A numerical study of the 3-periodic wave solutions to KdV-type equations

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1 Cooperated with Jianqing Sun and Yingnan Zhang
1 Introduction
   - Periodic wave solutions
   - KdV-type equations
   - Conditions for having $N$-periodic wave solutions

2 Numerical scheme with the Gauss-Newton’s method
   - Problem: $N = 3$
   - Gauss-Newton method
   - Initial guess: quasi-dispersion relation

3 Numerical results

4 Problem and discussion
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Introduction

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Introduction

Periodic wave solutions

\( N \) solitary wave solution:

Localized wave, decays to zero fast, determinant or Pfaff expressions, etc.

Example, the KdV equation \( u_t + 6uu_x + u_{xxx} = 0, \ u = 2(\ln f)_{xx} \)

\[
f = \sum_{j=1}^{N} \exp[\mu_j \eta_j + \sum_{j<k} A_{jk} \mu_j \mu_k]
\]  

(1.1)

\[
\mu_j = 0, 1
\]  

(1.2)

where

\[
\eta_j = \omega_j t + k_j x + \eta_j^0, \quad j = 1, 2, \cdots N.
\]  

(1.3)
A numerical study of the 3-periodic wave solutions to KdV-type equations

Introduction

Periodic wave solutions
$N$-periodic wave solution:

Periodic generalization of $N$-soliton solution or multiple collision of $N$ solitons.

Example, the KdV equation $u_t + 6uu_x + u_{xxx} = 0, u = 2(\ln\theta(\eta; 0|\tau))_{xx}$

$$\theta(\eta; s|\tau) = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_N=-\infty}^{\infty} \exp[i \sum_{j=1}^{N} (m_j + s_j)\eta_j$$

$$-\frac{1}{2} \sum_{j,k=1}^{N} (m_j + s_j)\tau_{j,k}(m_k + s_k),$$

where

$$\eta_j = \omega j + k_j x + \eta_j^0, \quad j = 1, 2, \cdots N.$$

(1.4)
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Introduction

Periodic wave solutions
References

- KdV equation: finite-gap or finite-genus equation.
  Novikov (Funct. Anal. Appl. 1974),
  Dubrovin (Funct. Anal. Appl. 1975),
  Its and Matveev (Funct. Anal. Appl. 1975),

- Inverse scattering method.
  Date (Progr. Theoret. Phys. Suppl. 1976)
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Periodic wave solutions

References

- Algebro-geometric approach.
  Dubrovin (Russian Math. Surveys. 1981),

- Direct method.

- Numerical method.
  Kalla and Klein (Nonlinearity 2012)

- Dose a soliton equation which exhibits $N$-soliton solutions also exhibits $N$-periodic wave solutions?

- All soliton equations which exhibit 3-soliton solutions also exhibit $N$-soliton solutions.

- Dose a soliton equation which exhibits 3-soliton solutions also exhibits $N$-periodic wave solutions?
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Nonlinear evolution equation

$$L(u, u_t, u_x, u_{xx}, \cdots) = 0,$$  \hspace{1cm} (1.6)

Dependent variable transformation

$$u = 2(\ln f)_{xx}$$ \hspace{1cm} (1.7)

$$F(D_t, D_x, \cdots, \lambda)f \cdot f = 0.$$ \hspace{1cm} (1.8)

$F$ is an even function of $D_t, D_x, \cdots$ and $\lambda$ is an integration constant.

$D$-operator

$$D_t^m D_x^n a(t, x) \cdot b(t, x) = \frac{\partial^m}{\partial s^m} \frac{\partial^n}{\partial y^n} a(t + s, x + y)b(t - s, x - y)|_{s=0, y=0},$$

$$m, n = 0, 1, 2, \cdots.$$ \hspace{1cm} (1.9)
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**Introduction**

KdV-type equations

- **KdV equation:** \( u_t + 6uu_x + u_{xxx} = 0, \ (D_x D_t + D_x^4 + \lambda)f \cdot f = 0 \)
- **Boussinesq equation:** \( u_{tt} - u_{xx} + \delta u_{xxxx} + 3\delta (u^2)_{xx} = 0, \ (D_t^2 - D_x^2 + \delta D_x^4 + \lambda)f \cdot f = 0 \)
- **Sawada-Koterra equation:** \( u_t + u_{5x} + 15(uu_{xx} + u^3)_x = 0, \ (D_x D_t + D_x^6 + \lambda)f \cdot f = 0 \)
- **Ito equation:** \( u_{tt} + u_{xxx} + 6u_x u_t + 3uu_{xt} + 3u_{xx} \int u_t dx = 0, \ (D_t^2 + D_x D_x^3 + \lambda)f \cdot f = 0 \)
- **Hietarinta equation:** \( (D_x^3 D_t + \sqrt{3} D_x^2 D_t^2 + D_x D_t^3 + a D_x^2 + b D_x D_t + c D_t^2)f \cdot f = 0 \)
- **KP equation:** \( (4u_t - u_{xxx} - 12uu_x)_x - 3u_{yy} = 0, \ (D_x^4 - 4D_x D_t + 3D_y^2 + \lambda)f \cdot f = 0 \)
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Riemann's $\theta$-function

$$\theta(\eta; s|\tau) = \sum_{m_1} \sum_{m_2} \cdots \sum_{m_N = -\infty}^\infty \exp[i \sum_{j=1}^N (m_j + s_j) \eta_j]$$

$$-\frac{1}{2} \sum_{j,k=1}^N (m_j + s_j) \tau_{j,k}(m_k + s_k)], \quad (1.10)$$

$$\eta_j = \omega_j t + k_j x + \cdots + \eta_j^0, \quad j = 1, 2, \cdots N. \quad (1.11)$$

The $N$-periodic wave solution is obtained via the transformation $u = 2(\ln \theta(\eta; 0|\tau))_{xx}$
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Introduction

Conditions for having \( N \)-periodic wave solutions

Lemma 1.1

For the Riemann’s \( \theta \)-function defined by (1.10), \( \theta(\eta; 0|\tau) \) is a \( N \)-periodic wave solution of the bilinear equation (1.8) if

\[
\sum_{m_1} \sum_{m_2} \cdots \sum_{m_N=-\infty}^{\infty} F\{2i \sum_{j=1}^{N} (m_j - \mu_j/2) \omega_j, 2i \sum_{j=1}^{N} (m_j - \mu_j/2) k_j, \cdots, \lambda\} \\
\times \exp\left[- \sum_{j,k=1}^{N} (m_j - \mu_j/2) \tau_{j,k}(m_k - \mu_k/2)\right] = 0, \tag{1.12}
\]

for all possible combinations \( \mu_1 = 0, 1, \mu_2 = 0, 1, \cdots, \mu_N = 0, 1, \)

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Numerical scheme with the Gauss-Newton’s method

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   - Problem: $N = 3$
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Numerical scheme with the Gauss-Newton’s method

Problem : $N = 3$

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Numerical scheme with the Gauss-Newton’s method

Problem: $N = 3$

\[ H_n(\omega_1, \omega_2, \omega_3, \tau_{12}, \tau_{13}, \tau_{23}, \lambda) = 0, \quad n = 1, 2, \ldots, 8 \] (2.1)

\[ H_n = \sum_{m_1} \sum_{m_2} \sum_{m_3 = -\infty}^{\infty} F\{2i \sum_{j=1}^{3} (m_j - \mu_j/2) \omega_j, 2i \sum_{j=1}^{3} (m_j - \mu_j/2) k_j, \ldots, \lambda\} \times \exp[- \sum_{j,k=1}^{3} (m_j - \mu_j/2) \tau_{j,k}(m_k - \mu_k/2)] \] (2.2)
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Numerical scheme with the Gauss-Newton’s method

Problem: \( N = 3 \)

\[
\begin{align*}
H_1 &\quad (\mu_1 = 0, \mu_2 = 0, \mu_3 = 0) \\
H_2 &\quad (\mu_1 = 1, \mu_2 = 0, \mu_3 = 0) \\
H_3 &\quad (\mu_1 = 0, \mu_2 = 1, \mu_3 = 0) \\
H_4 &\quad (\mu_1 = 0, \mu_2 = 0, \mu_3 = 1) \\
H_5 &\quad (\mu_1 = 1, \mu_2 = 1, \mu_3 = 0) \\
H_6 &\quad (\mu_1 = 1, \mu_2 = 0, \mu_3 = 1) \\
H_7 &\quad (\mu_1 = 0, \mu_2 = 1, \mu_3 = 1) \\
H_8 &\quad (\mu_1 = 1, \mu_2 = 1, \mu_3 = 1)
\end{align*}
\]

Nonlinear least square problem: minimize the objective function

\[
S(x) = \frac{1}{2} \sum_{n=1}^{8} H_n^2(x) = \frac{1}{2} H(x)^T H(x), \tag{2.3}
\]

where \( x = (\omega_1, \omega_2, \omega_3, \tau_{12}, \tau_{13}, \tau_{23}, \lambda)^T \), \( H = (H_1, H_2, H_3, H_4, H_5, H_6, H_7, H_8)^T \).
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Numerical scheme with the Gauss-Newton’s method

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Starting with an initial guess $x^{[0]}$, the Gauss-Newton method is

$$x^{[j+1]} = x^{[j]} - (J^T J)^{-1} J^T H |_{x=x^{[j]}}, \quad (2.4)$$

where $x^{[j]} \ (j \geq 1)$ is the $j$-th iterative output and $J$ is the Jacobian matrix of $H$, i.e.

$$J = \left[ \frac{\partial H_j}{\partial x_i} \right]_{i=1,\ldots,7; j=1,\ldots,8}. \quad (2.5)$$
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Numerical scheme with the Gauss-Newton’s method

Gauss-Newton method

Step 1: Set \( j = 0, \mathbf{x}^{[0]} = (\omega_1^{[0]}, \omega_2^{[0]}, \omega_3^{[0]}, 0, 0, \lambda^{[0]})^T. \)

Solve \( \omega_n^{[0]} (n = 1, 2, 3) \) from the quasi-dispersion relation

\[
F(i\omega_n^{[0]}, ik_n, \ldots, \lambda^{[0]}) = 0. \quad (2.6)
\]

Step 2: Set

\[
\mathbf{x}^{[j+1]} = \mathbf{x}^{[j]} - (J^T J)^{-1} J^T H |_{x = \mathbf{x}^{[j]}}. \quad (2.7)
\]

Step 3: If \( \|\mathbf{x}^{[j+1]} - \mathbf{x}^{[j]}\|_2 > \epsilon \) or \( \|H(\mathbf{x}^{[j+1]})\|_2 > \epsilon, \)

\( j = j + 1, \) go back to Step 2,

else, go to the next step.

Step 4: Take \( \mathbf{x} = \mathbf{x}^{[j+1]} \) as the final result.
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Numerical scheme with the Gauss-Newton's method

Initial guess: quasi-dispersion relation

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Numerical scheme with the Gauss-Newton’s method
Initial guess: quasi-dispersion relation

Dispersion relation
Solitary waves: \( \exp(kx + \omega t + \eta) \)

\[
\begin{align*}
D_x & \leftrightarrow k, \quad D_t \leftrightarrow \omega. \\
F(\omega, k, \cdots, 0) &= 0.
\end{align*}
\] (2.8) (2.9)

Example: KdV equation \((D_xD_t + Dx^4)f \cdot f = 0, \omega + k^3 = 0\)

2-solitons:

\[
f = 1 + \exp(k_1x + \omega_1t + \eta_1^0) + \exp(k_2x + \omega_2t + \eta_2^0) + \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} \exp[(k_1 + k_2)x + (\omega_1 + \omega_2)t + (\eta_1^0 + \eta_2^0)],
\] (2.10)

where \( \omega_i + k_i^3 = 0. \)
Periodic waves: \( \exp(i(\omega t + kx)) \),

\[
D_x \leftrightarrow ik, \quad D_t \leftrightarrow i\omega. \quad (2.11)
\]

\[
F(i\omega, ik, \cdots, \lambda) = 0. \quad (2.12)
\]
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Numerical scheme with the Gauss-Newton’s method

Initial guess: quasi-dispersion relation

\[
\begin{align*}
KdV & \quad -k \omega + k^4 + \lambda = 0, \\
BSQ & \quad -\omega^2 + k^2 + \delta k^4 + \lambda = 0, \\
SK & \quad -k \omega + k^6 + \lambda = 0, \\
Ito & \quad -\omega^2 + k^3 \omega + \lambda = 0, \\
Hietarinta & \quad k^3 \omega + \sqrt{3}k^2 \omega^2 + k \omega^3 - k \omega + \lambda = 0, \\
KP & \quad k^4_x + 4k_x \omega - 3k_y^2 + \lambda = 0.
\end{align*}
\]
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Numerical scheme with the Gauss-Newton’s method

Initial guess: quasi-dispersion relation

Why the dispersion relation could help choose the initial guess? Take $N = 1$,

\[
\tilde{H}_1 = \sum_{m_1 = -\infty}^{\infty} F\{i(2m_1)\omega_1, i(2m_1)k_1, \ldots, \lambda\} \\
\times \exp\left[-\frac{1}{4}(2m_1)^2\tau_{1,1}\right] = 0,
\]

\[
\tilde{H}_2 = \sum_{m_1 = -\infty}^{\infty} F\{i(2m_1 - 1)\omega_1, i(2m_1 - 1)k_1, \ldots, \lambda\} \\
\times \exp\left[-\frac{1}{4}(2m_1 - 1)^2\tau_{1,1}\right] = 0.
\]

A guidance to choose the initial guess

\[
F(i\omega_1, ik_1, \ldots, \lambda) = 0.
\] (2.19)
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Numerical results

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Numerical results

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$\tau_{11}$</th>
<th>$\tau_{22}$</th>
<th>$\tau_{33}$</th>
<th>$\lambda^{[0]}$</th>
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<td>$1.03 \times 2\pi$</td>
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<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
<td>$\tau_{13}$</td>
<td>$\tau_{23}$</td>
<td>$\lambda$</td>
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<td>3.1596</td>
<td>-0.3630</td>
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**Table 1:** Examples of KdV equation
A numerical study of the 3-periodic wave solutions to KdV-type equations

Numerical results

Figure 1:

Figure 2:
A numerical study of the 3-periodic wave solutions to KdV-type equations

Numerical results

Figure 3:

Figure 4:
A numerical study of the 3-periodic wave solutions to KdV-type equations

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<td>$2 \times \frac{2\pi}{100}$</td>
<td>$3 \times \frac{2\pi}{100}$</td>
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<td>$1.27 \times 2\pi$</td>
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<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
<td>$\tau_{13}$</td>
<td>$\tau_{23}$</td>
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<td>0.0643</td>
<td>-0.0008</td>
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Table 2: Examples of Boussinesq equation with $\delta = 1$
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Numerical results

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<td>$1.27 \times 2\pi$</td>
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<td>$\omega_1$</td>
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<td>$\tau_{13}$</td>
<td>$\tau_{23}$</td>
<td>$\lambda$</td>
</tr>
<tr>
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</table>

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</table>

**Table 3:** Examples of Boussinesq equation with $\delta = -1$
### Numerical results

- **Table 4: Examples of SK equation**

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$\tau_{11}$</th>
<th>$\tau_{22}$</th>
<th>$\tau_{33}$</th>
<th>$\lambda^{[0]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times \frac{2\pi}{50}$</td>
<td>$2 \times \frac{2\pi}{50}$</td>
<td>$3 \times \frac{2\pi}{50}$</td>
<td>$0.2 \times 2\pi$</td>
<td>$0.76 \times 2\pi$</td>
<td>$1.57 \times 2\pi$</td>
<td>-1</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
<td>$\tau_{13}$</td>
<td>$\tau_{23}$</td>
<td>$\lambda$</td>
</tr>
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<td>-0.0314</td>
<td>0.7755</td>
<td>0.3190</td>
<td>2.6275</td>
<td>-0.0005</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$k_1$</th>
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<th>$\tau_{22}$</th>
<th>$\tau_{33}$</th>
<th>$\lambda^{[0]}$</th>
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<td>$0.76 \times 2\pi$</td>
<td>$1.57 \times 2\pi$</td>
<td>-1</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
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<td>$\tau_{23}$</td>
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<td>0.0003</td>
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### Numerical results

<table>
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<th>$k_1$</th>
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<th>$\tau_{22}$</th>
<th>$\tau_{33}$</th>
<th>$\lambda^{[0]}$</th>
</tr>
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<tbody>
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<td>$1.03 \times 2\pi$</td>
<td>$1.62 \times 2\pi$</td>
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<td>$\omega_1$</td>
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<td>$\omega_3$</td>
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<td>$\lambda$</td>
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<td>0.1884</td>
<td>6.6600</td>
<td>-0.9032</td>
<td>0.5940</td>
<td>1.3623</td>
<td>0.0289</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$k_1$</th>
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<th>$k_3$</th>
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<th>$\tau_{22}$</th>
<th>$\tau_{33}$</th>
<th>$\lambda^{[0]}$</th>
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<tbody>
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<td>$3.12 \times \frac{2\pi}{10}$</td>
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<td>$1.03 \times 2\pi$</td>
<td>$1.62 \times 2\pi$</td>
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</tr>
<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
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<td>1.4324</td>
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</table>

<table>
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<tr>
<th>$k_1$</th>
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<th>$k_3$</th>
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<th>$\tau_{22}$</th>
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</thead>
<tbody>
<tr>
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<td>$1.73 \times \frac{2\pi}{10}$</td>
<td>$3.12 \times \frac{2\pi}{10}$</td>
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<td>$1.03 \times 2\pi$</td>
<td>$1.62 \times 2\pi$</td>
<td>-1</td>
</tr>
<tr>
<td>$\omega_1$</td>
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<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
<td>$\tau_{13}$</td>
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<td>0.0041</td>
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</table>

**Table 5: Examples of Ito equation**
A numerical study of the 3-periodic wave solutions to KdV-type equations

Numerical results

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<tr>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$\tau_{11}$</th>
<th>$\tau_{22}$</th>
<th>$\tau_{33}$</th>
<th>$\lambda^{[0]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1 \times \frac{2\pi}{10}$</td>
<td>$2 \times \frac{2\pi}{10}$</td>
<td>$3 \times \frac{2\pi}{10}$</td>
<td>$0.46 \times 2\pi$</td>
<td>$1.03 \times 2\pi$</td>
<td>$1.62 \times 2\pi$</td>
<td>$0.1$</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
<td>$\tau_{13}$</td>
<td>$\tau_{23}$</td>
<td>$\lambda$</td>
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<td>2.1998</td>
<td>5.7812</td>
<td>0.0901</td>
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</table>

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
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<th>$\tau_{22}$</th>
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</thead>
<tbody>
<tr>
<td>$1 \times \frac{2\pi}{10}$</td>
<td>$1.6 \times \frac{2\pi}{10}$</td>
<td>$2.5 \times \frac{2\pi}{10}$</td>
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<td>$1.03 \times 2\pi$</td>
<td>$1.62 \times 2\pi$</td>
<td>$0.1$</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
<td>$\tau_{13}$</td>
<td>$\tau_{23}$</td>
<td>$\lambda$</td>
</tr>
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<td>-1.1142</td>
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<td>0.7545</td>
<td>0.1113</td>
<td>6.9029</td>
<td>1.5724</td>
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</tbody>
</table>

**Table 6:** Examples of Hietarinta equation
Numerical results

<table>
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<tr>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
<th>$\lambda^{[0]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2\pi}{10}$</td>
<td>$2 \times \frac{2\pi}{10}$</td>
<td>$3 \times \frac{2\pi}{10}$</td>
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<td>$2 \times \frac{2\pi}{5}$</td>
<td>$3 \times \frac{2\pi}{5}$</td>
<td>1</td>
</tr>
<tr>
<td>$\omega_1$</td>
<td>$\omega_2$</td>
<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
<td>$\tau_{13}$</td>
<td>$\tau_{23}$</td>
<td>$\lambda$</td>
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<td>2.8434</td>
<td>1.5961</td>
<td>0.9225</td>
<td>-0.3456</td>
<td>-0.4759</td>
<td>3.1432</td>
<td>-6.0223</td>
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</tbody>
</table>

<table>
<thead>
<tr>
<th>$k_1$</th>
<th>$k_2$</th>
<th>$k_3$</th>
<th>$l_1$</th>
<th>$l_2$</th>
<th>$l_3$</th>
<th>$\lambda^{[0]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{2\pi}{10}$</td>
<td>$1.73 \times \frac{2\pi}{10}$</td>
<td>$3.12 \times \frac{2\pi}{10}$</td>
<td>$\frac{2\pi}{5}$</td>
<td>$1.73 \times \frac{2\pi}{5}$</td>
<td>$3.12 \times \frac{2\pi}{5}$</td>
<td>1</td>
</tr>
<tr>
<td>$\omega_1$</td>
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<td>$\omega_3$</td>
<td>$\tau_{12}$</td>
<td>$\tau_{13}$</td>
<td>$\tau_{23}$</td>
<td>$\lambda$</td>
</tr>
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<td>2.5546</td>
<td>-5.5904</td>
</tr>
</tbody>
</table>

**Table 7: Examples of KP equation**
A numerical study of the 3-periodic wave solutions to KdV-type equations

Problem and discussion

1 Introduction
   - Periodic wave solutions
   - KdV-type equations
   - Conditions for having $N$-periodic wave solutions

2 Numerical scheme with the Gauss-Newton’s method
   - Problem: $N = 3$
   - Gauss-Newton method
   - Initial guess: quasi-dispersion relation

3 Numerical results

4 Problem and discussion
How about $N = 4, 5, 6 \cdots$