Laguerre unitary ensemble perturbed by a pole in the potential

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Outline

- Background: Laguerre unitary ensemble
- Laguerre unitary ensemble perturbed by a pole
- Main results
- Proof
Laguerre unitary ensemble (LUE)

LUE is the set of all \( n \times n \) positive definite Hermitian matrices with the following probability measure

\[
\frac{1}{C_n} (\det M)^\alpha e^{-\text{Tr}M} dM, \quad \alpha > -1
\]

where \( C_n \) is a constant, \( \text{Tr}M \) is the trace of \( M \) and
\[
dM = \prod_{i=1}^n dM_{ii} \prod_{i=1}^{n-1} \prod_{j=i+1}^n d\text{Re}M_{ij}d\text{Im}M_{ij}.
\]

The density is described by the Laguerre measure and invariant under unitary conjugation of \( M \), hence the name of LUE.
The joint probability density function for the eigenvalues of LUE is given by

$$
\rho_n(\lambda_1, \lambda_2...\lambda_n) = \frac{1}{Z_n} \prod_{i=1}^{n} w(\lambda_i) \prod_{i<j} |\lambda_i - \lambda_j|^2,
$$

with \( w(x) = x^\alpha e^{-x}, x > 0 \).

Partition function

$$
Z_n = n! D_n[w(x)],
$$

where the Hankel determinants

$$
D_n[w(x)] = \text{det}(\mu_{j+k})_{j,k=0}^{n-1}, \quad \mu_i = \int_0^\infty x^i w(x) dx.
$$
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Correlation kernel (Christoffel-Darboux kernel)

\[ K_n(x, y) = \sqrt{w(x)w(y)} \sum_{k=0}^{n-1} P_k(x)P_k(y), \]

\[ \rho_n(x_1, x_2 \ldots x_n) = \frac{1}{n!} \det[K_n(x_i, x_j)]_{i,j=1}^n. \]

Density of eigenvalues

\[ \psi_n(x) = \frac{1}{n} K_n(x, x). \]

Statistics of eigenvalues for \( n \) large can be obtained by studying the asymptotics of Christoffel-Darboux kernel as \( n \to \infty \).
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Global regime: limiting density of eigenvalues of LUE

\[ \psi(x) = \lim_{n \to \infty} 4K_n(4nx, 4nx) = \frac{2}{\pi} \sqrt{\frac{1-x}{x}}, \quad 0 < x < 1. \]

![Figure: Marčenko-Pastur law](image)

Local limiting eigenvalues behavior in the bulk

\[ \lim_{n \to \infty} 4\psi(x)K_n\left(4n\left(x + \frac{u}{n\psi(x)}\right), 4n\left(x + \frac{v}{n\psi(x)}\right)\right) = \frac{\sin \pi(u - v)}{u - v}, \]

for \( x \in (0, 1). \)
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for \( x \in (0, 1) \).
Local limiting eigenvalues behavior at the soft edge

\[
\lim_{n \to \infty} 2^{4/3} n^{1/3} K_n\left(4n\left(1 + \frac{u}{(2n)^{2/3}}\right), 4n\left(1 + \frac{v}{(2n)^{2/3}}\right)\right) = \mathbb{A}(u, v),
\]

with \( \mathbb{A}(u, v) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}(v)\text{Ai}'(u)}{u - v} \).

At the hard edge

\[
\lim_{n \to \infty} \frac{1}{4n} K_n\left(\frac{u}{4n}, \frac{v}{4n}\right) = \frac{J_\alpha(\sqrt{u})\sqrt{v}J'_\alpha(\sqrt{v}) - J'_\alpha(\sqrt{u})\sqrt{u}J_\alpha(\sqrt{v})}{2(u - v)}.
\]

Universality: The limiting kernels depend only on general property of the density function, but not on the the specific model of random matrices.
Local limiting eigenvalues behavior at the soft edge

\[
\lim_{n \to \infty} 2^{4/3} n^{1/3} K_n(4n(1 + \frac{u}{(2n)^{2/3}}), 4n(1 + \frac{v}{(2n)^{2/3}})) = A(u, v),
\]

with \(A(u, v) = \frac{\text{Ai}(u)\text{Ai}'(v) - \text{Ai}(v)\text{Ai}'(u)}{u-v}\).

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with \( A(u, v) = \frac{Ai(u)Ai'(v) - Ai(v)Ai'(u)}{u-v} \).

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Universality: The limiting kernels depend only on general property of the density function, but not on the the specific model of random matrices.
Tracy-Widom distribution

- Gap probability near the soft edge \( \text{Tracy \& Widom ('94)} \)
  \[
  \Pr \left( 2(2n)^{1/3}(\lambda_{\text{max}} - 4n) < s \right) \rightarrow F(s).
  \]

- Via Fredholm determinant:
  \[
  F(s) = \det(I - A_s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \int_{(s, \infty)^k} \det[A(x_i, x_j)]_{i,j=1}^{k} \, dx_1 \ldots dx_k
  \]
  where \( A_s : L^2(s, +\infty) \rightarrow L^2(s, +\infty) \) with Airy kernel.

- Tracy-Widom distribution:
  \[
  F(s) = \exp \left( - \int_{s}^{+\infty} (x - s)y^2(x) \, dx \right)
  \]
  where \( y(x) \) is the the Hastings-Mcleod solution to \( P_2 \) equation
  \[
  \begin{cases}
  y'' = xy + 2y^3 \\
  y(x) \sim \text{Ai}(x), \quad x \to \infty.
  \end{cases}
  \]
As $s \to -\infty$, we have the large gap asymptotic

$$\ln F(s) = -\frac{1}{12}|s|^3 - \frac{1}{8}\ln |s| + \frac{1}{24}\ln 2 + \zeta'(-1) + o(1).$$

- The constant was first conjectured by Tracy & Widom ('94).
- The constant was proved by Deift, Its and Krasovsky ('11).
- University: Tracy-Widom distribution holds for general unitary random matrix ensembles.
- It also appears in a large family of combinatorial and statistical models: the length of the longest increasing subsequence of a random permutation. Baik, Deift & Johansson ('99)
Gap probability near the hard edge \( \text{Tracy & Widom ('94)} \)

\[
\lim_{n \to \infty} P_n(\lambda_{\min} > \frac{s}{4n}) = \exp\left(-\int_0^s \frac{\sigma(x)}{x} dx\right)
\]

where \( \sigma \) satisfies the Jimbo-Miwa-Okamoto \( \sigma \)-form of the Painlevé III equation

\[
(s\sigma'')^2 + \sigma'(\sigma - s\sigma')(4\sigma' - 1) - \alpha^2\sigma'^2 = 0
\]

with the boundary conditions

\[
\sigma(s) \sim \frac{1}{4^{\alpha+1}\Gamma(1+\alpha)\Gamma(2+\alpha)}s^{1+\alpha}, \ s \to 0.
\]
As $s \to +\infty$, we have the large gap asymptotic

$$\ln F(s) = -\frac{1}{4}s + \alpha s^{1/2} - \frac{\alpha^2}{4} \ln s + \ln \left( \frac{G(1 + \alpha)}{(2\pi)^{\alpha/2}} \right) + o(1)$$

$G(z)$ is the Barnes G-function:

$$G(z + 1) = G(z)\Gamma(z), \quad G(1) = 1.$$

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The constant was proved by Deift, Its and & Vasilevska ('11).
Laguerre unitary ensembles Perturbed by a pole:

\[ \frac{1}{Z_n} \left( \det M \right)\alpha e^{-\text{Tr} V(M)} dM, \quad \alpha > -1 \]

with \( V(x) = x + t/x, x > 0, t > 0 \).

Applications:

- Wigner time delay (Brouwer, Franhm, Beenakker '97, Texier, Majumdar '13)
- Statistics of zeros of Riemann zeta function (Berry and Schukla '08)
- Quantum field theory at finite temperature (Chen and Its '08)
**Theorem (Chen and Its ’08)**

Let \( w(x, t) = x^\alpha e^{-x-t/x}, x > 0, t > 0 \) and \( n \) be fixed,

\[
\pi_n(x) = \pi_{n+1}(x) + \alpha_n(t)\pi_n(x) + \beta_n(t)\pi_{n-1}(x),
\]

then \( y_n = \alpha_n - (2n + 1 + \alpha) \) satisfies the Painlevé III’

\[
y''_n(t) = \frac{[y'_n(t)]^2}{y_n} - \frac{y'_n(t)}{t} + (2n + 1 + \alpha)\frac{y_n(t)^2}{t^2} + \frac{y_n(t)^3}{t^2} + \frac{\alpha}{t} - \frac{1}{y_n(t)},
\]

and \( H_n(t) = t \frac{d}{dt} \ln D_n[w] \) satisfies \( \sigma \)-form of the Painlevé III,

\[
(tH''_n)^2 + 4(tH'_n - H_n)H'_n(H'_n - 1) - (\alpha H'_n - n)^2 = 0.
\]

**Large \( n \) asymptotics of the correlation kernel, the Hankel determinant and the recurrences?**
Theorem (Chen and Its ’08)

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y_n''(t) = \frac{[y_n'(t)]^2}{y_n} - \frac{y_n'(t)}{t} + (2n + 1 + \alpha) \frac{y_n(t)^2}{t^2} + \frac{y_n(t)^3}{t^2} + \frac{\alpha}{t} - \frac{1}{y_n(t)},
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and \( H_n(t) = t \frac{d}{dt} \ln D_n[w] \) satisfies \( \sigma \)- form of the Painlevé III,

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(tH_n'')^2 + 4(tH_n' - H_n)H_n'(H_n' - 1) - (\alpha H_n' - n)^2 = 0.
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Large \( n \) asymptotics of the correlation kernel, the Hankel determinant and the recurrences?
Global regime (limiting density of eigenvalues)

- For $t = 0$, we have the (Marcenko-Pastur law)

$$
\psi(x; t = 0) = \lim_{n \to \infty} 4K_n(4nx, 4nx) = \frac{2}{\pi} \sqrt{\frac{1 - x}{x}}, 0 < x < 1.
$$

- For $t > 0$, the equilibrium-measure with $V(x) = \frac{1}{n}(x + t/x)$

$$
b_n\psi(b_n x; t > 0) dx = \frac{b_n \sqrt{(x - a_n)(1 - x)(x + c_n)}}{2n\pi x^2} dx, x \in (a_n, 1)
$$

with $b_n \sim 4n$, $a_n \sim \frac{1}{4}(t/4n^2)^{2/3}$ and $c_n \sim \frac{1}{2}(t/4n^2)^{2/3}$.

- New limiting kernel in the double scaling limit where $n \to \infty$ and $t \to 0^+$ with proper speed?
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with $b_n \sim 4n$, $a_n \sim \frac{1}{4}(t/4n^2)^{2/3}$ and $c_n \sim \frac{1}{2}(t/4n^2)^{2/3}$.

New limiting kernel in the double scaling limit where $n \to \infty$ and $t \to 0^+$ with proper speed?
Lax pair for Painlevé III

\[ \Psi_\zeta(\zeta, s) = \left( A_0(s) + \frac{A_1(s)}{\zeta} + \frac{A_2(s)}{\zeta^2} \right) \Psi(\zeta, s) \]

\[ \Psi_s(\zeta, s) = \frac{B_1(s)}{\zeta} \Psi(\zeta, s) \]

with

\[ A_0(s) = \begin{pmatrix} 0 & 0 \\ i/2 & 0 \end{pmatrix}, \quad A_1(s) = \begin{pmatrix} -\frac{1}{4} + \frac{1}{2}r(s) & -i \frac{1}{2} \\ -iq(s) & \frac{1}{4} - \frac{1}{2}r(s) \end{pmatrix}, \]

\[ A_2(s) = -sB_1(s) \]

and

\[ B_1(s) = \begin{pmatrix} q'(s) & -ir'(s) \\ ip'(s) & -q'(s) \end{pmatrix}. \]
By the compatibility condition $\Psi_{\zeta s}(\zeta, s) = \Psi_{s\zeta}(\zeta, s)$, we have

$$v'' - \frac{v'^2}{v} + \frac{v'}{s} - \frac{v^2}{s^2} - \frac{\alpha}{s} + \frac{1}{v} = 0, \quad v(s) = sr'(s).$$

Let $y(s) = -4v(s^2)/s = -2 \frac{d}{ds} r(s^2)$, we get the Painlevé III,

$$y'' - \frac{y'^2}{y} + \frac{y'}{s} + \frac{y^2}{s} + \frac{16\alpha}{s} + \frac{64}{y} = 0.$$
Proposition. (Dai, Xu and Zhao ’14) For $\alpha > -1$, there exist analytic solutions $r(s)$, $v(s)$ on $(0, \infty)$ with the boundary behaviors

$$r(s) = \frac{1 - 4\alpha^2}{8} + O(s) + O(s^{1+\alpha}), \quad s \to 0;$$

$$r(s) = \frac{3}{2} s^\frac{2}{3} - \alpha s^\frac{1}{3} + O(1), \quad s \to \infty;$$

$$v(s) = O(s) + O(s^{1+\alpha}), \quad s \to 0;$$

$$v(s) = s^\frac{2}{3} - \frac{\alpha}{3} s^\frac{1}{3} + O(1), \quad s \to \infty.$$
Ψ- Kernel:

\[ K_\psi(u, v; s) = \frac{\psi_1(-v, s)\psi_2(-u, s) - \psi_1(-u, s)\psi_2(-v, s)}{2\pi i(u - v)} \]

where

\[ \frac{\partial}{\partial \zeta} \begin{pmatrix} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{pmatrix} = \begin{pmatrix} A_0(s) + \frac{A_1(s)}{\zeta} + \frac{A_2(s)}{\zeta^2} \end{pmatrix} \begin{pmatrix} \psi_1(\zeta, s) \\ \psi_2(\zeta, s) \end{pmatrix}. \]
Theorem (Dai, Xu and Zhao ('14)).

Let $t \to 0$ and $n \to \infty$ in the way such that $2nt \to s$, we have the double scaling limit

$$\lim_{n \to \infty} \frac{1}{4n} K_n \left( \frac{u}{4n}, \frac{v}{4n}; t \right) = K_\Psi (u, v; s)$$

uniformly for $u$, $v$ and $s$ in compact subsets of $(0, \infty)$. 

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Transitions from Bessel kernel to Airy kernel:

- The $\Psi$-kernel is approximated by the Bessel kernel as $s \to 0^+$

$$K_\Psi(u, v) = J_\alpha(u, v) + O(s).$$

- The $\Psi$-kernel is approximated by the Airy kernel as $s \to \infty$

$$s^{4/9} c^{-1} K_\Psi \left( s^{2/3} \left( 1 - \frac{u}{c s^{2/9}} \right), s^{2/3} \left( 1 - \frac{v}{c s^{2/9}} \right) \right) = A(u, v)$$
$$+ O \left( s^{-2/9} \right).$$

- Let $a_n = 2^{-4/3} n^{-1/3} t^{2/3}$, $s = 2nt$ and $t > 0$,

$$\lim_{n \to \infty} \frac{a_n}{c s^{2/9}} K_n \left( a_n \left( 1 - \frac{u}{c s^{2/9}} \right), a_n \left( 1 - \frac{v}{c s^{2/9}} \right); t \right) = A(u, v).$$
Theorem. (Dai, Xu and Zhao '14)

Let $w(x) = x^\alpha e^{-x - \frac{t}{x}}$, $t > 0$, $\alpha > -1$, then as $n \to \infty$ and $t \to 0$ such that $2nt \to s$, we have the asymptotic expansion

$$D_n[w; t] = D_n[w; 0] \exp\left\{ \left[ 1 + O\left( \frac{1}{n} \right) \right] \int_0^{2nt} \frac{1 - 4\alpha^2 - 8r(\xi)}{16\xi} d\xi \right\}$$

where $D_n[w; 0] = \frac{1}{n!} \prod_{j=1}^{n} j! \Gamma(j + \alpha)$. 
Asymptotics of the recurrence coefficients:

- Three-term recurrence relation

\[ x \pi_n(x) = \pi_{n+1}(x) + \alpha_n(t)\pi_n(x) + \beta_n(t)\pi_{n-1}(x). \]

- Theorem. (Dai, Xu and Zhao '14)

We have the asymptotic expansion for the recurrence coefficients

\[ \alpha_n(t) = 2n + \alpha + 1 + \frac{v(2nt)}{2n} \left( 1 + O\left(n^{-1/3}\right) \right), \]

\[ \beta_n(t) = n^2 + \alpha n + \frac{4\alpha^2 - 1 + 8r(2nt) - 8v(2nt)}{16} \left( 1 + O\left(n^{-1/3}\right) \right), \]

uniformly for \( t \in (0, d] \), \( d > 0 \) fixed.
Corollary As $t \to d > 0$ and $n \to \infty$, then $s = 2nt \to \infty$

$$\alpha_n(t) = 2n + \alpha + 1 + \frac{t^{2/3}}{2^{1/3}} \frac{1}{n^{1/3}} + O \left( \frac{1}{n^{2/3}} \right),$$

$$\beta_n(t) = n^2 + \alpha n + 2^{-4/3} t^{2/3} n^{2/3} + O \left( n^{1/3} \right).$$
Perturbed Gaussian unitary ensemble (Brightmore, Mezzadri and Mo ‘14)

\[
\frac{1}{Z_n} e^{-n \text{Tr} V(M)} dM
\]

\[V(x) = \frac{x^2}{2} + \frac{t_1^2}{x^2} - \frac{t_2}{x}, \quad t_1 \in R \setminus \{0\}, \quad t_2 \geq 0.
\]

- Asymptotics of the partition functions: PDE.
- For \( t_2 = 0 \), the PDE is reduced to Painlevé III.
Higher order singularity and Painlevé III hierarchy (Atkin, Claeys and Mezzadri ('15))

\[ \frac{1}{Z_n}(\det M)^\alpha e^{-n\text{Tr} V_k(M)}dM \]

with

\[ V_k = V(x) + \left(\frac{t}{x}\right)^k, \quad x > 0. \]

• \( V \) is such that the E-measure is

\[ \psi_V(x) = \sqrt{\frac{b-x}{x}}h(x), \quad x \in (0, b). \]

• Painlevé III hierarchy limit kernel.

• Painlevé III hierarchy asymptotics of the partition functions for the perturbed Laguerre weight

\[ V_k = x + \left(\frac{t}{x}\right)^k. \]
The distribution of the smallest eigenvalue in PLUE can be expressed in terms of the Fredholm determinant of Painlevé III kernels,

\[
\lim_{n \to \infty} \text{Prob} [\lambda_{\text{min}} > x/cn)] = \det[I - K_{x}^\text{PIII}],
\]

where \( K_{x}^\text{PIII} \) is the integral operator acting on \( L^2(0, x) \) with the Painlevé III kernel \( K_{\psi}(u, v; s) \).

Question

1. Is possible to find analogous expressions of the Tracy-Widom formulas for the determinants of the Painlevé III kernels?
2. Large gap asymptotics for the determinants?
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**Question**

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2. Large gap asymptotics for the determinants?
Theorem \text{Dai, Xu & Zhang(’17)}

For $\alpha > -1$, $x > 0$ and $s > 0$, let $K_{x}^{\text{PIII}}$ be the integral operator with kernel $K_{\psi}(u, v; s)$ acting on the function space $L^2((0, x))$, we have the large gap asymptotics $x \to +\infty$

\[
\ln \det [I - K_{x}^{\text{PIII}}] = -\frac{1}{4}x + \alpha x^{1/2} - \frac{\alpha^2}{4} \ln x
\]
\[
+ \int_{0}^{s} \frac{1}{2t} \left( r(t) + \frac{\alpha^2}{2} - \frac{1}{8} \right) dt + \ln \left( \frac{G(1 + \alpha)}{(2\pi)^{\alpha/2}} \right)
\]
\[
+ O \left( x^{-1/2} \right),
\]

where the function $r(t)$ is smooth solution to the Painlevé III equation.
Remark. As $s \to 0$, we recovered the large gap asymptotics of the Bessel kernel

$$\ln \det[I - K_x^{\text{Bes}}] = -\frac{1}{4} x + \alpha x^{1/2} - \frac{\alpha^2}{4} \ln x + \ln \left( \frac{G(1 + \alpha)}{(2\pi)^{\alpha/2}} \right) + o(1).$$

Remark. It is interesting to see the appearance of Painlevé III function in the large gap asymptotics

$$\int_0^s \frac{1}{2t} \left( r(t) + \frac{\alpha^2}{2} - \frac{1}{8} \right) dt.$$
Remark. As $s \to 0$, we recovered the large gap asymptotics of the Bessel kernel

$$\ln \det[I - K_{\text{Bes}}^x] = -\frac{1}{4}x + \frac{\alpha^2}{4} \ln x + \ln \left( \frac{G(1 + \alpha)}{(2\pi)^{\alpha/2}} \right) + o(1).$$

Remark. It is interesting to see the appearance of Painlevé III function in the large gap asymptotics

$$\int_{0}^{s} \frac{1}{2t} \left( r(t) + \frac{\alpha^2}{2} - \frac{1}{8} \right) dt.$$
Large gap asymptotics of the determinant of Painlevé II kernel ($\alpha = 0$)
- Integral of Hastings-McLeod solution of Painlevé II.
- Constant: Riemann zeta-function.

Painlevé II, XXXIV-kernel determinant
- Integral of Painlevé XXXIV function.
- Constant: Riemann zeta-function.
Large gap asymptotics of the determinant of Painlevé II kernel ($\alpha = 0$) \textit{Bothner & Its (’14)}
- Integral of Hastings-McLeod solution of Painlevé II.
- Constant: Riemann zeta-function.

Painlevé II, XXXIV-kernel determinant \textit{Dai & Xu (’17)}
- Integral of Painlevé XXXIV function.
- Constant: Riemann zeta-function.
Theorem

Dai, Xu & Zhang ('17)

For $\alpha > -1$, $x > 0$ and $s > 0$, let $K_{x}^{\text{PIII}}$ be the integral operator with kernel $K_{\psi}(u, v; s)$ acting on the function space $L^{2}((0, x))$, we have

$$
\det[I - K_{x}^{\text{PIII}}] = \exp \left( - \int_{0}^{\sqrt{s}} [a(\tau; x/s) - a(\tau; 0)] d\tau \right),
$$

where the function $a(\lambda; s)$ is the smooth solution to the coupled Painlevé III system with the asymptotic behavior as $\tau \to 0^{+}$

$$
a(\tau; x) = \frac{1 - 4\alpha^{2}}{8\tau} + O(\tau^{1+2\alpha}).
$$
Tracy-Widom type formula for Painlevé II, XXXIV-kernel determinant are obtained and expressed in terms of integral of solutions to the Coupled Painlevé II system.

Dai & Xu (’17)
In deriving Painlevé III limiting kernel:

- Asymptotics of the orthogonal polynomials $\pi_n(z)$ with respect to the perturbed Laguerre weight $w(x; t) = x^\alpha e^{-x-t/x}$.

Riemann-Hilbert problem for OPs (Fokas & Its ('92))

- $Y(z)$ is analytic on $\mathbb{C} \setminus [0, \infty)$.
- \[ Y_+(x) = Y_-(x) \begin{pmatrix} 1 & w(x; t) \\ 0 & 1 \end{pmatrix}, \quad x > 0. \]
- \[ Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}. \]
- Solution

\[
Y(z) = \begin{pmatrix} \pi_n(z) & \frac{1}{2\pi i} \int_0^\infty \frac{\pi_n(x)w(x; t)}{x-z} \, dx \\ -2\pi i \gamma_{n-1}^2 \pi_{n-1}(z) & -\gamma_{n-1}^2 \int_0^\infty \frac{\pi_{n-1}(x)w(x; t)}{x-z} \, dx \end{pmatrix}
\]
Proofs

- Painlevé III-kernel determinant
  - Its-Izergin-Korepin-Slavnov Riemann-Hilbert problem
  - Tracy-Widom formulas:
    First undress the IIKS Riemann-Hilbert problem, derive systems of differential equations for the solution to the RH problem and then analyze their Lax-compatibility.
  - Large gap asymptotics
  - Constant: the Painlevé III-kernel reduces to the Bessel kernel.

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Thank you!