Ordinary $q$-difference equations:
From $\phi_1$ to $\phi_1$ and beyond

Simon Ruijsenaars

School of Mathematics
University of Leeds, UK

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1. Introduction

- This lecture is concerned with ordinary linear differential and difference equations with meromorphic coefficients. From the perspective of mathematical physics, it is natural to write them in operator form as $D(z)u(z) = 0$. After a suitable similarity transformation

$$D(z) \rightarrow \mu(z)D(z)\mu(z)^{-1}, \quad \mu \text{ meromorphic},$$

they can then be viewed as Schrödinger equations for a particle on a line or ring (taking $z$ real), with the transformed operator viewed as the Hamiltonian.

- The $N$-variable case yields $N$-particle quantum models. For special coefficient choices, the pertinent ordinary differential/difference equations can be tied to rank-1 classical/quantum groups; these choices admit rank-$N$ generalizations giving rise to $N$ commuting differential/difference operators (quantum integrable systems).
1A. Meromorphic ODEs

Consider a second-order linear ODE with meromorphic coefficients. It can be written as

$$u'' + p(z)u' + q(z)u = 0, \quad p, q \text{ meromorphic} \quad \text{(ODE)}$$

Recall that for a point $z_0 \in \mathbb{C}$ at which $p$ and $q$ are analytic, there exists a 2-dimensional space of solutions of the form

$$\sum_{k=0}^{\infty} c_k(z - z_0)^k,$$

with the convergence radius related to the singularity of $p$ or $q$ that is nearest to $z_0$. Choosing a base $u_1(z), u_2(z)$, it is easy to check that any other solution $u(z)$ can be written as $c_1 u_2(z) - c_2 u_1(z)$, with $c_j$ given by the Wronskian ratio

$$c_j = W(u_j, u) / W(u_1, u_2), \quad j = 1, 2,$$

where

$$W(v, w) \equiv vw' - v'w.$$
Even when $p$ and $q$ are rational, the solutions to (ODE) need not be meromorphic. For example, for the special case

$$u'' + \frac{1}{z} u' = 0,$$

the general solution is

$$u(z) = c_1 + c_2 \ln z.$$

In Section 2 we briefly recall the Frobenius method to solve so-called Fuchsian ODEs. In this case the poles of $p$ and $q$ have order at most 1 and 2, resp. (regular singularities). We illustrate this with some details for the ODE solved by the hypergeometric function $\,_2F_1$.

A transformation $z \rightarrow w(z)$ turns an ODE into another ODE. For the difference equations studied next, this is only true for very special $w(z)$. 
1B. Meromorphic ordinary \( q \)-difference equations

Next, we consider a second-order linear \( q \)-difference equation of the form

\[
f(z)u(q^2z) + g(z)u(qz) + h(z)u(z) = 0, \quad |q| \neq 0, 1, \quad (q\Delta E)
\]

with \( f, g, h \) meromorphic. Now assume that there exists a base \( u_1, u_2 \) of \( \mathbb{C}^* \)-meromorphic solutions over the field \( \mathcal{P}_q \) of \( q \)-periodic \( \mathbb{C}^* \)-meromorphic functions (with \( \mathbb{C}^* := \mathbb{C} \setminus \{0\} \)). Then it is not hard to verify that any other \( \mathbb{C}^* \)-meromorphic solution \( u(z) \) can be written as

\[
u(z) = \mu_1(z)u_2(z) - \mu_2(z)u_1(z),
\]

with \( \mu_j \in \mathcal{P}_q \) given by the ratio of Casorati determinants

\[
m_j(z) = C(u_j, u)(z)/C(u_1, u_2)(z), \quad j = 1, 2,
\]

where

\[
C(v, w)(z) \equiv v(z)w(qz) - v(qz)w(z).
\]
Upon substituting 
\[ z \rightarrow \exp(ix), \]
the functions in \( \mathcal{P}_q \) give rise to elliptic functions with periods \( 2\pi, i\ln q \).

This substitution turns the multiplicative difference equation \((q \Delta E)\) into an additive difference equation

\[ P(x)f(x + 2is) + Q(x)f(x + is) + R(x)f(x) = 0, \]

with \( s = -\ln q \) and \( 2\pi \)-periodic meromorphic coefficients \( P, Q, R \).

More generally, requiring only that \( P, Q, R \) be meromorphic, this equation has a 2-dimensional solution space over the field of meromorphic functions with period \( is \); just as in the multiplicative case, any meromorphic solution \( f(x) \) can be written as a linear combination of two base functions with \( is \)-periodic meromorphic coefficients that are ratios of Casorati determinants.
A transformation $z \rightarrow w(z)$ turns $(q\Delta E)$ into another equation of this type only when $w(z) = \lambda z$ or (restricting to rational coefficients) $w(z) = 1/z$. Thus this class of equations is far more ‘rigid’ than ODEs.

**Key problem** for second-order (additive) difference equations: The solution space is $\infty$-dimensional, and the existence of a base follows from non-constructive arguments (such as the Runge approximation theorem). So what are the ‘simplest’/most ‘natural’ solutions?

For the case that $f, g, h$ in $(q\Delta E)$ are polynomials, there do exist ‘simple’/‘natural’ bases. This will be illustrated for the special case of first-degree polynomials in Section 3; in particular, the basic hypergeometric function $2\phi_1$ solves a $q$-difference equation of this type, which we study in some detail.

In Section 4 we collect some information about more general cases.
2. Fuchsian ODEs

2A. A recap of the general case

Let \( z_0 \) be a regular singularity for

\[
    u'' + p(z)u' + q(z)u = 0, \quad p, q \text{ meromorphic} \quad (\text{ODE})
\]

Thus we have

\[
    p = \alpha(z - z_0)^{-1} + O(1), \quad q = \beta(z - z_0)^{-2} + O((z - z_0)^{-1}), \quad z \to z_0,
\]

with \( \alpha \) and/or \( \beta \) nonzero. Then for \( |z - z_0| < r \), there exists at least one solution of the form

\[
    u = (z - z_0)^\lambda \sum_{n=0}^{\infty} c_n(z - z_0)^n, \quad c_0 \equiv 1.
\]

(The convergence radius \( r \) of the power series is \( \geq \) the distance to the pole of \( p \) or \( q \) nearest to \( z_0 \).)
Indeed, upon substitution of the RHS in (ODE) and comparison of coefficients of \((z - z_0)^{\lambda + k}\), one first gets for \(k = 0\) the indicial equation

\[ \lambda(\lambda - 1) + \lambda \alpha + \beta = 0, \]

with solutions

\[ e_{\pm} = \frac{1}{2}[(1 - \alpha) \pm ((\alpha - 1)^2 - 4\beta)^{1/2}]. \]

Using this for \(k > 0\), the coefficients are determined recursively whenever \(e_{+}\) and \(e_{-}\) do not differ by an integer. For the case

\[ e_{+} = e_{-} + \ell, \quad \ell \in \mathbb{N} := \{0, 1, 2, \ldots\}, \]

one gets one solution with \(\lambda = e_{+}\), but there need not be a second one for \(\lambda = e_{-}\). In that case reduction of order yields a second solution with a logarithmic term. For example, the Euler ODE

\[ z^2 u'' + \alpha zu' + \beta u = 0, \quad \alpha, \beta \in \mathbb{C}, \]

yields solution base \(z^{e_{+}}, z^{e_{-}}\) for \(e_{+} \neq e_{-}\), but for \((\alpha, \beta) = (0, 1/4)\) one gets a base \(z^{1/2}, z^{1/2} \ln z\).
2B. The ODE solved by $2F_1$

The special ODE

$$z(1 - z)u'' + (c - (a + b + 1)z)u' - abu = 0$$

yields regular singularities for $z = 0, 1$, for which

$$(\alpha, \beta)_0 = (c, 0), \quad (\alpha, \beta)_1 = (1 - c + a + b, 0),$$

so that

$$(e_+, e_-)_0 = (1 - c, 0), \quad (e_+, e_-)_1 = (c - a - b, 0).$$

In particular, for $e_-, 0 = 0$ the Ansatz (trial solution)

$$f(z) = 1 + \nu_1 z + \nu_2 z^2 + \cdots$$

yields
\[ z(1 - z)(2\nu_2 + 3 \cdot 2\nu_3 z + 4 \cdot 3\nu_4 z^2 + \cdots) \]
\[ + (c - (a + b + 1)z)(\nu_1 + 2\nu_2 z + 3\nu_3 z^2 + \cdots) \]
\[ - ab(1 + \nu_1 z + \nu_2 z^2 + \cdots) = 0. \]

Thus, vanishing of the coefficient of \( z^k \) gives

\[ \nu_{k+1}[(k + 1)k + (k + 1)c] = \nu_k[k(k - 1) + k(a + b + 1) + ab]. \]

As a result, we get the Gauss series representation

\[ \nu_1 = \frac{ab}{c}, \ldots, \nu_{k+1} = \frac{a(a + 1) \cdots (a + k)b(b + 1) \cdots (b + k)}{(k + 1)!c(c + 1) \cdots (c + k)} \]

for \( _2F_1(a, b; c; z) \), with convergence radius 1. Note that we need to exclude the choices \( c = 0, -1, -2, \ldots \), for which \( e_+,0 = e_-,0 + 1, 2, 3, \ldots \).
The above complex power series Ansatz for the 3 remaining exponents also yields a first-order recurrence for the coefficients, which can be explicitly solved.

The Gauss series solution for $|z| < 1$ can be analytically continued to the cut plane

$$\mathbb{C} \setminus \{z \in [1, \infty)\}.$$ 

There is a logarithmic branch point at $z = 1$ (generically), yielding a Riemann surface with an infinite number of sheets.

The substitution $z \rightarrow w = 1/z$ yields an ODE with a regular singularity at $w = 0$. Then we get an indicial equation with

$$(\alpha, \beta)_\infty = (1 - a - b, ab),$$

so that

$$(e_+, e_-)_\infty = (b, a).$$
3. The $q\Delta E$ solved by $2\phi_1$

In this section we focus on the special $q\Delta E$

$$f(z)u(q^2 z) + g(z)u(qz) + h(z)u(z) = 0, \quad |q| \in (0, 1) \quad (sq\Delta E)$$

with $f, g, h$ linear in $z$:

$$f(z) = q^c - q^{a+b+1} z,$$
$$g(z) = -q^c - q + (q^{a+1} + q^{b+1})z,$$
$$h(z) = q - qz.$$

As we shall see shortly, one of its solutions is

$$2\phi_1(q^a, q^b; q^c; q, z) = \sum_{k=0}^{\infty} \frac{(q^a; q)_k (q^b; q)_k}{(q^c; q)_k (q; q)_k} z^k, \quad |z| < 1,$$

where $c \neq 0, -1, -2, \ldots$. For $q \to 1$ this yields the (termwise) limit

$$\sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k k!} z^k = \, _2F_1(a, b; c; z).$$
To get the $\binom{2}{1}$-ODE from $(sq\Delta E)$ involves more work: We need to substitute on the LHS

$$q^h = (1 + \epsilon)^h = 1 + \epsilon h + \frac{\epsilon^2}{2} h(h - 1) + O(\epsilon^3),$$

$$u((1 + \epsilon)z) = u(z) + \epsilon zu' (z) + \frac{\epsilon^2 z^2}{2} u''(z) + O(\epsilon^3),$$

$$u((1 + \epsilon)^2 z) = u(z) + (2\epsilon + \epsilon^2) zu' (z) + 2\epsilon^2 z^2 u''(z) + O(\epsilon^3),$$

and calculate the coefficients of $\epsilon^0, \epsilon^1, \epsilon^2$. For $\epsilon^0$ and $\epsilon^1$ they vanish. But $\epsilon^2$ yields the coefficient

$$-abzu(z) + z(c - (a + b + 1)z)u'(z) + z^2(1 - z)u''(z),$$

from which we recover the ‘expected’ ODE.
For convergence of the power series we must require \( |z| < 1 \). But we can now use \((sq\Delta E)\) in the form

\[
u(z) = \frac{1}{q(z - 1)} \left(f(z)u(q^2 z) + g(z)u(qz)\right),
\]

to analytically continue to \( |z| < |q|^{-1} \), meeting only a simple pole for \( z = 1 \). This pole is present for all \( a, b, c \in \mathbb{C} \) (with \( c \neq 0, -1, -2, \ldots \)), since the series diverges for \( |z| > 1 \). Continuing recursively to \( |z| < |q|^{-2}, |q|^{-3}, \ldots \), we obtain a meromorphic solution with a sequence of simple poles at \( z = q^{-n}, n \in \mathbb{N} \).

However, we need a second independent \( \mathbb{C}^* \)-meromorphic solution to get a base for the space of all \( \mathbb{C}^* \)-meromorphic solutions (over the field of \( q \)-periodic \( \mathbb{C}^* \)-meromorphic functions). We proceed to address this problem.
The crux is to start (as in ‘Fuchs theory’) from a complex power series Ansatz

\[ u(z) = z^e \sum_{k=0}^\infty c_k z^k, \quad c_0 \equiv 1. \]

This yields (upon cancelling \( z^e \))

\[
(q^c - q^{a+b+1} z) q^{2e}(1 + c_1 q^2 z + \cdots )
+ (q^c + q - (q^{a+1} + q^{b+1}) z) q^{e}(1 + c_1 qz + \cdots )
+ q(1 - z)(1 + c_1 z + \cdots ) = 0.
\]

Vanishing of the constant term yields

\[ q^c q^{2e} - (q^c + q) q^{e} + q = 0. \]

Setting \( y := q^e \), we now get just as in the ODE case the indicial (or characteristic) equation
\[ y^2 - (1 + q^{1-c})y + q^{1-c} = 0. \]

This yields solutions
\[ y_- = 1 \Rightarrow e_- = 0, \]
\[ y_+ = q^{1-c} \Rightarrow e_+ = 1 - c. \]

More generally, vanishing of the coefficients of \( z, z^2, \ldots, z^k \), yields a first-order recurrence
\[
[q^c q^2 e q^{2k} - (q^c + q) q^e q^k - q] c_k
= [q^{a+b+1} q^2 e q^{2k-2} - (q^{a+1} + q^{b+1}) q^e q^{k-1} + q] c_{k-1}.
\]

Since we have normalized the solution by requiring \( c_0 = 1 \), this recurrence determines \( c_1, c_2, \ldots \).
In particular, choosing $e = 0$ and multiplying by $q^{-1}$, we get

$$(1 - q^k)(1 - q^{c+k-1})c_k = (1 - q^{a+k-1})(1 - q^{b+k-1})c_{k-1},$$

which yields the solution $2\phi_1(q^a, q^b; q^c; q, z)$. Choosing next $e = 1 - c$, we get similarly

$$(1 - q^k)(1 - q^{-c+k+1})c_k = (1 - q^{a-c+k})(1 - q^{b-c+k})c_{k-1},$$

yielding a second solution

$$z^{1-c} 2\phi_1(q^{a-c+1}, q^{b-c+1}; q^{2-c}; q, z).$$

(Hence we now need to exclude the choices $c = 2, 3, \ldots$) As before, the $2\phi_1$-factor extends from $|z| < 1$ to a meromorphic function with simple poles at $z = q^{-n}, n \in \mathbb{N}$. 

Simon Ruijsenaars (University of Leeds)  
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Problem: We were aiming for a second $\mathbb{C}^*$-meromorphic solution, but the factor $z^{1-c}$ has a logarithmic branch point at $z = 0$ (generically).

This snag can be obviated as follows. Consider the quotient

$$Q(h; q, z) := \frac{(q^{-h}z; q)_\infty (q^{1+h}z^{-1}; q)_\infty}{(z; q)_\infty (qz^{-1}; q)_\infty}.$$ 

Writing out the infinite products, one easily checks that $Q$ satisfies

$$Q(h; q, qz) = q^h Q(h; q, z).$$

Therefore, the function $Q(h; q, z)z^{-h}$ is $q$-periodic, and so we do obtain a second $\mathbb{C}^*$-meromorphic solution

$$Q(1 - c; q, z) _2\phi_1(q^{a-c+1}, q^{b-c+1}; q^{2-c}; q, z).$$

N. B. The function $Q(h; q, e^{ix})$ is a ratio of theta functions.
By contrast to the ODE case, it is equally easy to obtain solutions corresponding to \( z_0 = \infty \): The Ansatz

\[ u(z) = z^{-e}(1 + d_1 z^{-1} + \ldots) \]

gives, upon multiplication by \( z^{e-1} \),

\[
(q^c z^{-1} - q^{a+b+1} z) q^{-2e} (1 + d_1 q^{-2} z^{-1} + \ldots) \\
-((q^c + q) z^{-1} - q^{a+1} - q^{b+1}) q^{-e} (1 + d_1 q^{-1} z^{-1} + \ldots) \\
+ q(z^{-1} - 1)(1 + d_1 z^{-1} + \ldots) = 0.
\]

Letting \( y := q^{-e} \), the constant term gives the indicial equation

\[
y^2 - (q^{-b} + q^{-a}) y + q^{-a-b} = 0.
\]

Just as for the \( _2F_1 \)-ODE, this entails the exponents

\[
(y_+, y_-) = (q^{-b}, q^{-a}) \Rightarrow (e_+, e_-) = (b, a).
\]
Like in the case of $z_0 = 0$, vanishing of the $z^{-k}$-coefficient yields a first-order recurrence that can be solved, giving rise to a reparametrized $_2\phi_1$. Explicitly, the resulting solutions are

$$z^{-b} _2\phi_1(q^{-c+b+1}, q^b; q^{-a+b+1}; q, q^{c-a-b+1} z^{-1})$$

and

$$z^{-a} _2\phi_1(q^{-c+a+1}, q^a; q^{-b+a+1}; q, q^{c-a-b+1} z^{-1}).$$

The power series converge for $|z| > |q^{c-a-b+1}|$, and now we can use $(sq\Delta E)$ to obtain $\mathbb{C}^*$-meromorphic continuations with simple poles for

$$z = q^{c-a-b+1+m}, \quad m \in \mathbb{N}.$$ 

In this case, the pole sequence is spawned by the zero $\hat{z} = q^{c-a-b-1}$ of the coefficient $f(z)$ of $u(q^2 z)$: As before, it implies the presence of a simple pole at $q^2 \hat{z}$. 
Just as in the case $z_0 = 0$, these solutions associated with $z_0 = \infty$ can be traded for $\mathbb{C}^*$-meromorphic solutions

$$Q(-b; q, z) \, _2\phi_1(q^{-c+b+1}, q^b; q^{-a+b+1}; q, q^{c-a-b+1}z^{-1})$$

and

$$Q(-a; q, z) \, _2\phi_1(q^{-c+a+1}, q^a; q^{-b+a+1}; q, q^{c-a-b+1}z^{-1}).$$

From the general theory it now follows that the meromorphic solution $\, _2\phi_1(q^a, q^b; q^c; q, z)$ is a linear combination of these two solutions with $q$-periodic $\mathbb{C}^*$-meromorphic coefficients.

In fact, however, these coefficients are $z$-independent, and they are explicitly known. They follow after considerable analysis of a Barnes type integral representation for $\, _2\phi_1(q^a, q^b; q^c; q, z)$, introduced by Watson a century ago, cf. Gasper/Rahman, Basic hypergeometric series, Section 4.3.
4. A sketch of the general case

- Assume the coefficients $f, g, h$ in

$$f(z)u(q^2z) + g(z)u(qz) + h(z)u(z) = 0, \quad |q| \in (0, 1), \quad (q\Delta E)$$

are polynomials of degree $N$ with $h(0) \neq 0$. Then the Ansatz

$$u(z) = z^e \sum_{k=0}^{\infty} c_k z^k, \quad c_0 \equiv 1,$$

again leads to a second-degree indicial equation with (generically) two exponent solutions $e_+, e_-$. 

- Vanishing of the coefficients of $z^{e+1}, z^{e+2}, \ldots$, successively determines the coefficients $c_1, c_2, \ldots$, but the $N$th order recurrence that arises cannot be explicitly solved when $N > 1$. (That is, one cannot obtain a ‘closed formula’ for $c_k$, as in the first-order case.)
It follows from general theory that the power series has a
convergence radius determined by the zero of \( h(z) \) that has
minimal distance to the origin.

Using \( (q\Delta E) \) in the form

\[
    u(z) = -\frac{1}{h(z)} \left( f(z)u(q^2z) + g(z)u(qz) \right),
\]

we can continue the power series to a meromorphic function with
simple pole sequences \( z_k q^{-n}, n \in \mathbb{N}, \) with \( z_1, \ldots, z_N \) the (distinct) zeros of \( h(z) \).

Multiplying \( (q\Delta E) \) by \( z^{-N} \) and using the Ansatz

\[
    u(z) = z^{-e} \sum_{k=0}^{\infty} d_k z^{-k}, \quad d_0 \equiv 1,
\]

we can similarly determine two solutions that are analytic for
\( |z| > R \); the power series can then be continued to
\( \mathbb{C}^* \)-meromorphic functions with \( N \) (generically) simple pole
sequences.
Using $Q(h; q, z)$ as before, we can switch to $\mathbb{C}^*$-meromorphic solutions, if need be.

For $q$-difference equations of order $M > 2$ and with polynomial coefficients, one can still proceed analogously. Then one gets an indicial equation that is of degree $M$ in $q^e$, yielding $M$ exponents. From this can obtain a base of $M$ $\mathbb{C}^*$-meromorphic solutions over the field of $q$-periodic $\mathbb{C}^*$-meromorphic functions, and a second base corresponding to power series in $z^{-1}$.

In a recent paper, the second-order theory is applied to $q$-difference equations whose $q \to 1$ limit leads to the Heun ODE (in rational form):