Asymptotics via difference equation methods:
Charlier polynomials

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Outline

1. Charlier polynomials
2. Non-oscillatory region asymptotics
3. An intermediate region
4. At turning points
Charlier polynomials

The Charlier polynomials $C_n^{(a)}$, $a > 0$. Orthogonality

$$\sum_{k=0}^{\infty} C_n^{(a)}(k) C_m^{(a)}(k) \frac{a^k}{k!} = e^a a^n n! \delta_{mn}.$$ 

An explicit expression $C_n^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} \binom{x}{k} k! (-a)^{n-k}$;

Generating function $e^{-aw} (1 + w)^x = \sum_{n=0}^{\infty} C_n^{(a)}(x) \frac{w^n}{n!}$.

- Known zero distribution: uniform distribution in $y$ for $C_n^{(a)}(ny)$, $y \in (0, 1)$.
- With higher accuracy, two turning points $y \sim 1 - 2 \sqrt{\frac{a}{n}}$ and $y \sim 1 + 2 \sqrt{\frac{a}{n}}$.

\[
\begin{align*}
\text{Saturated region:} & \quad y \in (0, 1 - 2 \sqrt{\frac{a}{n}}), \\
\text{band:} & \quad y \in (1 - 2 \sqrt{\frac{a}{n}}, 1 + 2 \sqrt{\frac{a}{n}}), \\
\text{void:} & \quad y \in (1 + 2 \sqrt{\frac{a}{n}}, +\infty).
\end{align*}
\]
Historic remarks

▶ Methods apply to derive asymptotic formulas, such as Darboux’s method and steepest descent method.
▶ 1985, asymptotics for $C_n^{(a)}(x)$ when $x < 0$, by Maejima and Van Assche via probabilistic arguments.
▶ Bo and Wong, 1994, uniform asymptotics of $C_n^{(a)}(ny)$, integral method.
▶ 2001, Dunster, $C_n^{(a)}(x) = n! L_n^{(x-n)}(a)$, differential equation with respect to parameter $a$.
▶ 2010, Ou and Wong, Riemann-Hilbert approach, for $C_n^{(a)}(x)$, global uniformity.
▶ Other methods, generalizations in several respects. · · · · · ·
▶ The polynomials can, in a sense, serve as a touchstone for new tools and techniques developed.
▶ Motivations, Heun’s equation. Focus on difference equation methods.
Non-oscillatory region

Three-term recurrence formula for monic polynomials

\[ xC_n^{(a)}(x) = C_{n+1}^{(a)} + (n + a)C_n^{(a)}(x) + anC_{n-1}^{(a)}(x), \quad n = 0, 1, \ldots, \]

with fixed \( a > 0 \), and initial data \( C_{-1}^{(a)} = 0 \) and \( C_0^{(a)} = 1 \).

A natural re-scaling \( x = ny \) for \( C_n^{(a)}(x) \), and the non-oscillatory region is the unbounded domain away from \( y \in [0, 1] \).

Denote \( a_n = n + a \) and \( b_n = an \), and introduce

\[ C_n^{(a)}(x) = \prod_{k=1}^{n} w_k(x). \]

We see that

\[ w_1(x) = x - a, \quad w_{k+1}(x) = x - a_k - \frac{b_k}{w_k(x)}, \quad k = 1, 2, \ldots. \]

It is observed and proved that

\[ w_k = (x - k) \left\{ 1 + \frac{(1 - a)x - k}{(x - k)^2} + O \left( \frac{1}{n^2} \right) \right\}, \]

uniform for \( k = 1, 2, \ldots, n \), and in \( \text{Dist}(y, [0, 1]) > r \) for \( \forall r > 0 \).
Asymptotic formulas in \( y = x/n \)

Now we have (applying the trapezoidal rule, etc.)

\[
C_n^{(a)}(ny) = n^n \sqrt{\frac{y}{y-1}} \exp \left( -\frac{a}{y-1} \right) \exp \left\{ n \left[ y \log \frac{y}{y-1} - 1 \right] \right\} \\
\times (y - 1)^n \left[ 1 + O \left( \frac{1}{n} \right) \right],
\]

holding uniformly for large \( n \) and \( y \) keeping a constant distance from \([0, 1]\). The logarithms and square roots take principal branches.

Formal derivation might give a formula for fixed \( y \in (0, 1) \), namely,

\[
C_n^{(a)}(ny) \sim 2n^n \sqrt{\frac{y}{1-y}} \exp \left( -\frac{a}{y-1} \right) \exp \left\{ n \left[ y \log \frac{y}{1-y} - 1 \right] \right\} \\
\times (y - 1)^n \cos \left( ny \pi + \frac{\pi}{2} \right).
\]
With extra help from integral method $\Rightarrow$

**Proposition**

*There is a large-\(n\) asymptotic approximation*

\[
C_n^{(a)}(ny) = (-1)^n e^{\frac{a}{1-y}} \frac{\Gamma(n-ny)}{\Gamma(-ny)} + \varepsilon_1,
\]

*holding uniformly in \(|y| < \delta\) with arbitrary constant \(\delta \in (0, 1)\), where*

\[
|\varepsilon_1| \leq \frac{M_1}{n} \left| e^{\frac{a}{1-y}} \frac{\Gamma(n-ny)}{\Gamma(-ny)} \right|,
\]

*with \(M_1\) being a constant.*
Uniform asymptotic approximation away from $y = 1$

For arbitrary positive constant $\delta$, it holds globally uniformly

$$C_n^{(a)}(ny) = (-1)^n e^{\frac{a}{1-y}} \frac{\Gamma(n - ny)}{\Gamma(-ny)} + \varepsilon_1,$$

where

$$|\varepsilon_1| \leq \frac{M_1}{n} \left| e^{\frac{a}{1-y}} \frac{\Gamma(n - ny)}{\Gamma(-ny)} \right|, \quad |y - 1| \geq \delta, \quad n \to \infty.$$
An intermediate region

Write \( C_{n}^{(a)}(x) = (2a)^{\frac{n}{2}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} P_{n}(x) \) \( \Rightarrow \) the symmetric canonical form DE

\[ P_{n+1}(x) - (A_{n}x + B_{n}) P_{n}(x) + P_{n-1}(x) = 0, \]

where

\[ A_{n} = \frac{1}{\sqrt{2a}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} = \frac{1}{\sqrt{n}} \sum_{s=0}^{\infty} \frac{\alpha_{s}}{n^{s}}, \quad B_{n} = -\frac{n + a}{\sqrt{2a}} \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2} + 1\right)} = \sqrt{n} \sum_{n=0}^{\infty} \frac{\beta_{s}}{n^{s}}, \]

with \( \alpha_{0} = \frac{1}{\sqrt{a}}, \alpha_{1} = -\frac{1}{4\sqrt{a}}, \beta_{0} = -\frac{1}{\sqrt{a}}, \beta_{1} = \frac{1}{4\sqrt{a}} - \sqrt{a} \), and \( \beta_{s} = -\alpha_{s} - a\alpha_{s-1} \) for \( s = 1, 2, \ldots \).

Now we introduce a new local variable \( t \),

\[ x = n \left(1 + \frac{t}{\sqrt{n}}\right). \]

Intermediate means \( |t \pm 2\sqrt{a}| \gg 1 \) and \( |t| \ll O(\sqrt{n}) \).
An idea recently brought in by Huang-Cao-Wang, 2017, with modifications here. The symmetric difference equation possesses a pair of non-vanishing asymptotic solutions of the form

\[ P_n(x) \sim \exp \left( \sqrt{n} \phi_{-1}(t) + \phi_0(t) + \sum_{k=1}^{\infty} \frac{\phi_k(t)}{n^{k/2}} \right), \]

where \( \phi_k(t) \), to be determined, are functions independent of \( n \).

\[ t\text{-plane: } y = 1 + \frac{t}{\sqrt{n}} \]
Write
$$x = n \left(1 + \frac{t}{\sqrt{n}}\right) = (n + 1) \left(1 + \frac{t_+}{\sqrt{n+1}}\right) = (n - 1) \left(1 + \frac{t_-}{\sqrt{n-1}}\right).$$

$$t_+ = t - \frac{1}{\sqrt{n}} - \frac{t}{2n} + \cdots \quad \text{and} \quad t_- = t + \frac{1}{\sqrt{n}} + \frac{t}{2n} + \cdots.$$

Substituting $P_n(x) \sim e^{\sum_{k=-1}^{\infty} \frac{\phi_k(t)}{n^{k/2}}}$ into the equation, and equalizing the coefficients of like powers, we have a first order differential equation for each $\phi_k$. For $\phi_{-1}$,
$$e^{-\phi_{-1}'}(t) + e^{\phi_{-1}'}(t) = \frac{t}{\sqrt{a}}.$$

Two choices $\Rightarrow$ two solutions. One is
$$\phi_{-1}(t) = t \log \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a} + C_{-1},$$

such that $e^{\phi_{-1}'}(t) = \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}}$.

Accordingly, and recursively, the equation for $\phi_0(t)$:
$$\phi_0'(t) = \frac{a}{\sqrt{t^2 - 4a}} + \frac{1}{2} \sqrt{t^2 - 4a} - \frac{t}{2(t^2 - 4a)} + \frac{1}{2} C_{-1}.$$
An asymptotic solution

\[ P_n(x) \sim C \exp \left( \sqrt{n} \left[ t \log \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a} \right] - \frac{1}{4} \log(t^2 - 4a) + \frac{t}{4} \sqrt{t^2 - 4a} \right), \]

where \( x = n(1 + \frac{t}{\sqrt{n}}) \), and \( C = C(n, t) = (2a) \frac{n}{2} \frac{\Gamma(n+1)}{\Gamma(n/2)} e^{\sqrt{n}C-1+C_{-1}t+C_0}. \)

Alternatively, \( e^{\phi'_{-1}(t)} = \frac{t+\sqrt{t^2-4a}}{2\sqrt{a}} \implies \) the other asymptotic solution

\[ \tilde{P}_n(x) \sim \tilde{C} \exp \left( -\sqrt{n} \left[ t \log \frac{t - \sqrt{t^2 - 4a}}{2\sqrt{a}} + \sqrt{t^2 - 4a} \right] - \frac{1}{4} \log(t^2 - 4a) - \frac{t}{4} \sqrt{t^2 - 4a} \right). \]

Asymptotically, \( C_n^{(a)}(x) \sim A(x)P_n(x) + B(x)\tilde{P}_n(x) \) as \( n \to \infty. \)

\( A(x) = \left( \frac{\Gamma(x+1)}{a^x} \right)^{1/2}, \quad B(x) = 0, \quad C_{-1} = 0 \) and

\( C_0 = -\frac{3}{4} \log 2 - \frac{1}{4} \log \pi + \frac{a}{2}, \) as a result of matching with the non-oscillatory asymptotics.
Airy-type approximation at turning points

Apply the turning point theory of Wang and Wong. Still, the difference equation is

\[ P_{n+1}(x) - (A_n x + B_n) P_n(x) + P_{n-1}(x) = 0. \]

Differ from Wang and Wong in assumptions:

\[ A_n \sim n^{-\theta} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \quad B_n \sim \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}. \]

Types of asymptotic solutions described by the characteristic equation \( \lambda^2 - (\alpha_0 y + \beta_0) \lambda + 1 = 0 \), \( y = n^\theta x \), having two roots that coincide when \( \alpha_0 y + \beta_0 = \pm 2 \implies \) two turning points \( y \).

However, in the Charlier case, the \( P_n \) term is dominant, and the characteristic equation degenerates to \( \alpha_0 y + \beta_0 = 0 \), \( y = n^{-1} x \), giving raise to a single critical point at \( y = 1 \).

\[ A_n \sim n^{-\frac{1}{2}} \sum_{s=0}^{\infty} \frac{\alpha_s}{n^s}, \quad B_n \sim n^{\frac{1}{2}} \sum_{s=0}^{\infty} \frac{\beta_s}{n^s}. \]
That is why we use $x = n \left( 1 + \frac{t}{\sqrt{n}} \right)$. Two turning point in $t$, $t_c = \pm 2\sqrt{a}$. Solutions of the form

$$Q_n(x) = \chi \left( n^{1/3} \eta + n^{-1/6} \Phi \right) \sum_{k=0}^{\infty} \frac{A_k(\eta)}{n^{k/2}} + n^{-1/6} \chi' \left( n^{1/3} \eta + n^{-1/6} \Phi \right) \sum_{k=1}^{\infty} \frac{B_k(\eta)}{n^{k/2}},$$

where $\chi$ solves the Airy equation, $\eta(t)$ is a conformal mapping at $t = t_c$ such that $\eta(t_c) = 0$.

Steps
To determine $\eta(t)$, $\Phi(t)$ and $A_0(\eta)$, and then (theoretically) determine all coefficients iteratively.
Regarding $\zeta$ and $\mu$ as a free variables, we take a closer look at $\chi(n^{1/3}\zeta + n^{-1/6}\mu)$.

**Lemma**

Assume that $\chi$ solves the Airy equation. Then the equality holds

$$\chi \left( n^{1/3}\zeta + n^{-1/6}\mu \right) = \chi \left( n^{1/3}\zeta \right) X(n; \zeta, \mu) + n^{-1/6} \chi' \left( n^{1/3}\zeta \right) Y(n; \zeta, \mu),$$

where the coefficients

$$X(n; \zeta, \mu) = \sum_{k=0}^{\infty} \frac{X_k(\zeta, \mu)}{n^{k/2}} \quad \text{and} \quad Y(n; \zeta, \mu) = \sum_{k=0}^{\infty} \frac{Y_k(\zeta, \mu)}{n^{k/2}},$$

with

$$X_0(\zeta, \mu) = \frac{1}{2} \left( e^{\sqrt{\zeta}\mu} + e^{-\sqrt{\zeta}\mu} \right), \quad Y_0(\zeta, \mu) = \frac{1}{2\sqrt{\zeta}} \left( e^{\sqrt{\zeta}\mu} - e^{-\sqrt{\zeta}\mu} \right),$$

$$X_k(\zeta, \mu) = \frac{1}{2\sqrt{\zeta}} \int_0^\mu sX_{k-1}(\zeta, s) \left( e^{\sqrt{\zeta}(\mu-s)} - e^{\sqrt{\zeta}(s-\mu)} \right) ds, \quad k = 1, 2, \ldots$$

and

$$Y_k(\zeta, \mu) = \frac{1}{2\sqrt{\zeta}} \int_0^\mu sY_{k-1}(\zeta, s) \left( e^{\sqrt{\zeta}(\mu-s)} - e^{\sqrt{\zeta}(s-\mu)} \right) ds, \quad k = 1, 2, \ldots,$$

in which $\sqrt{\zeta}$ takes the principal branch.
Re-expanding of everything, and substituting into the difference equation, will determine analytic structures. For instance, equalizing the $O(1)$ orders gives

\[ e^{-\eta'(t)\sqrt{\eta(t)}} + e^{\eta'(t)\sqrt{\eta(t)}} = \frac{t}{\sqrt{a}}, \]

remind us of the equation for $\phi_{-1}(t)$.

\[ \text{Solve out} \]
\[ \frac{2}{3} (\eta(t))^{3/2} = t \log \frac{t+\sqrt{t^2-4a}}{2\sqrt{a}} - \sqrt{t^2-4a}, \quad t \in (-\infty, 2\sqrt{a}] \text{ at } t_c = 2\sqrt{a} \text{ appealing to the initial condition } \eta(t_c) = 0. \]

\[ \text{Readily seen that } \eta(t) \text{ is a conformal mapping at } t = t_c. \]

Another solution $\tilde{Q}_n(x)$, Asymptotically,

\[ C^{(a)}_n(x) \sim A(x)Q_n(x) + B(x)\tilde{Q}_n(x) \text{ as } n \to \infty. \]

Coefficients determined by a matching process. For example, at $t = 2\sqrt{a}$,

\[ C^{(a)}_n(x) \sim (2a)^{n/2} \frac{\Gamma(n+1)}{\Gamma(\frac{1}{2})} e^{\frac{a}{2}} \left( \frac{\Gamma(x+1)}{ax} \right)^{1/2} \left( \frac{4a\eta}{t^2-4a} \right)^{1/4} \text{Ai}(n^{1/3}\eta + n^{-1/6}\Phi). \]
Summarize

- Non-oscillatory asymptotics, with the aid of integral methods, the domain of uniformity fills all except the red circle. Connected to the initial data.
- Intermediate region lies in between large circle and small circles. Not connected to the initial data in this case. Matching process needed.
- Turning point asymptotics, with in the small discs. Not connected to the initial data in general. Matching process needed.

\[ t\text{-plane: } y = 1 + \frac{t}{\sqrt{n}} \]

\[ y\text{-plane} \]

\[ \text{via integral method} \]

\[ \text{non-oscillatory region} \]