Some $q$-congruences with parameters

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Introduction

It is well known that Ramanujan’s and Ramanujan-type formulae for $1/\pi$, such as

$$
\sum_{n=0}^{\infty} \frac{(4n)(2n)^2}{2^{8n}3^{2n}} (8n + 1) = \frac{2\sqrt{3}}{\pi}, \tag{1.1}
$$

may lead to Ramanujan-type supercongruences. See:


In the example (1.1), the result reads

$$
\sum_{k=0}^{p-1} \frac{(4k)(2k)^2}{2^{8k}3^{2k}} (8k + 1) \equiv p \left( \frac{-3}{p} \right) \pmod{p^3} \quad \text{for any prime } p > 3, \tag{1.2}
$$

where $\left( \frac{-3}{.} \right)$ denotes the Jacobi–Kronecker symbol.
Recently, the author and Zudilin [April 2018, arXiv:1803.07146] prove a $q$-analogue of (1.2) by using a creative microscoping method. Specifically, to prove the following $q$-analogue of (1.2):

$$\sum_{k=0}^{n-1} \frac{(q; q^2)_k^2(q; q^2)_k^2}{(q^2; q^2)_2k(q^6; q^6)_2k}[8k + 1]q^{2k^2} \equiv q^{-(n-1)/2}[n] \left(\frac{-3}{n}\right) \pmod{[n]\Phi_n(q^2)},$$

(1.3)

where $n$ is coprime with 6, we first establish a $q$-congruence with an extra parameter $a$:

$$\sum_{k=0}^{n-1} \frac{(aq; q^2)_k(q/a; q^2)_k(q; q^2)_k^2}{(q^2; q^2)_2k(aq^6; q^6)_k(q^6/a; q^6)_k}[8k + 1]q^{2k^2} \equiv q^{-(n-1)/2}[n] \left(\frac{-3}{n}\right) \pmod{[n](1 - aq^n)(a - q^n)}.$$

(1.4)
Here, the \textit{\textit{q}-shifted factorial} is defined by\[ (a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}) \] for \( n \geq 1 \) and \( (a; q)_0 = 1 \), while the \textit{\textit{q}-integer} is defined as \([n] = [n]_q = 1 + q + \cdots + q^{n-1}\). It is easy to see that we recover (1.3) from (1.4) by taking the limit as \( a \to 1 \).

We shall prove some \(q\)-congruences modulo \( \Phi_n(q)^2 \) by the creative microscoping method. Some of our results confirm the corresponding conjectures in

For any rational number $x$ and positive integer $m$ such that the denominator of $x$ is relatively prime to $m$, we let $\langle x \rangle_m$ denote the least non-negative residue of $x$ modulo $m$. Recall that the $n$-th cyclotomic polynomial $\Phi_n(q)$ is defined as

$$
\Phi_n(q) := \prod_{1 \leq k \leq n \atop \gcd(n,k)=1} (q - \zeta^k),
$$

where $\zeta$ is an $n$-th primitive root of unity. Our first result can be stated as follows.

**Theorem 1.** Let $d$, $n$ and $r$ be positive integers with $\gcd(d, n) = 1$ and $n$ odd. Let $s \leq n - 1$ be a nonnegative integer with $s \equiv \langle -r/d \rangle_n + 1 \pmod{2}$. Then

$$
\sum_{k=s}^{n-1} \frac{(q^r; q^d)_k (q^{d-r}; q^d)_k (q^d; q^{2d})_k q^{dk}}{(q^d; q^d)_k (q^d; q^d)_k+ (q^{2d}; q^{2d})_k} \equiv 0 \pmod{\Phi_n(q)^2}. \quad (1.5)
$$

We point out that, when $n$ is an odd prime, the above theorem confirms Conjecture 6.1 of Guo and Zeng [Acta Arith. 171 (2015), 309–326].
Here we shall give a simple proof of the above theorem by establishing the following more general congruence with two extra parameters.

**Theorem 2.** Let $d$, $n$ and $r$ be positive integers with $\gcd(d, n) = 1$ and $n$ odd. Let $s \leq n - 1$ be a nonnegative integer with $s \equiv \langle -r/d \rangle_n + 1 \pmod{2}$. Then

$$\sum_{k=s}^{n-1} \frac{(aq^r; q^d)_k(bq^{d-r}; q^d)_k(q^d; q^{2d})_kq^{dk}}{(q^d; q^d)_{k-s}(q^d; q^d)_{k+s}(abq^{2d}; q^{2d})_k} \equiv 0 \pmod{(1 - aq^{r+d\langle -r/d \rangle_n})(1 - bq^{d-r+d\langle (r-d)/d \rangle_n})}.$$  

(1.6)

It is easy to see that both $r + d\langle -r/d \rangle_n$ and $d - r + d\langle (r-d)/d \rangle_n$ are divisible by $n$, and so the limit of $(1 - aq^{r+d\langle -r/d \rangle_n})(a - q^{d-r+d\langle (r-d)/d \rangle_n})$ as $a, b \to 1$ has the factor $\Phi_n(q)^2$. On the other hand, all the denominators on the left-hand side of (2.1) as $a, b \to 1$ are relatively prime to $\Phi_n(q)$ since $\gcd(d, n) = 1$ and $n$ is odd. Thus, letting $a, b \to 1$ in (2.1), we are led to (1.5).
Another main result is

**Theorem**

**Theorem 3.** Let $d$, $n$ and $r$ be positive integers with $\gcd(d, n) = 1$ and $n$ odd. Then, modulo $(1 - aq^{r+d\langle-r/d\rangle n})(1 - bq^{d-r+d\langle(r-d)/d\rangle n})$,

$$
\sum_{k=0}^{n-1} \frac{(aq^r; q^d)_k(bq^{d-r}; q^d)_k(x; q^d)_k q^{dk}}{(q^d; q^d)_k(abq^{2d}; q^{2d})_k} \equiv (-1)^{\langle-r/d\rangle n} \sum_{k=0}^{n-1} \frac{(aq^r; q^d)_k(bq^{d-r}; q^d)_k(-x; q^d)_k q^{dk}}{(q^d; q^d)_k(abq^{2d}; q^{2d})_k}.
$$

(1.7)

It is clear that when $a, b \to 1$ and $n$ is an odd prime, the above theorem confirms Conjecture 7.3 of Guo and Zeng [J. Number Theory 145 (2014), 301–316]. Moreover, letting $b = 1/a$ and $x = -1$ in (3.5) and noticing that

$$
\frac{(-1; q^d)_k}{(q^{2d}; q^{2d})_k} = \frac{2}{(q^d; q^d)_k(1 + q^{dk})},
$$

we get the following conclusion.
Corollary

Let \( d, n \) and \( r \) be positive integers with \( \gcd(d, n) = 1 \) and \( n \) odd. Then, modulo \( (1 - a q^{r+d\langle -r/d \rangle n})(a - q^{d-r+d\langle (r-d)/d \rangle n}) \),

\[
\sum_{k=0}^{n-1} \frac{2(q^r; q^d)_k (q^{d-r}/a; q^d)_k q^{dk}}{(q^d; q^d)_k (q^d; q^d)_k (1 + q^{dk})} \equiv (-1)^{\langle -r/d \rangle n}.
\]

Similarly as before, letting \( a \to 1 \) in (1.8), we get

\[
\sum_{k=0}^{n-1} \frac{2(q^r; q^d)_k (q^{d-r}; q^d)_k q^{dk}}{(q^d; q^d)_k (q^d; q^d)_k (1 + q^{dk})} \equiv (-1)^{\langle -r/d \rangle n} \pmod{\Phi_n(q^2)},
\]

which was originally conjectured by Guo, Pan, and Zhang [J. Number Theory 174 (2017), 358–368, Conjecture 3.3].

Using the following proposition, we confirm two conjectures of Guo and Zudilin [March 2018, arXiv:1803.01830, Conjectures 4.15 and 4.16] by Theorem 3.2.
Let $d$, $n$ and $r$ be positive integers with $r < d$ and $\gcd(d, n) = 1$. Then
\[ r + d \langle -r/d \rangle_n = n \langle r/n \rangle_d. \]

The other results of this paper are closely related to a famous supercongruence of Rodriguez-Villegas (see [14, 17]):
\[ \sum_{k=0}^{p-1} \frac{(2k)^2}{16^k} \equiv (-1)^{(p-1)/2} \pmod{p^2} \text{ for any odd prime } p. \quad (1.9) \]

The author and Zeng [11] prove a $q$-analogue of (1.9):
\[ \sum_{k=0}^{p-1} \frac{(q; q^2)_k^2}{(q^2; q^2)_k^2} \equiv (-1)^{(p-1)/2} q^{(1-p^2)/4} \pmod{[p]^2} \text{ for odd prime } p, \quad (1.10) \]

which has been generalized by Guo, Pan, and Zhang and Ni and Pan. We shall give some new parameter-generalizations of (1.10), such as
Theorem 4. Let \( n \) be a positive odd integer. Then

\[
\sum_{k=0}^{n-1} \frac{(aq; q^2)_k (q/a; q^2)_k}{(q^2; q^2)^2_k} x^k \equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \pmod{(1 - aq^n)(a - q^n)}.
\]

Theorem

Let \( n \) be a positive odd integer. Then, modulo \((1 - aq^n)(a - q^n)\),

\[
\sum_{k=0}^{n-1} \frac{(aq; q^2)_k (q/a; q^2)_k}{(q^2; q^2)^2_k} x^k \equiv \sum_{k=0}^{(n-1)/2} \left[\frac{(n - 1)/2}{k}\right]^2 q^{k^2 - nk} (-x)^k (x; q^2)(n-1)/2-k.
\]

Theorem

Let \( n \) be a positive odd integer and let \( 0 \leq s \leq (n - 1)/2 \). Then

\[
\sum_{k=0}^{(n-1)/2} \frac{(aq; q^2)_k (q/a; q^2)_k+s}{(q^2; q^2)^{k+s}(q^2; q^2)^{k+s}} \equiv (-1)^{(n-1)/2} q^{(1-n^2)/4} \pmod{(1 - aq^n)(a - q^n)}.
\]
Proof of Theorem 2

Theorem 2. Let $d$, $n$ and $r$ be positive integers with $\gcd(d, n) = 1$ and $n$ odd. Let $s \leq n - 1$ be a nonnegative integer with $s \equiv \langle -r/d \rangle_n + 1 \pmod{2}$. Then

\[
\sum_{k=s}^{n-1} \frac{(aq^r; q^d)_k (bq^{d-r}; q^d)_k (q^d; q^{2d})_k q^{dk}}{(q^d; q^d)_{k-s} (q^d; q^d)_{k+s} (abq^{2d}; q^{2d})_k} \equiv 0
\]

(mod $(1 - aq^{r+d\langle -r/d \rangle_n})(1 - bq^{d-r+d\langle (r-d)/d \rangle_n})$). \hfill (2.1)

For $a = q^{-r-d\langle -r/d \rangle_n}$, the left-hand side of (2.1) is equal to
\[
\sum_{k=s}^{n-1} \frac{(q-r_1^d; q^d)_k (bq^{d-r}; q^d)_k (q; q^{2d})_k q^{dk}}{(q^d; q^d)_{k-s} (q^d; q^d)_{k+s} (bq^{2d-r_1^d-r}; q^{2d})_k} \\
= \frac{(q-r_1^d; q^d)_s (bq^{d-r}; q^d)_s (q; q^{2d})_s q^{dk}}{(q^d; q^d)_{2s} (bq^{2d-r_1^d-r}; q^{2d})_s} \\
\times 4\phi_3 \left[ \begin{array}{c} q^{(s-r_1^d)}d, bq^{(s+1)d-r}, q^{(s+1/2)d}, -q^{(s+1/2)d} \\ \sqrt{bq}^{(s+1)d-(r_1^d+r)/2}, -\sqrt{bq}^{(s+1)d-(r_1^d+r)/2}, q^{(2s+1)d}; q^d, q^d \end{array} \right],
\]

where \( r_1 = \langle -r/d \rangle_n \) and the basic hypergeometric series \( r+1\phi_r \) is defined as

\[
r+1\phi_r \left[ \begin{array}{c} a_1, a_2, \ldots, a_{r+1} \\ b_1, b_2, \ldots, b_r \end{array} ; q, z \right] = \sum_{k=0}^{\infty} \frac{(a_1; q)_k (a_2; q)_k \cdots (a_{r+1}; q)_k z^k}{(q; q)_k (b_1; q)_k (b_2; q)_k \cdots (b_r; q)_k}.
\]
If \( s > r_1 \), then \( (q^{-r_1d}; q^d)_s = 0 \) and so the right-hand side of (2.2) is equal to 0. If \( s \leq r_1 \), then by the assumption, we have \( s - r_1 \equiv 1 \pmod{2} \), and therefore by Andrews’ terminating \( q \)-analogue of Watson’s formula:

\[
\phi_3 \left[ q^{-n}, a^2 q^{n+1}, b, -b; q, q \right] = \begin{cases} 
0, & \text{if } n \text{ is odd}, \\
\frac{b^n(q, a^2 q^2 / b^2; q^2)_{n/2}}{(a^2 q^2, b^2 q; q^2)_{n/2}}, & \text{if } n \text{ is even},
\end{cases}
\]

we conclude that the right-hand side of (2.2) is still equal to 0. This proves that

\[
\sum_{k=s}^{n-1} \frac{(aq^r; q^d)_k (bq^{d-r}; q^d)_k (q^d; q^{2d})_k q^{dk}}{(q^d; q^d)_k (q^d; q^d)_k - s (abq^{2d}; q^{2d})_k} \equiv 0 \pmod{1 - aq^{r+d\langle -r/d \rangle_n}}.
\]
Since $\langle -r/d \rangle_n \equiv \langle -(d-r)/d \rangle_n \pmod{2}$ for odd $n$, by symmetry we have

$$\sum_{k=s}^{n-1} \frac{(aq^r; q^d)_k (bq^{d-r}; q^d)_k (q^d; q^{2d})_k q^{dk}}{(q^d; q^d)_{k-s} (q^d; q^d)_{k+s} (abq^{2d}; q^{2d})_k} \equiv 0 \pmod{(1 - bq^{d-r+d\langle r-d/d \rangle_n})}.$$ 

The proof of (2.1) then follows from the fact that the polynomials $(1 - aq^{r+d\langle -r/d \rangle_n})$ and $(1 - bq^{d-r+d\langle (r-d)/d \rangle_n})$ are relatively prime.
Proof of Theorem 3

We first establish the following lemma.

**Lemma**

Let $n$ be a positive integer and

$$F_n(x, b, q) = \sum_{k=0}^{n} \frac{(q^{-n}; q)_k (b; q)_k (x; q)_k}{(q; q)_k (bq^{1-n}; q^2)_k} q^k$$

Then

$$F_n(x, b, q) = (-1)^n F_n(-x, b, q). \quad (3.1)$$
Proof. Recall that the $q$-binomial theorem can be stated as follows:

$$
(x; q)_N = \sum_{k=0}^{N} \binom{N}{k} (-x)^k q^{k(k-1)/2},
$$

(3.2)

where the $q$-binomial coefficients $\binom{n}{k}$ are defined by

$$
\binom{n}{k} = \binom{n}{k}_q = \begin{cases} 
\frac{(q; q)_n}{(q; q)_k (q; q)_{n-k}} & \text{if } 0 \leq k \leq n, \\
0 & \text{otherwise.}
\end{cases}
$$
Thus, the coefficient of $x^j$ ($0 \leq j \leq n$) in $F_n(x, b, q)$ is given by

\[
(-1)^j q^{j(j-1)/2} \sum_{k=j}^{n} \frac{(q^{-n}; q)_k (b; q)_k}{(q; q)_k (bq^{1-n}; q^2)_k} \left[ \begin{array}{c} k \\ j \end{array} \right] q^k
\]

\[
= \frac{(-1)^j q^{j(j-1)/2}}{(q; q)_j} \sum_{k=j}^{n} \frac{(q^{-n}; q)_k (b; q)_k}{(q; q)_{k-j} (bq^{1-n}; q^2)_k} q^k
\]

\[
= \frac{(-1)^j q^{j(j+1)/2}(q^{-n}; q)_j (b; q)_j}{(q; q)_j (bq^{1-n}; q^2)_j} \sum_{k=j}^{n} \frac{(q^{j-n}; q)_{k-j} (bq^j; q)_{k-j}}{(q; q)_{k-j} (bq^{2j+1-n}; q^2)_{k-j}} q^{k-j}.
\]

(3.3)

Moreover, letting $a = q^{j-n}$ and $b \rightarrow bq^j$ in Andrews’ $q$-analogue of Gauss’ $2F_1(-1)$ sum:

\[
\sum_{k=0}^{\infty} \frac{(a; q)_k (b; q)_k q^{k(k+1)/2}}{(q; q)_k (abq; q^2)_k} = \frac{(aq; q^2)_\infty (bq; q^2)_\infty}{(q; q^2)_\infty (abq; q^2)_\infty},
\]
we get

\[
\sum_{k=0}^{n-j} \frac{(q^j-n; q)_k (bq^j; q)_k q^k (k+1)/2}{(q; q)_k (bq^{2j+1-n}; q^2)_k} = \frac{(q^{j+1-n}; q^2)_\infty (bq^{j+1}; q^2)_\infty}{(q; q^2)_\infty (bq^{2j+1-n}; q^2)_\infty}
\]

\[
= \begin{cases} 
\frac{(q^{j+1-n}; q^2)_{(n-j)/2}}{(bq^{2j+1-n}; q^2)_{(n-j)/2}}, & \text{if } j \equiv n \pmod{2}, \\
0, & \text{otherwise.}
\end{cases}
\]

Finally, replacing \( b \) and \( q \) by \( b^{-1} \) and \( q^{-1} \) respectively in (3.4) and making some simplifications, we see that the summation on the right-hand side of (3.3) is equal to 0 for \( j \not\equiv n \pmod{2} \). This proves (3.1). \( \square \)
Theorem 3. Let $d$, $n$ and $r$ be positive integers with $\gcd(d, n) = 1$ and $n$ odd. Then, modulo $(1 - a q^{r + d \langle -r/d \rangle_n}) (1 - b q^{d - r + d \langle (r-d)/d \rangle_n})$, 

$$
\sum_{k=0}^{n-1} \frac{(aq^r; q^d)_k (bq^{d-r}; q^d)_k (x; q^d)_k q^{dk}}{(q^d; q^d)_k (ab q^{2d}; q^{2d})_k}
$$

$$
\equiv (-1)^{-r/d}_n \sum_{k=0}^{n-1} \frac{(aq^r; q^d)_k (bq^{d-r}; q^d)_k (-x; q^d)_k q^{dk}}{(q^d; q^d)_k (ab q^{2d}; q^{2d})_k}.
$$

(3.5)

Proof. Let $a = q^{-r-d \langle -r/d \rangle_n}$ or $b = q^{-d+r-d \langle (r-d)/d \rangle_n}$. Then, by the previous lemma, the two sides of (3.5) are equal. This completes the proof. \qed
Some conjectures

For the following remarkable congruence of Sun and Tauraso [22, (1.9)]:

\[ \sum_{k=0}^{p^a-1} \binom{2k}{k} \equiv \left( \frac{p^a}{3} \right) \pmod{p^2}, \tag{4.1} \]

where \( \left( \frac{\cdot}{3} \right) \) is the Legendre symbol. We have [8, Conjecture 4.1]

**Conjecture**

*Let \( n \) be a positive integer. Then*

\[ \sum_{k=0}^{n-1} q^k \left[ \frac{2k}{k} \right] \equiv \left( \frac{n}{3} \right) q^\frac{n^2-1}{3} \pmod{\Phi_n(q^2)}, \]

\[ \sum_{k=0}^{n-1} q^{2k+1} \left[ \frac{2k}{k} \right] \equiv \left( \frac{n}{3} \right) q^\frac{n^2+2n\left( \frac{n}{3} \right)}{3} \pmod{\Phi_n(q^2)} \text{ if } \gcd(n, 3) = 1, \]

\[ \sum_{k=0}^{n-1} \frac{2q^k}{1 + q^k} \left[ \frac{2k}{k} \right] \equiv \left( \frac{n}{3} \right) q^\frac{n^2-n\left( \frac{n}{3} \right)}{3} \pmod{\Phi_n(q^2)} \text{ if } \gcd(n, 3) = 1. \]
Some other conjectures can be found in:
For example,

**Conjecture**

Let $n$ be a positive integer with $n \equiv 1 \pmod{4}$. Then

$$\sum_{k=0}^{(n-1)/2} [4k + 1] \frac{(q; q^2)^3_k}{(q^2; q^2)^3_k} q^{k(n^2-2nk-n-2)/4} \equiv 0 \pmod{\Phi_n(q^2)}.$$

Note that $k(n^2-2nk-n-2)/4$ is a two-variable polynomial of degree 3. Congruences or identities of this form are very rare!

For the case where $q = 1$ and $n = p^r$ is an odd prime power, the above congruence was conjectured by Z.-W. Sun [21, Conjecture 5.9], and is still open.
Thank you!


J. Guillera, WZ pairs and \( q \)-analogues of Ramanujan series for \( 1/\pi \), preprint, April 2018, arXiv:1803.08477.


V.J.W. Guo, Proof of a \( q \)-congruence conjectured by Tauraso, Int. J. Number Theory, DOI: 10.1142/S1793042118501713


