n} t^{\text{des}} \pi \, \rho^{(2-13)} \pi \, q^{(31-2)} \pi = \sum_{\pi \in S_n} t^{\text{des}} \pi \, \rho^{(31-2)} \pi \, q^{(2-13)} \pi \\
= \sum_{\pi \in S_n} t^{\text{des}} \pi \, \rho^{(2-31)} \pi \, q^{(31-2)} \pi = \sum_{\pi \in S_n} t^{\text{des}} \pi \, \rho^{(31-2)} \pi \, q^{(2-31)} \pi.
\]

\[
\pi \mapsto \pi^{-1} := \pi^{-1}(1)\pi^{-1}(2) \cdots \pi^{-1}(n),
\]
\[
\pi \mapsto \pi^r := \pi(n) \cdots \pi(2)\pi(1),
\]
\[
\pi \mapsto \pi^c := (n + 1 - \pi(1))(n + 1 - \pi(2)) \cdots (n + 1 - \pi(n)),
\]
\[
\pi \mapsto \pi^{rc} := (n + 1 - \pi(n)) \cdots (n + 1 - \pi(2))(n + 1 - \pi(1)).
\]
Final remarks

- Another direction to extend the results presented here is to place $\mathfrak{S}_n$ in the broader context of Coxeter groups, and consider the so-called types B and D Narayana polynomials (see the survey of Athanasiadis). This approach was shown fruitful for permutations in a recent work of Eu, Fu, Hsu and Liao.


Thanks for listening!
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- It would be appealing to establish a multivariate generating function (in the spirit of Shin-Zeng’s Lemma) that specializes to the $(2413, 3142)$-avoiding permutations or $(1342, 2413)$-avoiding permutations, and consequently giving us $q$-analogues of (40) or (41).
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